Efficient broadcasting from several originators in a $d-$ dimensional grid

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Abstract

Given an undirected graph representing a network of processors, and source nodes (called originators) containing messages that must be broadcast to all other nodes we want to find a scheme that accomplishes the task in as small number of time steps as possible. The general problem is known to be NP-complete ([11] unless the structure of the graph is known in advance. Let G be a $d-$dimensional grid, and let $f(d;k)$ be the time needed to broadcast one message in G from k different originators.

Our main results are:

1. Exact values of $f(1;k)$
2. Optimal bounds on $f(2;k)$
3. Bounds on $f(d;k)$.

The results are extensions of earlier results of Farley and Hedetniemi [5] , Van-Scoy and Brooks [12] and Roditty and Shoham [10].

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1 Introduction

1.1 The model

In this paper we deal with the problem of broadcasting in a communication network, which is represented by a connected graph. The graphs we deal with here are d-dimensional grids. In a broadcasting problem $k \geq 1$ vertices of a connected simple graph, called *originators*, initiate a mutual message that should be transmitted to all other vertices of the graph according to the following rules:

1. A vertex may send a message to an adjacent vertex only

2. Time is discrete. At a given time a vertex may either:

   (a) receive a message

   (b) send a message to one neighbor

   (c) be idle

1.2 Previous work

The problem of broadcasting in communication networks was dealt in many papers (see [9] ,[7] or [12]). The problem of broadcasting from several originators, using the above rules, was investigated in [6], [1] , and [2] in which efficient algorithms of broadcasting from several originators in a tree were presented.

Farley and Hedetniemi [5] showed optimal broadcasting of one message from any vertex in a $2-$dimensional grid. Van-Scoy and Brooks [12] extended their result to broadcasting of m messages from a corner of a 2 and 3 dimensional grid. These results were extended also to a $d-$ dimensional grid by Roditty and Shoham [10], where they also showed a broadcasting algorithm from any originator in the $d-$ dimensional grid.

2 Basic Definitions

A $d-$ dimensional grid graph $G_{n_1,n_2,...,n_d}$ is a graph with $n_1 \times n_2 \times ... \times n_d$ vertices ,denoted $G^d(\otimes n_i)$, each labeled with a unique element of $\{(i_1, i_2, ..., i_d) \mid 1 \leq i_j \leq n_j \ , \ 1 \leq j \leq d\}$, and with edges connecting vertices which differ by 1 in exactly one label component. If $n_1 = n_2 = ... = n_d = n$ we denote the grid by $G^d(n)$. In particular, $G^1(n)$ is a path on n vertices.

If $u, v \in G^d(\otimes n_i)$, $u = (a_1, a_2, ..., a_d)$, $v = (b_1, b_2, ..., b_d)$ then,

$$d(u,v) = \sum_{i=1}^{d} |a_i - b_i|$$

Let A be a set of vertices in a connected graph G. Define,

$$d(u, A) = \min\{d(u,v)|v \in A\}.$$

Finally we define $f(d;k)$ to be the minimum time needed to broadcast one message from k different originators in a $d-$dimensional grid, where in our model all the originators know the same message at $t = 0$.

For other basic graph theoretical concepts one may look at [8] and [13].

3 Results

Our main results are:

1. Exact values of $f(1;k)$ (theorem 4.1).
2. Optimal bounds on $f(2;2)$ (theorem 4.3) and $f(2;k)$ (theorem 4.4) .
3. Bounds on $f(d;k)$ (theorem 4.5).

4 Broadcast schemes and lower bounds

4.1 1-dimension grid (Paths)

We shall view the paths as lines with consecutive positive integer values on the vertices.

We start with two lemmas , which are true in any communication network.

The first lemma has been proved in [7] and the second is a consequence of it.

Lemma 4.1 *Let G be a connected graph and u some originator. Then, at time $t \geq 1$ a message can reach at most one vertex at distance t from u.*

Lemma 4.2 *Let G be a connected graph and let $A = \{x_1, x_2, ..., x_k\}$ be a set of any k different originators in G. Then, at time $t \geq 1$ a message can reach at most k vertices at distance t from members of A.*

A notation we are going to use throughout the paper is that of the eccentricity of the graph. It is well known that the eccentricity of a x in respect for G, $ecc(x)$ is the length of the furthest vertex from x in G, i.e.,

$$ecc(x) = \max\{d(u, x)|u \in V(G)\}$$

Proposition 4.1 *Let x be any originator in $G^1(n)$. Let u, v be the end-vertices of the path.*
Then,

$$f(1;1) = \begin{cases} ecc(x) & \text{if } d(u,x) = d(x,v) \\ ecc(x)+1 & \text{otherwise.} \end{cases}$$

Proof: Let $D = ecc(x)$

For the upper bound we introduce the following trivial algorithm: First x transmits the message to its neighbor in the direction of the end-vertex which is at distance D from it. (if both end-vertices are at the same distance, towards one of them, arbitrarily). From $t = 2$ on, both x and its informed neighbor pass the message in the opposite directions from each other, and so on. Clearly if $d(u, x) \neq d(x, v)$ the whole task takes D time units, otherwise it takes $D + 1$ time units. The lower bounds are derived from lemma 4.1.

□

Proposition 4.2 *Let $A = \{x_1, x_2\}$ be any two originators in $G^1(n)$ with $a \leq x_1 < x_2 \leq b$. Let,*

$$D = \max\{d(u, A) | u \in V(G)\}$$

Then,

$$f(1;2) = \begin{cases} D & \text{if the number of vertices at distance } D \text{ from } A \text{ is at most two} \\ D+1 & \text{otherwise.} \end{cases}$$

Proof:

Let $mid_1 = \lfloor \frac{(x_1+x_2)}{2} \rfloor$ and $mid_2 = \lceil \frac{(x_1+x_2)}{2} \rceil$ (mid_1 and mid_2 may be the same vertex)

We divide the path into 4 segments: $(a, x1), (x1, mid_1), (mid_2, x2), (x2, b)$ so there are 4 distances at which D may be achieved, respectfully.

- $mid_1 \neq mid_2$. This case is in fact the case of having paths, $P(u \rightarrow mid_1)$ and $P(mid_2 \rightarrow v)$. Since $d(x_1, mid_1) = d(mid_2, x_2)$ then it follows that if $D = d(x_1, mid_1)$ then x_1 cannot reach mid_2 faster than x_2. Applying the trivial algorithm and using the lower bound of lemma 4.1 we get that $f(1;2)$ is either D or $D+1$ depending on whether D is achieved once or twice, for each originator.

- mid_1 and mid_2 are the same vertex. Consider the case where all four mentioned distance are D. In this case we get a lower bound of $D+1$ from lemma 4.2 and the upper bound by applying the trivial algorithm.

  Otherwise, there are only 2 (or less) vertices at distance D from the two originators. We can easily "assign" to each originator one vertex at distance D and apply the trivial algorithm that finishes after D time units.

$\square$

Following the idea of the last proposition we extend it into k different originators.

Theorem 4.1 *Let $A = \{x_1, x_2, ..., x_k\}$ be any k originators in $G^1(n)$ with $a \leq x_1 < x_2 < ... < x_k \leq b$. Let,*

$$D = \max\{d(u, A) | u \in V(G)\}$$

Then,

$$f(1;k) = \begin{cases} D & \textit{if } D \textit{ is achieved at most twice for every two consecutive originators} \\ D{+}1 & \textit{otherwise.} \end{cases}$$

Proof:

For the upper bound we establish the following algorithm which is a natural generalization of the trivial algorithm for one or two originators. We divide responsibilities between the originators in the following manner: for every pair of consecutive vertices we get two segments defined as in proposition 4.2. Then, x_i starts transmitting in $t = 1$ towards the further between x_{i-1} and x_{i+1} (with $a = x_0$ and $b = x_{k+1}$). It is easily seen that the same bounds of 4.2 hold.

□

4.2 2-dimensional grid

Although the following theorem is a result in [5] we give here our proof in order to keep the paper almost self-contained.

Theorem 4.2 *Let $x = (a, b)$ be any originator in G_{n_1,n_2} . Let $D = ecc(x)$ Then,*

$$f(2;1) = \begin{cases} D & \textit{if there is one vertex at distance } D \textit{ from } x \\ D{+}1 & \textit{if there are 2 vertices at distance } D \textit{ from } x \\ D{+}2 & \textit{otherwise.} \end{cases}$$

Proof: Naturally a vertex at distance D from x should be a corner. We divide the proof into several cases depending upon the number of corners at distance D from x.

Case i: There is exactly one corner at distance D from x.

W.l.o.g. suppose the upper-right corner is the furthest vertex from the originator. Then, the lower bound is obvious. To obtain the upper bound we present the following algorithm:

We start transmitting along the column a. Suppose a vertex (a, b_1) , $b_1 < b$ receives the message at time t from (a, b_1+1). Then it transmits it at time $t+1$ to (a, b_1-1), and at time $t+2$ it transmits the message to $(a-1, b_1)$.

If $b_1 = 1$, then at time $t+1$ it transmits the message to $(a-1, b_1)$.

As one can easily see, the whole procedure terminates within D time units, as required.

Case ii: There are exactly two corners at distance D from x.
The lower bound $D+1$ is given by lemma 4.1. To obtain the upper bound we may assume w.l.o.g. that the corners $(1,1)$ and $(n_1, 1)$ are the furthest vertices from the originator. We apply the same algorithm as in the previous case, however the time needed to accomplish broadcast is now $D+1$.

Case iii: There are four corners at distance D from x.
Applying the former algorithm yields an upper bound of $D+2$. To see that this is also the lower bound we observe that after one time unit, whichever direction we take (up, down, right or left) the new vertex that gets the message from x is now at distance $D+1$ from two corner- vertices, and at distance $D-1$ from the other two. If we could broadcast the message to the corners which are at distance D from x at D more time units (all together $D+1$ time units) it would contradict lemma 4.1. Hence, we need at least $D+2$ time units. $\square$

Before proving our next results we need a definition.

Definition 4.1 *Given a grid, G, a set of originators A, we define the Control Area (CA) of an originator $x_i \in A$ as follows: A vertex u belongs to $CA(x_i)$ if $d(u, A) = d(u, x_i)$ and i is the minimal index for which the equality holds.*

Before presenting our main theorems we need the following essential lemma.

Lemma 4.3 *Let $A = \{x_1, x_2, \cdots, x_k\}$ be a set of k different originators in $G^d(\otimes n_i)$. Then, for each $x_s \in A$, and $u \in CA(x_s)$ all vertices in* **any** *possible path, $P(u \to x_s)$ of length $d(u, x_s)$, belong to $CA(x_s)$.*

Proof: Let $u \in CA(x_s)$ and $w \in P(u \to x_s)$ with $|P(u \to x_s)| = d(u, x_s)$. We claim that $w \in CA(x_s)$.

We have ,

$$d(u, x_s) = d(u, w) + d(w, x_s) \ . \tag{1}$$

Assume the contrary that $\exists r \neq s$ s.t. $w \in CA(x_r)$. Then there are two possibilities:
(i)

$$d(w, x_r) < d(w, x_s) \ . \tag{2}$$

Substituting (2) in (1) yields : $d(u, x_r) < d(u, x_s)$, contradicting the fact that $u \in CA(x_s)$.
(ii)

$$d(w, x_r) = d(w, x_s) \tag{3}$$

and $r < s$. Again substituting (3) into (1) yields, $d(u, x_r) \leq d(u, w) + d(w, x_r) = d(u, w) + d(w, x_s) \leq d(u, x_s)$. Since $r < s$ it would imply $u \in CA(x_r)$. A contradiction .

□

An immediate corollary is:

Corollary 4.1 *Let $A = \{x_1, x_2, \cdots, x_k\}$ be a set of k different originators in $G^d(\otimes n_i)$. Then, for each $1 \leq s \leq k$, $CA(x_s)$ is a connected subgraph of $G^d(\otimes n_i)$.*

Now we present our main algorithm that accomplishes broadcasting from 2 originators in a 2-dimensional grid. The algorithm is highly efficient and in fact takes only 2 time units more than the trivial lower bound. The idea shown here will be then easily adopted to a grid with a higher dimension.

Theorem 4.3 *Let $A = \{x_1, x_2\}$ be a set of any two originators in G_{n_1,n_2} and $D = max\{d(u, A)\}$. Then, either*

$$D \leq f(2;2) \leq D + 2 \tag{4}$$

or

$$D + 1 \leq f(2;2) \leq D + 3 \tag{5}$$

Proof: In order to prove (4) and (5) we prove first:

$$D \leq f(2;2) \leq D + 3 \ . \tag{6}$$

The lower bound is obvious. To obtain the upper bound we start with a BFS search from each of the originators to define $CA(x_1)$ and $CA(x_2)$. Each originator is responsible for broadcasting the message to the vertices in its control area only.

After dividing the grid into two control areas, we choose the main and secondary directions for each of the originators, where the directions are the horizontal and the vertical directions. Within these directions we choose whether we would like to start by broadcasting left, right, up or down, respectively. For the proof of (6) we may assume that the choice is arbitrary, so w.l.o.g suppose that for some originator its main direction is the horizontal-right direction, and its secondary direction is the vertical-up one.

Main Algorithm:

1. **Algorithm for an originator**
   Let $x_1 = (i, j)$. Then, x_1 starts transmitting along the j axis, i.e., at $t = 1$ it sends the message to $(i+1, j)$, and at $t = 2$ it sends the message to $(i - 1, j)$
   At $t = 3$ it starts transmitting on the secondary direction, the i axis, first to $(i, j + 1)$ and then at $t = 4$ to $(i, j - 1)$. Note that some of the transmissions may not take place if $deg_{CA(x_1)}(x_1) \leq 3$.

2. **Algorithm for a vertex located on the same row as the originator**
   Let $u = (i_1, j)$. Upon receiving the message from its neighbor on one side (w.l.o.g., $(i_1 - 1, j)$ it transmits the message to $(i_1 + 1, j)$.
   At the next time unit u transmits the message to $(i_1, j + 1)$ and after that to $(i_1, j - 1)$. (Again some of the transmissions might not occur if $deg_{CA(x_1)}(u) \leq 3$.)

3. **Algorithm for other vertices**
   Each of the remaining vertices will transmit the message, in one direction only, the secondary direction (the vertical in our example), either up or down, depending on the neighbor from which it has received the message.

Proof of correctness of (6) : It is easily seen that since each vertex, except the ones in the j's axis, in $CA(x_1)$, receives the message only from a neighbor on its vertical axis, no vertex receives the message from two different neighbors, which might have caused a collision.

Also, by lemma 4.3 all vertices in the CA's receive the message.

Proof of the upper bound of (6) : Let x be an originator and u a vertex in its control area. We prove here that regardless of the main and secondary directions chosen for x the message will reach u within at most $d(u, x) + 3$ time units.

Let $x = (i, j), u = (i_1, j_1)$. Let $d_x = |i - i_1|$ and $d_y = |j - j_1|$. Suppose the main direction of x is the horizontal direction. The message will reach the vertex (i_1, j) in d_x or $d_x + 1$ time units, depending upon whether x starts transmitting to its left or to its right. Then, within at most one delay in u (which must continue transmitting in the main direction), the message will reach u in at most d_y or $d_y + 1$ additional time units. Hence, we have a total of ,at most, three delays until the message reaches u, namely, u receives the message within at most $d(u) + 3$ time units.

$\square$

For the proof of (4) and (5) we present the way we choose the directions for each originator. We shall choose the directions according to the number of vertices which are at distance D from $A = \{x_1, x_2\}$.

There are two cases:

1. There are at most two vertices, say y_1, y_2, such that $d(y_1, A) = d(y_2, A) = D$, and one of the following possibilities hold:

   (a) $y_1 \in CA(x_1)$ and $y_2 \in CA(x_2)$

   (b) $d(y_1, x_2) > d(y_1, x_1) = D$ and $d(y_2, x_1) = d(y_2, x_2) = D$.

   (c) $d(y_1, x_2) = d(y_1, x_1) = d(y_2, x_1) = d(y_2, x_2) = D$.

2. Either there are more than two vertices at distance D from A, or there are exactly two vertices at distance D from A, say y_1, y_2, such that both of them are at distance D from the same originator, and at a greater distance from the other.

Case (1) will implies (4), while case (2) implies (5).

Case i: Proof of (4)

Case (1)(a)

Note that in this case each originator is responsible for only one vertex at distance D from it. We will then choose directions for each originator according to the location of that vertex. For example if $x_1 = (i, j)$ and $y_1 = (i_1, j_1)$, $i_1 \geq i$, $j_1 \geq j$. Then the directions of x_1 would be main: horizontal-right , secondary: vertical-up.

By choosing the directions in that manner it is easily seen that y_1 receives the message after at most $D + 1$ time units. Now, every other vertex, u will receive the message after at most $d(u) + 3$ time units. But since the other vertices are at distance at most $D - 1$ from x_1 it follows that broadcasting to them will end within at most $D + 2$ time units.

Case (1)(b)

In this case we change the definition of $CA(x_2)$ in a way that all vertices u that are at the same distance from both originators, causing them normally to belong to $CA(x_1)$ will instead be assigned to $CA(x_2)$. In particular, $y_2 \in CA(x_2)$. Thus, we can apply the procedure of the previous case obtaining the same result.

Case (1)(c)

In this case both y_1 and y_2 belong to $CA(x_1)$ so we choose our directions in a way that we still finish broadcasting from x_1 in at most $D + 2$ time units.

We distinguish between the following cases:

(a) : y_1, y_2 are located in the same direction compare to x_1

W.l.o.g. suppose that the directions are: main: horizontal/right secondary: vertical/up. Then we have at most one delay in reaching each of the vertices y_1, y_2.

(b): y_1, y_2 have only one mutual direction compare to x_1

W.l.o.g. assume that the direction is vertical/up. Then this will be the main direction while for the horizontal direction we choose either right or left. Then one of the vertices is reached within at most $D+1$ time units, while the other, in at most $D+2$ time units.

(c): y_1, y_2 are antipodals compare to x_1

W.l.o.g. let, $x_1 = (i,j)$ and $y_1 = (i_1, j_1)$, $i_1 < i$, $j_1 < j$, $y_2 = (i_2, j_2)$, $i_2 > i$, $j_2 > j$.

We shall prove that : $y_1 = (1,1)$ and $y_2 = (n_1, n_2)$.

We have

$$d(y_1, x_1) = d(y_1, x_2) = d(y_2, x_1) = d(y_2, x_2) = D \ . \tag{7}$$

Assume the contrary there exits $v = (i_2, j_2 + 1)$. Then it follows from (7) that $d(x_1, v) = D + 1$ and hence,

$$d(x_2, v) < D \ . \tag{8}$$

Thus from (7) and (8) we get,

$$2D + 1 = d(y_1, v) \leq d(y_1, x_2) + d(x_2, v) < 2D \ ,$$

a contradiction. So $y_1 = (1,1)$ and $y_2 = (n_1, n_2)$.

Now, w.l.o.g. assume the directions of x_1 are , main: vertical/up and secondary: horizontal/left. Then the message will reach y_1 through $w_1 = (i_1, n_1)$ (with at most one delay) and y_2 through $w_2 = (i_1, 1)$ (with at most two delays), since $deg(w_1) = deg(w_2) = 3$.

Case ii: Proof of (5)

If there are at least three vertices at distance D from A then by lemma 4.2 we obtain a lower bound $D+1$. Otherwise, if there are exactly two such vertices in $CA(x_1)$ then by lemma 4.1 we obtain a lower bound $D+1$. On the other hand, since x_2 is at distance at least $D+1$ from both of them, it needs ,again, at least $D+1$ time units to deliver the message to them. So we have proved the lower bound. The upper bound in obtained by the broadcasting algorithm.

This completes the proof of the theorem.

□

Following the idea of the last theorem we present a natural extension of it into k different originators.

Theorem 4.4 *Let* $A = \{x_1, x_2, ..., x_k\}$ *be a set of any* k *originators in* G_{n_1,n_2} *. Let,* $D = max\{d(u, A)\}$*. Then,*

$$D \leq f(2; k) \leq D + 3 \ .$$

Proof: Again the lower bound is obvious. To obtain the upper bound we use the same idea as in theorem 4.3. First we do BFS from each of the originators to determine the CA of each originator. Lemma 4.3 and its corollary 4.1 guarantee that all vertices belong to some CA in a way that each CA is a connected subgraph of G_{n_1,n_2}. We can therefore apply the algorithm presented in theorem 4.3 using arbitrary choices for main and secondary directions.

□

4.3 d-dimensional grid

Using the same arguments we used in the previous theorem we conclude with the following generalization:

Theorem 4.5 *Let* $A = \{x_1, x_2, ..., x_k\}$ *be any* k *originators in* $G^d(n)$*, and let* $D = max\{d(u, A)\}$ *. Then,*

$$D \leq f(d; k) \leq D + 2d - 1 \ .$$

Proof: The lower bound is obvious. To obtain the upper bounds we present the following algorithm: First we do BFS from each of the originators to determine the CA of each originator. Every originator, then, makes an ordering over the d dimensions.

Let $x = (a_1, a_2, ..., a_d)$ be an originator. we define a_1 to be its main direction , a_2 to be the second and so on. within a direction we can choose to go up the numerical value and then down.

Every vertex, upon receiving the message , first passes the message on along the coordinate from which it received the message, and then sends the message using the remaining directions.

For example, if a vertex $(v_1, v_2, ...v_d)$ received the message from a neighbor on the i-th dimension, $(v_1, v_2, .v_i - 1..v_d)$ it passes it on to $(v_1, v_2, .v_i + 1..v_d)$ and afterwards to the vertices $(v_1, v_2, .v_{i+1} + 1..v_d)$ and $(v_1, v_2, .v_{i+1} - 1..v_d)$

on the direction $i+1$ up to the d'th direction.
Let $u = (b_1, b_2, ..., b_d)$ be any vertex belonging to $CA(x)$. At worst case, x and u share no common coordinates.

It takes at most one delay to reach $(b_1, a_2, ..., a_d)$, depending whether $b_1 < a_1$ or the opposite. Then, after at most two more delays the message gets to $(b_1, b_2, ..., a_d)$. It is clear that every vertex u in fact receives the message after at most $d(u,x) + 2d - 1$ time units, so if D is the distance of the furthest vertex from A then, broadcast is complete after at most $D+2d-1$ time units. $\square$

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