Broadcasting Multiple Messages in Trees

A. Averbuch Y. Roditty B. Shoham
Department of Computer Science
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, Tel Aviv 69978
Israel

e-mail: jr@math.tau.ac.il

Abstract

In this paper simultaneous broadcasting from several originators in a tree, is presented.

1 Introduction

Efficient broadcasting is a key component in achieving high performance (throughput) from parallel and distributed processing.

Broadcasting has become popular in recent years with the introduction of new, very fast networks and their extensive use by parallel and distributed systems. Utilizing efficiently the underlying network in a massively parallel/distributed system is a major challenge. Existing solutions for broadcasting and its implementation are either inefficient or too expensive in terms of the host operating system overhead involved. Broadcasting is a crucial component in having efficient message passing in parallel/distributed multiprocessors.

We define broadcasting from an *originator(s)* source to be the process of passing one (many) unit(s) of information from that source to a set of predefined destinations which are connected via a network. This is accomplished by a series of transitions over the network. The messages are distributed over the network and spread using the communication network, where each vertex transmits a message to its neighbors upon receiving it regardless the activities in other vertices (beside the vertex that receives the message that has to be idle).

The broadcasting problem usually is described by the following rules:

1. A processor may send a message to an adjacent processor only.
2. Time is discrete. At a given time each processor will do exactly one of the following:
   (a) receive a message,

(b) send a message to one neighbor,

(c) be idle.

Formally, we can view the communication network as a finite connected undirected graph where the set of vertices are considered as processors and each edge which connects two vertices assumed to be a direct communication link between these vertices. Then, we define broadcasting from a vertex v (*the originator*) as transmitting a message (messages) from v to every vertex in $V \backslash \{v\}$ using the above rules. This problem, that was introduced in [13], is a variation of the gossiping problem (see [10]). For the basic definitions one may see [9]

We define the *broadcast number* of $v \in V(G)$, $G = G(V, E)$, denoted by $b(v)$, as the minimum time required to broadcast one message from v.

The *broadcast center* of G, denoted by $BC(G)$, is defined, as in [13], to be the set of all vertices having minimum broadcasting time.

The problem of broadcasting in a general graph is known to be NP-complete (see [8] and prof in [13]). On the other hand, in a tree with equal weights it is known to be linear [13]. Recently a generalization of broadcasting in trees was obtained [2], where, positive weights were assigned to the edges/vertices of the tree. In other topologies except the tree there are known results in a two and three dimensional grid [14] the complete graph and hypercube (see [3], [4] and [5]). Recently planar graphs were treated as well [11]. A conjecture concerning minimum broadcasting time starting from a given vertex in a $d-$dimensional grid ($d \geq 3$), was posed there[14]. In a recent paper [12] this conjecture was validated.

In [7] the problem of identifying the originators, in a tree, from which broadcasting multiple messages can be completed in a given time was treated.

In this paper we deal with the problem of broadcasting multiple messages from several originators in a tree within minimum time in one of the following cases:

(a) All the originators broadcast simultaneously the same message, or

(b) Each originator broadcast a distinguished message.

2 Simultaneous broadcasting from several originators

2.1 Definition of the problem

Before presenting the problem we need the following notations:

1. Let $T = T(V, E)$, where $V = V(T)$, $E = E(T)$ be a tree, $n = |V(T)|$.

2. Let $M \subseteq V(T)$ and $v \in V(T)$. Define $d(v, M) = \min_{u \in M} d(v, u)$, where $d(v, u)$ is the distance between v and u. If $v \in M$ we put $d(v, M) = 0$.

3. Let $M, P \subseteq V(T)$. Denote by $d(P, M) = \min_{u \in P}\{d(u, M)\}$. Observe that if $P \cap M \neq \emptyset$ then, $d(P, M) = 0$.

Assume that we have a tree T on n vertices and let $v_1, \ldots, v_k \in V(T)$, $1 \leq k \leq n$ be the originators. The purpose is to broadcast $m \geq 1$ messages in a minimum time in one of the following cases:

(a) All the originators broadcast simultaneously the same message, or

(b) Each originator broadcast a distinguished message.

Our aim is to evaluate the minimum time to broadcast simultaneously $m \geq 1$ messages in a tree from multiple originators.

2.2 All the originators broadcast simultaneously the same message

In this section we examine the broadcasting time in the case where several originators broadcast simultaneously the **same** message in a tree. In this case each originator has to broadcast only to a subset of the vertices where the union of the subsets is the entire set of vertices.
To the end of the article we use the following notations:

1. Let $A_1, A_2, \ldots A_x$, where $x = |BC(T)|$, be the induced sub-trees of $T \setminus < BC(T) >$.

   We define:

2. $\Delta^{max}(A_i) = \max\{\Delta(A_i) \mid \Delta(A_i) \text{ is the maximum degree in } A_i\}$.

3. $b(K, m, T)$ to be the time to broadcast $m \geq 1$ messages from a given set of originators $K \subseteq V(T)$, in a tree T.

4. $T' = T \setminus \{A_i \mid \text{if } K \cap A_i \neq \emptyset\}$.

5. $\Delta^{max}(T) = \max\{\Delta^{max}(A_i),\ \Delta(T')\}$.

Main Theorem:

Let T be a tree, $K = \{v_1, \ldots, v_k\}$ a set of originators and A_i, $1 \leq i \leq x$, be the induced sub-trees of $T \setminus < BC(T) >$, with $x = |BC(T)|$. Then,

$$b(K, m, T) \leq b(K, 1, T) + (m-1)\Delta^{max}(T).$$

Before proofing the main theorem recall the following essential result appeared in [13]:

Lemma 2.1 *Let T be a tree, $u \notin BC(T)$. Then,*

$$b(u) \leq b(T) + d(u, BC(T)).$$

Proof of main theorem:
The proof is divided into several cases as follows:

Case 1: There exists $1 \leq j \leq x$ such that $A_j \cap K = \emptyset$.

In this case we show that

$$b(K,m,T) \leq b(T) + d(K, BC(T)) + (m-1)\Delta^{max}(T). \tag{1}$$

By lemma 2.1 for $u \notin BC(T)$ it follows that $b(u) \leq b(T) + d(u, BC(T))$. Therefore, the broadcasting time of one message in a tree T with $|K|$ originators is at most $b(T) + d(K, BC(T))$. Hence, we have that $b(K,1,T) \leq b(T) + d(K, BC(T))$.

Now, to prove (1) observe the $j'th$ message must be delayed $\Delta^{max}(T)$ time units after the $(j-1)'th$ message was transmitted, otherwise, the vertex of degree $\Delta^{max}(T)$ is incapable to send the $(j-1)'th$ message to all its adjacent vertices prior to receiving the $j'th$ message.

(For example, see Fig. 1A where v_0 and v_1 are the originators of the same messages. Since v_0 nor v_1 are not vertices of A_1, namely, A_1 does not contain any originator, then, v_1 sends the message to a vertex in $BC(A_1)$ which transmit the message to the rest of the vertices of A_1.)

Case 2: For all, $1 \leq i \leq x$, $A_i \cap K \neq \emptyset$.

In order to proof this case we need the following notations:

1. $b^K(T) = b(T) + d(K, BC(T))$
2. $b^K(A_i) = b(A_i) + \min\{d(v, BC(A_i)) \mid v \in K\}$.
3. $b^K(A_1, A_2, \ldots, A_x) = \max\{b^K(A_1),\ b^K(A_2),\ \ldots,\ b^K(A_x)\}$

First we establish the upper bound to $b(K,1,T)$.

By 2.1 and the definition of $b^K(T)$ we have,

$$b(K,1,T) \leq b^K(T). \tag{2}$$

On the other hand

$$b(K,1,T) \leq b^K(A_1, A_2, \ldots, A_x). \tag{3}$$

Indeed, there exists A_j such that $b^K(A_1, A_2, \ldots, A_x) = b^K(A_j)$ so that obviously (3) holds. Therefore, from (2) and (3),

$$b(K,1,T) \leq \min\left\{b^K(A_1, A_2, \ldots, A_x),\ b^K(T)\right\}.$$

Note that if, in addition, we can transmit the message from an originator $v_s \notin A_j$ to a subset of $V(A_j)$ without increasing the value of $b^K(A_1, A_2, \ldots, A_x)$ then, $b^K(A_i)$ may be reduced.

Now, the existence of an originator in each A_i, $1 \leq i \leq x$, produce the following upper bound:

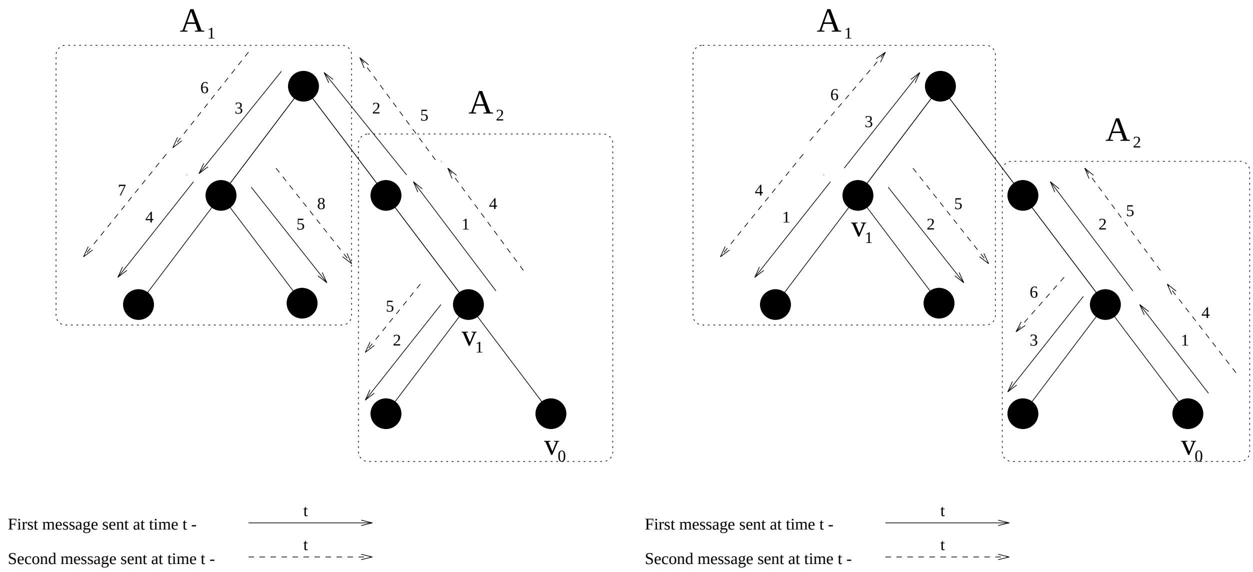
$$b(K,m,T) \leq b(K,1,T) + (m-1)\Delta^{max}(A_i). \tag{4}$$

Indeed, this bound is obtained by broadcasting each message in each sub-tree, A_i, with a delay of $\Delta^{max}(A_i)$ time units, after the previous message was transmitted.

One can easily observe that the bound of (4) is always lees than equal the bound in (1).

For example see 1B where v_0 and v_1 are the originators of the same messages. Since $v_0 \in A_2$ and $v_1 \in A_1$ (namely each sub-tree contains at least one originator) then, v_0 and v_1 broadcast the messages simultaneously within A_2 and A_1, respectively.

□



A: There exists a tree, A_1 with no originator in its vertices.

B: Each sub-tree contains at least one originator.

Figure 1: Broadcasting the same messages from two originators.

Remark:

Since broadcasting depends upon the structure of the tree, the lower bound we give here is the obvious bound which is:

$$\lceil \frac{n}{x} \rceil - 1.$$

2.3 Each originator broadcast a distinguished message

In this section we examine the broadcasting time in the case where several originators broadcast simultaneously different messages in a tree. In this case each originator has to broadcast its messages to all vertices of T.

We assume that the vertices (except for the originators) can not store more than one message.

Let $K = \{v_1, \ldots, v_r\}$ be a set of originators, $|K| = r$, and $a_i = d(v_i, BC(T))$. Without loss of generality we assume $a_i \leq a_{i+1}$, $1 \leq i \leq r$.

Before establishing the broadcasting algorithm we define a labeling of the originators as follows:

$$f : K \longrightarrow \{0, 1, \ldots, |K|\},$$

such that,

$$f(v_i) = r - i.$$

Each originator, v_i, $1 \leq i \leq r$, has $m(v_i)$ messages at time $t = 0$.
Broadcasting is done as follows:
At time $t = 1$ the originator v_r starts broadcasting its $m(v_r)$ messages, using the broadcasting algorithm presented in [13].

Once all $m(v_r)$ messages received by the entire tree, the originator v_{r-1} continues by broadcasting its $m(v_{r-1})$ messages, and so one until v_1 completes the broadcasting of its messages.

We note that based upon the structure of the tree, it might be that originator v_{a+1} may start broadcasting even before originator v_a completes its procedure. However, we are able to give the following upper bound,

Lemma 2.2
Let T be a tree, $K = \{v_1, \ldots, v_r\}$ be a set of originators, $|K| = r$, and $m = \sum_{i=1}^{r} m(v_i)$. Then,

$$b(K, m, T) \leq \sum_{i=1}^{r} (b(T) + d(v_i, BC(T)) + (m(v_i) - 1)\Delta)$$

□

Since the suggested broadcasting algorithm depends upon the structure of the tree, in many cases this bound is achieved. For example, in Fig. 2A the broadcasting time and the upper bound coincide, where the broadcasting time from the originator v_1 is 4, the broadcasting time from the originator v_2 is also 4 and the time to broadcast from v_1 **and** v_2 is 8.

In Fig. 2B, another example is demonstrated for broadcasting messages from two originators, s_0 and s_1 and three messages. Again, the broadcasting time and the upper bound coincide. However, in the example presented in Fig. 2C it is shown that broadcasting three messages from three originators s_0, s_1 and s_2, is done in a 17 time units which is less than the upper bound that equals 20 (The upper bound is the summation of the time to broadcast from s_0, s_1 and s_2).

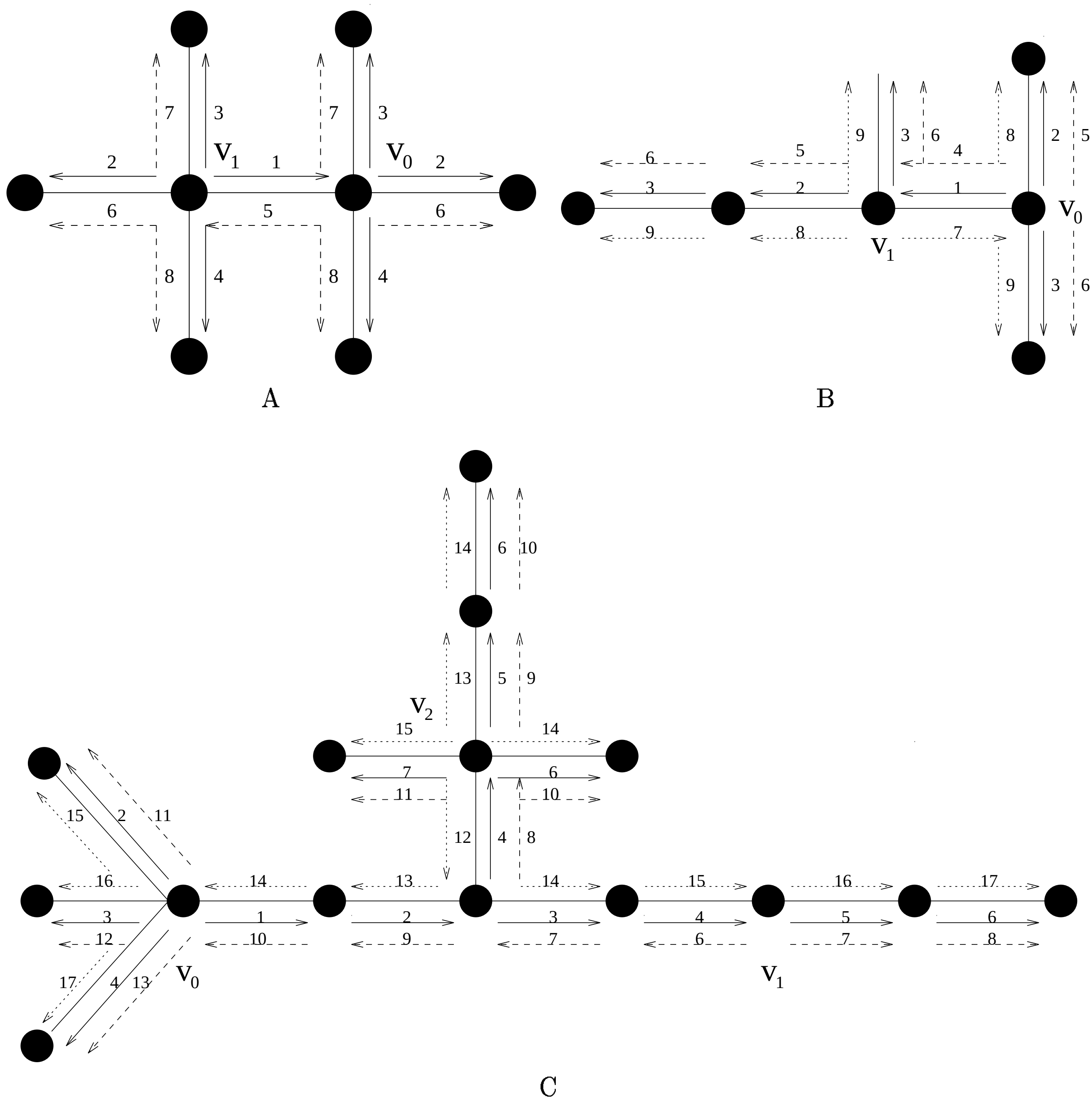


Figure 2: Broadcasting different messages from several originators.

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