Efficient Line Broadcasting in a d-Dimensional Grid

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Abstract

The results of Fujita and Farley in [6] and Kane and Peters in [7] on line broadcasting in paths and cycles are extended into a d-dimensional grid, obtaining optimal algorithms in most cases.

1 Introduction

Efficient broadcasting is a key component to achieve high performance (throughput) from parallel and distributed processing.

The motivation of this work was triggered by our interest in the investigation of how to perform optimal query on a distributed database on diverse MIMD multiprocessor architectures [1]. In [1] scheduling and evaluation of queries with minimal cost on diverse multiprocessor architectures was investigated.

Broadcasting has become popular in recent years with the introduction of new, very fast networks and their extensive use by parallel and distributed systems. Utilizing efficiently the underlying network in a massively parallel/distributed system is a major challenge. Existing solutions for broadcasting and its implementation are either inefficient or too expensive in terms of the host operating system overhead involved. Broadcasting is a crucial component in having efficient message passing in parallel/distributed multiprocessors.

Let $G = G(V,E)$ be a connected graph. A communication network is defined as a connected graph. Given a set $A \subset V(G), A \neq \emptyset$, we define *broadcasting (multicasting)* from A to be the process of transmitting one (many) unit(s) of information from A to all the vertices in $G \setminus A$ ($B \subset (G \setminus A)$). The vertices in A are called *originators*. This is accomplished by a series of transitions over G. The messages are distributed over G and spread through the edges, where each vertex transmits a message to its neighbors upon receiving it regardless the activities in other vertices (beside the vertex that receives the message that has to be idle). This procedure is called *local*.

In *line* broadcasting an informed vertex may call any other vertex using the communication links of any simple path between the two vertices with the restriction that no link is used in more than one call in a given time unit. It is obvious that if local broadcasting is in action then the number of informed vertices can be at most double each time unit, so that at least $\lceil log_2 n \rceil$ time units are needed to broadcast in a network of n vertices. In general it is not possible to inform all vertices in a network in that time. However, Farley [4] showed that there is a minimum-time line broadcast scheme for any originator in any connected network.

The question we deal here is:

How much line broadcasting is needed to complete a minimum-time broadcasting from any arbitrary originator in a given connected graph?

Farley [4] ,which introduced the topic, gave a constructive proof that minimum-time line broadcasting is possible in any tree ,and thus in any connected network. His construction gives an upper bound of $(n-1)\lceil log_2 n \rceil$ on the total length (total number of edges) needed to line broadcasting in a minimum time in any network of n vertices. Kane and Peters [7] determined the value of the minimum total length for minimum-time line broadcasting in any cycle on n vertices. When $n = 2^k$ they gave an exact value, while for other choices of n an upper bound was presented, which is about $\frac{1}{3}$ of Farley's upper bound. Not much work has been done along this line since then (for further information see [7]).

In this paper we extend the results of [6] and [7] to a $d-$dimensional grid. In any case, our upper bound is about $c_1 n + c_2 n^{\frac{1}{d}}$, where $c_1,\ c_2$ are constants which depends on d.

A $d-$dimensional grid graph, $G(n_1, n_2, \ldots, n_d)$, is a graph with $n_1 \times n_2 \times \ldots \times n_d$ vertices, each of them is labeled by a unique element of $\{(i_1, i_2, \ldots, i_d) \mid 0 \leq i_j \leq n_j - 1,\ 1 \leq j \leq d\}$, and with edges connecting vertices which differ by 1 in exactly one label component. If, in particular, we connect vertices which differ by $1 (\mathrm{mod}\ n_i)$ we get a $d-$dimensional "torus". When $d = 2$ this is the well known torus. Denote by $G^d(n)$ a $d-$dimensional grid, $n_i = n,\ i = 1, \ldots, d$. Note that a $d-$dimensional hypercube is a particular case of the $d-$dimensional grid graph, where only two vertices reside in each dimension.

The cost to broadcast a message from a vertex is the number of edges the message passes through. Denote by $F_O^d(n_1, n_2, \ldots, n_d)$ the optimal cost to broadcast a message from a vertex O in $G^d(n_1, n_2, \ldots, n_d)$. When the dimensions are equal to n we denote the optimal cost to broadcast from O by $F_O^d(n)$.

Given an originator and broadcast time t, an optimal broadcasting tree, called hereinafter OBT, is the largest possible tree for which the originator can broadcast a message in time at most t. The OBT for broadcasting time $\log_2 n$ that contains n vertices can be divided into two OBTs each of $\frac{n}{2}$ vertices. Obviously, it can be divided into four OBTs each of $\frac{n}{4}$ vertices (see [3, 5]).

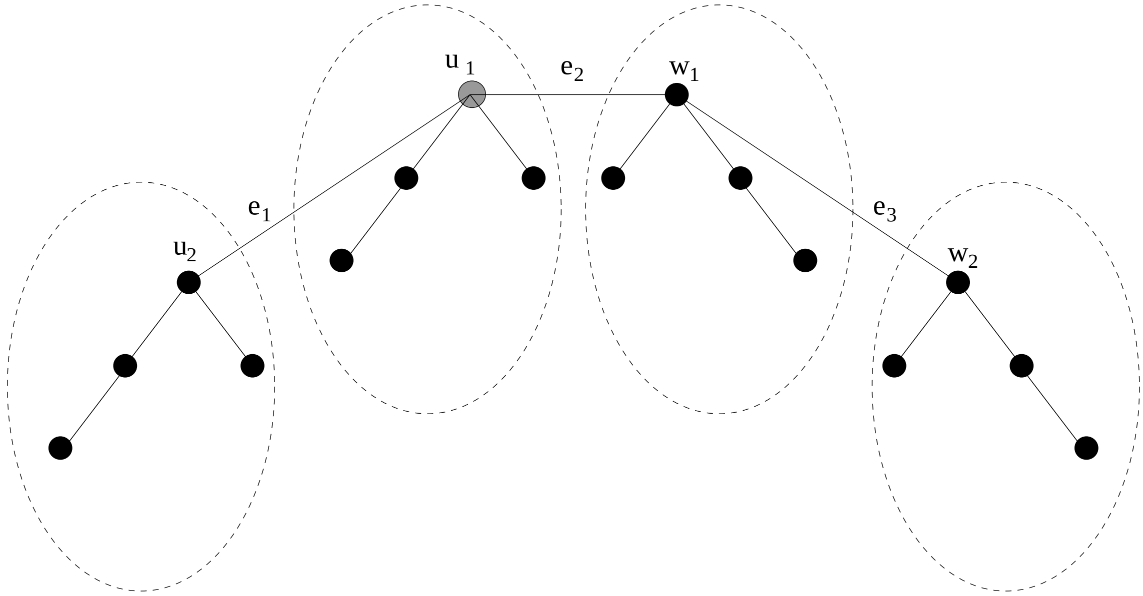

Figure 1: An optimal broadcasting tree (OBT) for $n = 16$.

For other graph theory concepts consult Harary [8] or West [12].

2 Results

In this section we present a broadcast scheme for a d-dimensional grid. We start with a d-dimensional grid that is composed of $n_i = 4$ for $i = 1 \dots d$ (4^d vertices). Then, the broadcast scheme is extended to the case where in each dimension we have $n_i = 4^k,\ k \geq 1$, which is a positive integer. Finally we present the broadcast scheme for any d-dimensional grid.

The main results are:

Theorem 2.1
Let $d \geq 2,\ D = \{(i_1, i_2, \dots, i_d) \mid deg(i_1, i_2, \dots, i_d) = 2d,\ (i_1, i_2, \dots, i_d) \in \{1, 2\}\}$. *Then,*

$$F^d_{\{(i_1,i_2,\dots,i_d)\in D\}}(4) = 4^d - 1.$$

Theorem 2.2
Let $d \geq 3,\ D = \{(i_1, i_2, \dots, i_d) \mid deg(i_1, i_2, \dots, i_d) = d,\ (i_1, i_2, \dots, i_d) \in \{0, 3\}\}$. *Then,*

$$F^d_{\{(i_1,i_2,\dots,i_d)\in D\}}(4) = 4^d + d^2 - 2.$$

Theorem 2.3
Let d and k be integers such that $d \geq 3,\ k \geq 2$. *Then:*

$$4^{dk} + d \times 4^k + kd(2kd - 2d - 3) - d \ \leq\ F^d_{\{(i_1,i_2,\dots,i_d)\mid(i_1,i_2,\dots,i_d)\in\{0,4^k-1\}\}}(4^k) \ \leq$$

$$\frac{(4^d - 1)[4^{dk-1} + \frac{4^{k-1}}{9}(5d^2 - 12)] - \frac{4^{k-1}}{3}(5d^2 - 3)}{4^{d-1} - 1} - \frac{d^2}{9}(5 + 6k) + \frac{1}{3}\ . \tag{1}$$

Remark

1. Observe that the lower bound in Theorem 2.3 is

$$n + dn^{\frac{1}{d}} + O(dk)$$

while the upper bound is about

$$n + d^2 n^{\frac{1}{d}} + O(dk)$$

where $n = 4^{dk}$.

2. The cost of a line broadcasting in a $d-$dimensional "torus" (we connect vertices which differ by 1(mod n) where n is the number of vertices along each dimension) from any vertex is: Let d and k be integers such that $d \geq 2,\ k \geq 1$. Then:

$$\sum_{i=1}^{k} \left[4^{d(i-1)} \times (4^d - 1) \times 4^{k-i}\right] =$$

$$\left(\frac{4^d - 1}{4^d - 4}\right) \times 4^{dk} - \left(\frac{4^d - 1}{4^d - 4}\right) \times 4^k.$$

This bound is about $cn - cn^{\frac{1}{d}}$, where $c = c(d),\ c > 1$.

Observe that for $k = 1$ we obtain an optimal broadcasting.

2.1 Broadcasting in a d-dimensional grid with $n_i = 4, \quad i = 1 \ldots d$

Before proving the main results we present an essential propositions for treatment of $G^2(4)$-grid.

Proposition 2.1

$$F^2_{\{(i,j)\ \mid\ (i,j) \in \{0,3\}\}}(4) = 18.$$

Proof: Since we have exactly 4 vertices of degree two, namely the corners of the $G^2(4)$ grid, we may assume that the originator is $(0,0)$.

Since $|V| = 16$ we have $F^2_{(0,0)}(4) \geq 15$. We have four time units to complete the broadcast and the originator must be active during each time step. Also, $deg((0,0)) = 2$. Therefore, it follows that $F^2_{(0,0)}(4) \geq 17$.

Next we show that $F^2_{(0,0)}(4) \geq 18$. In order to get $F^2_{(0,0)}(4) = 17$ we need to transmit the message from $(0,0)$ to two vertices from $\{\ (0,2),\ (2,0),\ (1,1)\ \}$. In addition, $(0,0)$ has to transmit the message to $(1,0)$ and $(0,1)$. However, in order to complete the broadcast without any additional transmission of length at least two, two vertices out of the four that received the message at time $t = 2$ need to transmit the message to vertex $(1,0)$ or $(0,1)$ (from $(0,0)$ and from $(2,0)$ or from $(0,2)$). Therefore, $F^2_{(0,0)}(4) \geq 18$. In Fig. 2 we present a broadcasting scheme with $F^2_{(0,0)}(4) = 18$ which completes the proof. ■

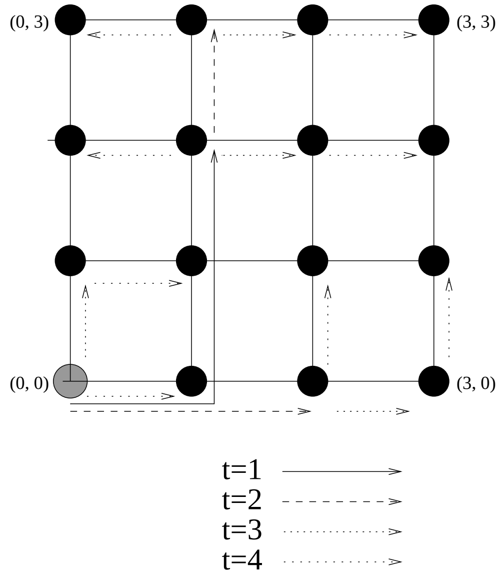

Figure 2: Broadcasting scheme for $G^2(4)$ where the originator is a vertex of degree 2 $((0,0))$.

Proposition 2.2
If $A = \{(0,1),\ (0,2),\ (3,1),\ (3,2),\ (1,0),\ (2,0)\ (1,3),\ (3,2)\}$ *then,*

$$F^2_{\{(i,j)\in A\}}(4) = 16.$$

Proof: Since $|V| = 16$ then $F^2_{\{(i,j)\in A\}}(4) \geq 15$. Moreover, $F^2_{(i,j)\in A}(4) \geq 16$ since the originator has to transmit to 4 vertices and has only three adjacent vertices (degree 3). However, in Fig. 3 we present a broadcasting scheme with $F^2_{(0,0)}(4) = 16$ which completes the proof. ■

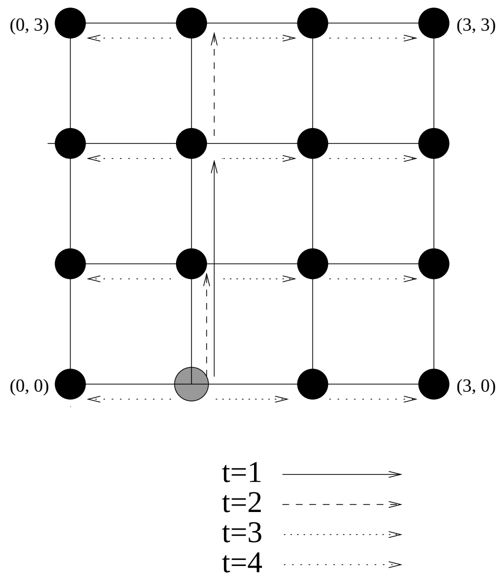

Figure 3: Broadcasting scheme for $G^2(4)$ where the originator is a vertex of degree 3 $((0,1))$.

Proposition 2.3
Let $B = \{(1,1),\ (1,2),\ (2,1),\ (2,2)\}$. *Then,*

$$F^2_{\{(i,j)\in B\}}(4) = 15.$$

Proof: W.l.o.g assume the originator is $(1,1)$.

Since $|V| = 16$, it follows that $F^2_{\{(i,j)\in B\}}(4) \geq 15$. However, in Fig. 4 we present a spanning tree of the 2-dimensional grid which is an OBT for 16 vertices, rooted in $(1,1)$. Therefore, $F^2_{(1,1)}(4) = 15$. ■

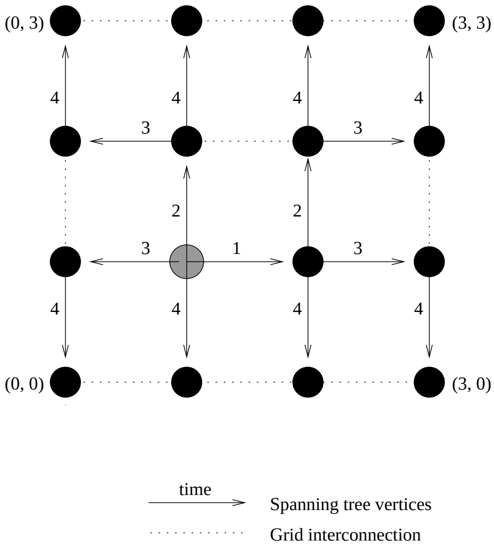

Figure 4: Broadcasting scheme for $G^2(4)$ grid where the originator is a vertex of degree 4 $((1,1))$.

The results for the d-dimensional grid are divided into the following cases:

1. The originator is a vertex of degree $2d$:

Proof of Theorem 2.1: Since we have exactly 2^d vertices of degree $2d$, namely the internal vertices of the $G^d(4)$, we may assume that the originator is $O = (1,1,\ldots,1)$. $|V| = 4^d$, therefore, $F^2_{\{(i_1,i_2,\ldots,i_d)\in D\}}(4) \geq 4^d - 1$. To see that $F^d_{\{(i_1,i_2,\ldots,i_d)\in D\}}(4) \leq 4^d - 1$ we show that $G^d(4)$ contains an OBT as a spanning tree with the originator O. Recall that the OBT for n vertices can be divided into four OBTs each contains $\frac{n}{4}$ vertices (see Fig. 1). We divide $G^d(4)$ into four sub-grids $G^{d-1}(4)$ denoted A_j, $j = 0,1,2,3$ as follows: $A_j = \{(i_1, i_2, \ldots, i_{d-1}, j) \mid i_s \in \{0,1,2,3\},\ s = 1,\ldots,d-1\}$ (for example see Fig. 5). Now to have the OBT, as described in Fig. 1, we choose:

$u_1 = (1,1,\ldots,1,0) \in A_0$

$u_2 = (1,1,\ldots,1,1) \in A_1$

$$w_1 = (1,1,\ldots,1,2) \in A_2$$
$$w_2 = (1,1,\ldots,1,3) \in A_3$$

such that

$$(u_1,u_2) = e_1$$
$$(u_2,w_1) = e_2$$
$$(w_2,w_1) = e_3$$

The vertices $u_1,\ u_2,\ w_1,\ ,w_2$ of the OBT are the roots of the OBTs of the four $G^{d-1}(4)$. In order to build the OBT in each $G^{d-1}(4)$ we divide it into four $G^{d-2}(4)$ parts and proceed with the same algorithm as done in $G^d(4)$. This process is repeated until we left with $G^2(4)$ in which we build the OBT as presented in lemma 2.3.

Now the proof is proceeded by induction on d which is the dimension of the grid that the spanning tree built is an OBT. For $G^3(4)$ we have an OBT since the above spanning tree composed of four spanning trees for each $A_i,\ i = 0,1,2,3$ and three vertices combined those trees (see Fig. 1). By the induction hypothesis we assume that an OBT exists for $G^{d-1}(4)$. The $G^d(4)$ grid is divided into four $G^{d-1}(4)$, each contains the required OBT. The OBTs are connected via e_1, e_2 and e_3 to an OBT which is a spanning tree for the $G^d(4)$. ■

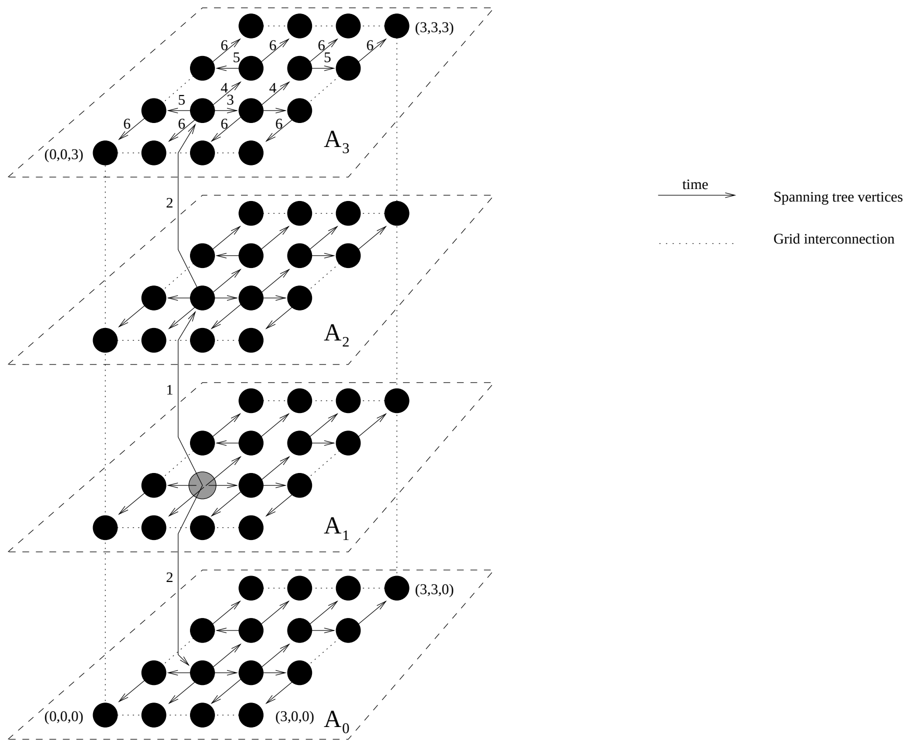

Figure 5: Broadcasting scheme for $G^3(4)$ where the originator is a vertex of degree 6 ((1,1,1)).

2. The originator is a vertex of degree d:

Proof of Theorem 2.2: Since we have exactly 2^d vertices of degree d, namely the corners of $G^d(4)$ we may assume that the originator is $O = (0, 0, \ldots, 0)$.

First we show the lower bound. We have a total time $2d$. So it follows that in order to accomplish broadcasting within that time, once a vertex receives the message it has to be busy to the reset of the time left ([9]). Also $deg(O) = d$. So that we must have at least d additional edges to pass at least twice. An edge that we pass at least twice is called a "double passed" edge. Our main goal is to minimize the number of "double passed" edges. To do this, the originator must send the message to vertices which are originators of some OBT, since doing so the message is spread as quick as possible, no vertex is idle until he finishes transmitting and no edges are "double passed" edges (except for the "double passed" edges in the case where the originator O transmit the message to an originator of some OBT).

Given two vertices u, v, we define a "jump" between u and v to be the way of transmitting a message along the path $v \longrightarrow v$, such that no vertex in that path, but u, receives the message during the "jump". The value of a "jump" is the the number of edged that are "double passed" during that transmitting from u to v. We can decompose $G^d(4)$ into d optimal broadcasting trees with decreasing sizes, starting with a tree of size 2^{2d-1} down to a tree of size 2^d. Hence, we have d "jumps" of total size

$$\sum_{j=3}^{d}\left(2 + (\sum_{i=1}^{j-1} 1) - 1 + 1 + (\sum_{i=1}^{j-1} 1) - 1\right) + 3 = d^2 - 1.$$

Since we have 2^{2d} vertices, the broadcasting cost is $2^{2d} - 1 + d^2 - 1 = 4^d + d^2 - 2$.

To obtain the upper bound we present a broadcasting scheme with cost of at most $4^d + d^2 - 2$.

We divide $G^d(4)$ into four sub-grids $G^{d-1}(4)$ denoted A_j, $j = 0, 1, 2, 3$ as follows: $A_j = \{(i_1, i_2, \ldots, i_{d-1}, j) \mid i_s \in \{0, 1, 2, 3\},\ s = 1, \ldots, d-1\}$ (see for example Fig. 6). Now to have the spanning tree which is an optimal broadcasting tree in which the broadcasting time is as required ($d - 1$ in each OBT), we choose:

$u_1 = (0, 0, \ldots, 0, 0) \in A_0$

$u_2 = (1, 1, \ldots, 1, 1) \in A_1$

$w_1 = (1, 1, \ldots, 1, 2) \in A_2$

$w_2 = (1, 1, \ldots, 1, 3) \in A_3$

such that

$p(u_1, u_2) = P_1$

$p(u_1, w_1) = P_2$

$(w_2, w_1) = e_3$

Where $p(u_i, u_j)$ is the minimum length of a path from u_i to u_j.

The vertices u_1, u_2, w_1, w_2 are the roots of the four optimal broadcasting trees which are spanning trees of $G^{d-1}(4)$. In order to build the spanning tree in each $G^{d-1}(4)$ we divide it into four $G^{d-2}(4)$ parts and proceed the same algorithm as done in $G^d(4)$. This process is repeated until we left with $G^2(4)$ in which we build the spanning tree as presented in lemma 2.1 and in lemma 2.3.

The cost to transmit a message from v to u in a d-dimensional grid is composed of four parts, denoted $C_i(d)$, $i = 1, 2, 3, 4$.

The cost to broadcast within $G^d(4)$ is computed as follows:

$t = 1$: The transmission at time $t = 1$ is done from u_1 to w_1 using the path P_2. The length of P_2 is $C_1(d) = 2 + (d - 1)$.

$t = 2$:

The transmission at time $t = 2$ is done n two parallel paths as follows: path P_2.

1. Transmission from u_1 to u_2 using the path P_1. Namely, transmission to a internal vertex in A_1. The length of P_1 is $C_2(d) = 1 + (d - 1)$.
2. Transmission from w_1 to w_2 using e_3, namely transmission to an internal vertex in A_3. The cost of this transmission is $C_3(d) = 1$.

$3 \leq t \leq 2d$: We broadcast the message in parallel within each A_i.

The broadcast within the A_i, in which the originator is an internal vertex of degree $2(d-1)$ (u_2, w_1 and w_2, respectively), is done as described in lemma 2.1 (note that each A_i is a $G^{d-1}(4)$ grid). Therefore, the cost to broadcast in A_1, A_2, A_3 is $C_4(d) = 3 \times (4^{(d-1)} - 1)$.

The broadcast within A_1 (which is a $G^{d-1}(4)$ grid) from the originator $(0, 0, \ldots, 0)$ is done by dividing it into four $G^{d-2}(4)$ sub-grids, and again broadcasting in each $G^{d-2}(4)$ as in the above. This process is repeated until we get $G^2(4)$. The cost to broadcast within $G^2(4)$ is 18, as described in lemma 2.1.

Therefore,

$$F^d_{(0,0,\ldots,0)}(4) \leq \sum_{i=3}^{d} \left[\left(\overbrace{2 + (i-1)}^{C_1(i)} \right) + \left(\overbrace{1 + (i-1)}^{C_2(i)} \right) + \overbrace{1}^{C_3(i)} + \overbrace{3\left(4^{(i-1)} - 1\right)}^{C_4(i)} \right]$$

$$+ \overbrace{18}^{F^2_{(0,0)} \text{ (after the last iteration)}} = 4^d + d^2 - 2$$

■

2.1.1 Example:

For $G^3(4)$ we have $F^3_{(0,0,0)}(4) = 71$. We get the upper bound, as presented in Fig. 6 by dividing the $G^3(4)$ grid into four $G^2(4)$ grids, denoted A_1, A_2, $, A_3$, A_4. The cost to broadcast from $(0,0,0)$ is the cost to transmit the message to $(1,1,2) \in A_2$, to $(1,1,1) \in A_1$ and to $(1,1,3) \in A_3$ plus the cost to broadcast within each $G^2(4)$ grid. Hence, $F^3_{(0,0,0)} = 4 + 3 + 1 + 3 \times 15 + 18 = 71$.

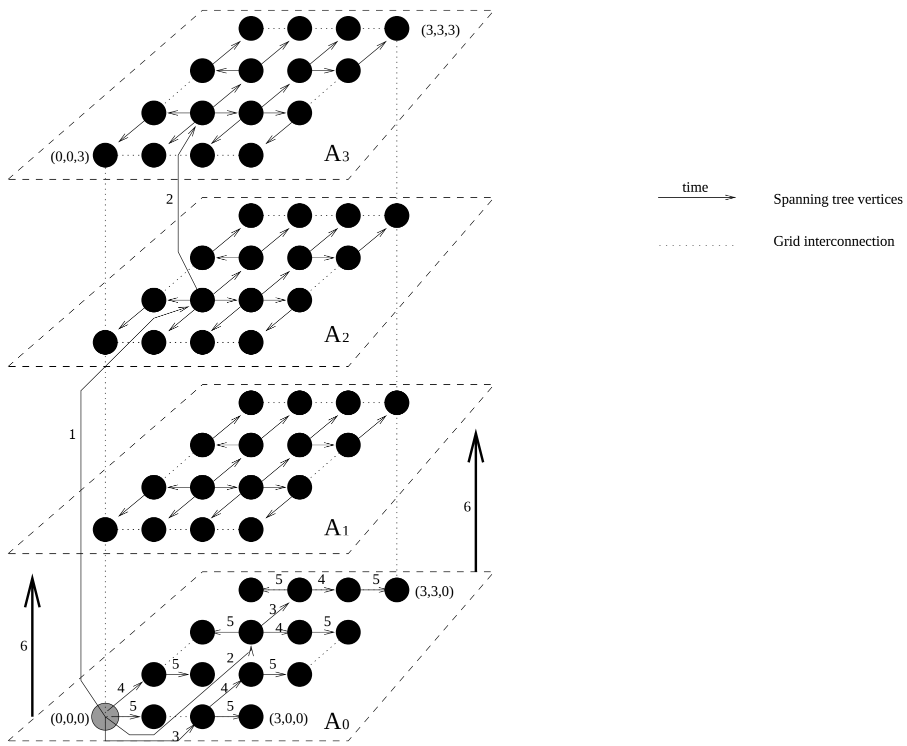

Figure 6: Broadcasting scheme for $G^3(4)$ grid where the originator is a vertex of degree 3 ((0,0,0)).

2.2 Broadcasting in the d-dimensional, $d \geq 2$, grid with $n_i = 4^k$, $i = 1 \ldots d$, and k a positive integer

In this section we present a broadcast scheme for broadcasting one message from a corner vertex in a d-dimensional grid, $G^d(4^k)$.

Proof of Theorem 2.3:

Again, the broadcast is done from a corner vertex. Since we have exactly 2^d vertices of degree d, namely the corners of the $G^d(4^k)$, we may assume that the originator, O, is $(0,0,\ldots,0)$.

First we show the lower bound. We have a total time $2dk$. Since $|V| = 4^{dk}$ it follows that in order to accomplish broadcasting within that time, once a vertex receives the message it has to be busy to the reset of the time left ([9]). Also $deg(O) = d$. So that we must have at least $2dk - d$ "double passed edges" (see proof of Theorem 2.2). A vertex, say u, that receives the message at time t, $1 < t < 2kd - 2d$ must have at least $2dk - t - 2d$ additional "double passed edges" since $deg(u) \leq 2d$. Therefore, it follows that the minimum number of "double passed edges" is $(2dk - d) + \sum_{t=2}^{2kd-2d-1}(2dk - t - 2d)$. Hence, $F^d_{(0,0,\dots,0)}(4) \geq 4^{dk} - 1 + kd(2kd - 2d - 1) + 1$. Moreover, the distance from the originator to the antipodal vertex is $d \times (4^k - 1)$. However, we have only $2dk$ time units to complete the broadcast. Therefore, it follows that $F^d_{(0,0,\dots,0)}(4^k) \geq 4^{dk} - 1 + kd(2kd - 2d - 1) + 1 + d \times (4^k - 1) - 2dk = 4^{dk} + d \times 4^k + kd(2kd - 2d - 3) - d$.

To obtain the upper bound we present a broadcasting scheme with cost that yields the upper bound in (1). The idea of the proof is based upon a reduction of broadcasting in $G^d(4^k)$ into broadcasting in $G^d(4^{k-1})$.

We look at $G^d(4^k)$ as a grid $G^d(4)$ in which each vertex is a $G^d(4^{k-1})$. We denote the "new vertices" by $A^1_{a_1,a_2,\dots,a_d}, a_i \in \{0,1,2,3\}$ where $|\{A^1_{a_1,a_2,\dots,a_d}\}| = 4^d$. The same devision is done with each $G^d(4^{k-1})$ where here the "vertices" are $G^d(4^{k-2})$, denoted $A^2_{a_1,a_2,\dots,a_d}$ ($|\{A^2_{a_1,a_2,\dots,a_d}\}| = 4^d \times 4^d$) and so on until each "vertex" is a grid $G^d(4)$ ($|\{A^{k-1}_{a_1,a_2,\dots,a_d}\}| = 4^{d(k-1)}$).

Broadcasting Scheme:

Now starting from $O = (0,0,\dots,0)$ we shell describe the broadcasting scheme. The first stage of the procedure is to multicast the message from O to the vertices:

$$\{S^1_{i_1,i_2,\dots,i_d},\ i_j \in \{a_j 4^{k-1} + 4^{k-2} + 4^{k-3} + \ldots + 4^0\},\ j\ \in \{b_j 4^{k-1} + 4^{k-2} + 4^{k-3} + \ldots + 4^0\},$$
$$a_j, b_j \in \{0,1,2,3\},\ j = 1,2,\dots d\}$$

except the vertex O. Observe that $|\{S^1_{i_1,i_2,\dots,i_d}\}| = 4^d - 1$.

Now the broadcast is done in each member of $A^1_{a_1,a_2,\dots,a_d}$, but $A^1_{0,0,\dots,0}$. W.l.o.g we shell describe the procedure in $A^1_{1,0,\dots,0}$ where the originator is $S^1_{1,0,\dots,0}$. Recall that $A^1_{1,0,\dots,0}$ is divided into 4^k d-dimensional grids such that each vertex is a $G^d(4^{k-2})$. The originator $S^1_{1,0,\dots,0}$ multicast the message to

$$\{S^2_{i_1,i_2,\dots,i_d},\ i_j \in \{a_j 4^{k-2} + 4^{k-3} + \ldots + 4^0\},\ j\ \in \{b_j 4^{k-2} + 4^{k-3} + \ldots + 4^0\},$$
$$a_j, b_j \in \{0,1,2,3\},\ j = 1,2,\dots d\}$$

which are the originators of each such $G^d(4^{k-2})$. We continue this procedure until we get that each "vertex" is a grid $G^d(4)$. The broadcast within $G^d(4)$ is done as described in Theorem 2.1.

The broadcasting scheme within $A^1_{0,0,\dots,0}$ is as described above where $A^1_{0,0,\dots,0}$ is a grid $G^d(4^{k-1})$ and the originator is $O = (0,0,\dots,0)$. The broadcast in $A^{k-1}_{0,0,\dots,0}$ is done as in Theorem 2.2.

At the end of the procedure all the 4^{dk} vertices receive the message since this procedure is repeated $k-1$ times until we get $G^d(4)$. The duration of each step is $2d$ time units (multicast to 4^d vertices). The broadcast within $G^d(4)$ is done in $2d$ time units (lemma 2.1). Therefore, the procedure is finished within $2dk$ time units as required.

The broadcasting cost is calculated as follows:
Denote by $C^i_{a_1,a_2,\ldots,a_d}$ the cost to multicast within $A^i_{a_1,a_2,\ldots,a_d}$ where the originator is a vertex of degree $2d$, excluding $A^i_{0,0,\ldots,0}$, where the originator is a vertex of degree d.

The cost to broadcast in the different stages is presented in the following table:

4^d-1	Number of transmissions within $A^i_{a_1,a_2,\ldots,a_d}$.
4^{k-i}	Length of each transmission within $A^i_{a_1,a_2,\ldots,a_d}$.
$C^i_{a_1,a_2,\ldots,a_d}$	$(4^d-1)\times 4^{k-i}$
$4^{d(i-1)}-1$	Number of grids of cost $C^i_{a_1,a_2,\ldots,a_d}$.
4^d+d^2-2	Number of transmissions within $A^i_{0,0,\ldots,0}$.
4^{k-i}	Length of each transmission within $A^i_{0,0,\ldots,0}$.
$2d^2\sum_{j=1}^{k-i}4^{j-1}$	The cost to multicast within $S^i_{a_1,a_2,\ldots,a_d}$.
$4^{d(k-1)}-1$	Number of $G^2(4)$ grids where their broadcasting cost is $C^k_{a_1,a_2,\ldots,a_d}$.
4^d-1	Number of transmissions within $A^k_{a_1,a_2,\ldots,a_d}$ ($C^k_{a_1,a_2,\ldots,a_d}$).
4^d+d^2-2	Number of transmissions within $A^k_{0,0,\ldots,0}$ which is $G^2(4)$.

Therefore,

$$F^d_{\{(i_1,i_2,\ldots,i_d)|(i_1,i_2,\ldots,i_d)\in\{0,4^k-1\}\}}(4^k)\le$$

$$\sum_{i=1}^{k-1}\left[(4^{d(i-1)}-1)\times(4^d-1)\times 4^{k-i}+(4^d+d^2-2)\times 4^{k-i}+2d^2\sum_{j=1}^{k-i}4^{j-1}\right]+$$

$$(4^{d(k-1)}-1)\times(4^d-1)+(4^d+d^2-2)$$

$$=\frac{(4^d-1)[4^{dk-1}+\frac{4^{k-1}}{9}(5d^2-12)]-\frac{4^{k-1}}{3}(5d^2-3)}{4^{d-1}-1}-\frac{d^2}{9}(5+6k)+\frac{1}{3}\,.$$

■

2.3 Final Remark

A generalization for a d-dimensional grid where n is any positive integer is obtained as follows:

First we find the minimal positive integer k such that $4^k\ge n$. If $4^k=n$ the broadcast is done as in Theorem 2.3. Otherwise, similar to the broadcasting scheme presented in Theorem 2.3 we divide the d-dimensional grid into sub-grids such that each vertex, excluding on the grid surface, resembles a grid $G^d(4^{k-2})$. A vertex resides on the grid surface resembles a grid $G^d(n-4^{k-1})$. The broadcasting within the vertices excluding on the grid surface is done as in Theorem 2.3. For vertices reside on the surface we continue the process until we get a grid $G^d(a)$, $a\le 4$. The broadcast within $G^d(a)$ is done as described in Proposition 2.1.

Finally, concerning Theorem 2.3 we conjecture:

Conjecture:

The value of $F^d_{\{(i_1,i_2,\ldots,i_d)|(i_1,i_2,\ldots,i_d)\in\{0,4^k-1\}\}}(4^k)$ is about $n+d^2n^{\frac{1}{d}}+dk$.

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