ELSEVIER

Discrete Applied Mathematics 113 (2001) 129–141

DISCRETE APPLIED MATHEMATICS

Efficient line broadcasting in a d-dimensional grid

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Received 3 March 1999; revised 29 February 2000; accepted 26 June 2000

Abstract

The results of Fujita and Farley (Discrete Appl. Math. 75 (1997) 255–268) and Kane and Peters (Discrete Appl. Math. 83 (1998) 207–228) on line broadcasting in paths and cycles are extended into a d-dimensional grid, obtaining optimal algorithms in most cases. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Efficient broadcasting is a key component to achieve high performance (throughput) from parallel and distributed processing.

The motivation of this work was triggered by our interest in the investigation of how to perform optimal query on a distributed database on diverse MIMD multiprocessor architectures [2]. In [2] scheduling and evaluation of queries with minimal cost on diverse multiprocessor architectures was investigated.

Broadcasting has become popular in recent years with the introduction of new, very fast networks and their extensive use by parallel and distributed systems. Utilizing efficiently the underlying network in a massively parallel/distributed system is a major challenge. Existing solutions for broadcasting and its implementation are either inefficient or too expensive in terms of the host operating system overhead involved. Broadcasting is a crucial component in having efficient message passing in parallel/distributed multiprocessors.

Let $G=G(V,E)$ be a connected graph. A communication network is defined as a connected graph. Given a set $A\subset V(G)$, $A\neq\emptyset$, we define *broadcasting* (*multicasting*) from A to be the process of transmitting one (many) unit(s) of information from

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PII: S0166-218X(00)00277-8

A to all the vertices in $G \setminus A$ ($B \subset (G \setminus A)$). The vertices in A are called *originators*. This is accomplished by a series of transitions over G. The messages are distributed over G and spread through the edges, where each vertex transmits a message to its neighbors upon receiving it regardless of the activities in other vertices (beside the vertex that receives the message that has to be idle). This procedure is called *local*.

In *line* broadcasting, an informed vertex may call any other vertex using the communication links of any simple path between the two vertices with the restriction that no link is used in more than one call in a given time unit. It is obvious that if local broadcasting is in action then the number of informed vertices can be at most double in each time unit, so that at least $\lceil \log_2 n \rceil$ time units are needed to broadcast in a network of n vertices. In general, it is not possible to inform all vertices in a network in that time. However, Farley [5] showed that there is a minimum-time line broadcast scheme for any originator in any connected network.

The question we deal here is:

How much line broadcasting is needed to complete a minimum-time broadcasting from any arbitrary originator in a given connected graph?

Farley [5], who introduced the topic, gave a constructive proof that minimum-time line broadcasting is possible in any tree, and thus, in any connected network. His construction gives an upper bound of $(n-1)\lceil \log_2 n \rceil$ on the total length (total number of edges) needed to line broadcasting in a minimum time in any network of n vertices. Kane and Peters [10] determined the value of the minimum total length for minimum-time line broadcasting in any cycle on n vertices. When $n = 2^k$ they gave an exact value, while for other choices of n an upper bound was presented, which is about $\frac{1}{3}$ of Farley's upper bound. Recently, Averbuch et al. [1] established linear, in n, algorithms for line-broadcasting in certain classes of trees such as a full binary tree and a binomial tree.

In this paper, we extend the results of [7,10] to a d-dimensional grid.

A d-dimensional grid graph, $G(n_1, n_2, \ldots, n_d)$, is a graph with $n_1 \times n_2 \times \cdots \times n_d$ vertices, each of them is labeled by a unique element of $\{(i_1, i_2, \ldots, i_d) \mid 0 \leqslant i_j \leqslant n_j - 1, 1 \leqslant j \leqslant d\}$, and with edges connecting vertices which differ by 1 in exactly one label component. If, in particular, we connect vertices which differ by $1 \pmod{n_i}$ we get a d-dimensional "torus". When $d = 2$ this is the well-known torus. Denote by $G^d(n)$ a d-dimensional grid, $n_i = n$, $i = 1, \ldots, d$. Note that a d-dimensional hypercube is a particular case of d-dimensional grid graph, where only two vertices reside in each dimension.

The cost to broadcast a message from a vertex is the number of edges the message passes through. Denote by $F_O^d(n_1, n_2, \ldots, n_d)$ the optimal cost to broadcast a message from a vertex O in $G^d(n_1, n_2, \ldots, n_d)$. When the dimensions are equal to n we denote the optimal cost to broadcast from O by $F_O^d(n)$.

Given an originator and broadcast time t, an optimal broadcasting tree, called hereinafter OBT, is the largest possible tree for which the originator can broadcast a message in time at most t. The OBT for broadcasting time $\log_2 n$ that contains n vertices

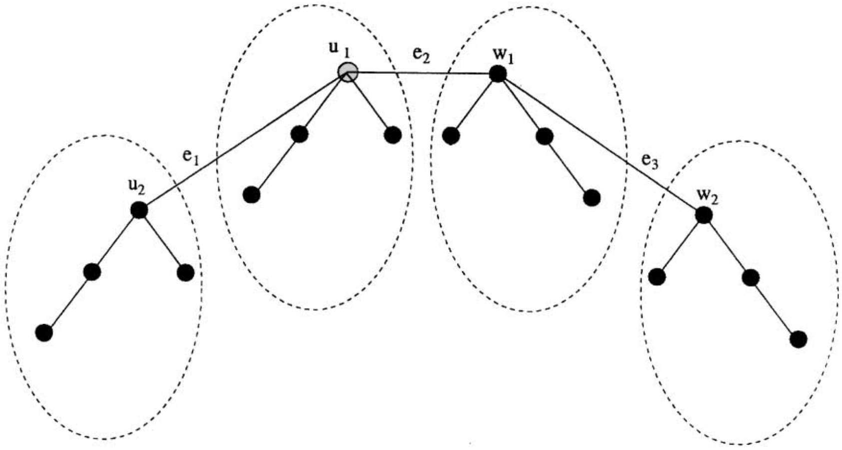

Fig. 1. An optimal broadcasting tree (OBT) for $n = 16$.

can be divided into two OBTs each of $n/2$ vertices. Obviously, it can be divided into four OBTs each of $n/4$ vertices (see [4,6]) (Fig. 1).

For other graph theory concepts consult Harary [8] or West [13].

2. Results

In this section, we present a broadcast scheme for a d-dimensional grid. We start with a d-dimensional grid that is composed of $n_i = 4$ for $i = 1,\dots,d$ (4^d vertices). Then, the broadcast scheme is extended to the case where in each dimension we have $n_i = 4^k$, $k \geqslant 1$, which is a positive integer. Finally, we present the broadcast scheme for any d-dimensional grid.

The main results are:

Theorem 2.1. *Let* $d \geqslant 2$, $D = \{(i_1,i_2,\dots,i_d) \mid deg(i_1,i_2,\dots,i_d) = 2d, (i_1,i_2,\dots,i_d) \in \{1,2\}\}$. *Then,*

$$F^d_{\{(i_1,i_2,\dots,i_d)\in D\}}(4) = 4^d - 1.$$

Theorem 2.2. *Let* $d \geqslant 2$, $D = \{(i_1,i_2,\dots,i_d) \mid deg(i_1,i_2,\dots,i_d) = d, (i_1,i_2,\dots,i_d) \in \{0,3\}\}$. *Then,*

$$4^d + \frac{d(d-1)}{2} - 1 \leqslant F^d_{\{(i_1,i_2,\dots,i_d)\in D\}}(4) \leqslant 4^d + \frac{d(d+1)}{2} - 1.$$

Theorem 2.3. *Let d and k be integers such that* $d \geqslant 2, k \geqslant 2$. *Then*:

$$4^{dk} + 4^{dk-d} - 1 \leqslant F^d_{\{(i_1,i_2,\dots,i_d)\mid(i_1,i_2,\dots,i_d)\in\{0,4^k-1\}\}}(4^k) \leqslant 4^{dk} + \left(3 + \frac{1}{4^{d-1}-1}\right)4^{dk-d} + d^2 \cdot 4^k. \quad (1)$$

Remark

1. Observe that the difference between the lower and the upper bounds in Theorem 2.2 is $\frac{1}{2}\log_2 n$, where, $n = 4^d$. In addition, the assertion of the theorem trivially holds for $d = 1$.
2. The lower bound in Theorem 2.3 is
$$n + \frac{n}{n^{1/k}} - 1,$$
while the upper bound is about
$$n + 3\frac{n}{n^{1/k}} + d^2 n^{1/d},$$
where $n = 4^{dk}$.

However, if a d-dimensional "torus" is considered (we connect vertices which differ by $1 \pmod{4^k}$ where 4^k is the number of vertices along each dimension), then we have the following result.

Lemma 2.1. *Let $d \geqslant 2$, $k \geqslant 1$ be integers and $n=4^{dk}$. Then the cost of line-broadcasting from any vertex in the torus on n vertices is*

$$\sum_{i=1}^{k} [4^{d(i-1)} \times (4^d - 1) \times 4^{k-i}]$$
$$= \left(\frac{4^d - 1}{4^d - 4}\right) \times 4^{dk} - \left(\frac{4^d - 1}{4^d - 4}\right) \times 4^k = c(n - n^{1/d}),$$

where $c = c(d)$, $c > 1$.

Observe that for $k = 1$ we obtain an optimal broadcasting, namely, the cost is $4^d - 1 = n - 1$.

2.1. Broadcasting in a d-dimensional grid with $n_i = 4$, $i = 1, \ldots, d$

Before proving the main results we present an essential propositions for treatment of $G^2(4)$-grid.

Proposition 2.1.

$$F^2_{\{(i,j)|(i,j)\in\{0,3\}\}}(4) = 18.$$

Proof. Since we have exactly four vertices of degree two, namely the corners of the $G^2(4)$ grid, we may assume that the originator is $(0, 0)$.

Since $|V| = 16$ we have $F^2_{(0,0)}(4) \geqslant 15$. We have four time units to complete the broadcast and the originator must be active during each time step. Also, $deg((0, 0))=2$. Therefore, it follows that $F^2_{(0,0)}(4) \geqslant 17$.

Next, we show that $F^2_{(0,0)}(4) \geqslant 18$. In order to get $F^2_{(0,0)}(4) = 17$, we need to transmit the message from $(0, 0)$ to two vertices from $\{(0, 2), (2, 0), (1, 1)\}$. In addition, $(0, 0)$

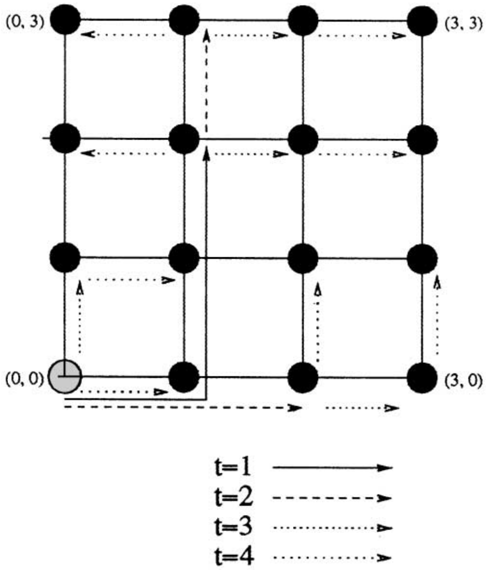

Fig. 2. Broadcasting scheme for $G^2(4)$ where the originator is a vertex of degree 2 ((0,0)).

has to transmit the message to $(1,0)$ and $(0,1)$. However, in order to complete the broadcast without any additional transmission of length at least two, two vertices out of the four that received the message at time $t=2$ need to transmit the message to vertex $(1,0)$ or $(0,1)$ (from $(0,0)$ and from $(2,0)$ or from $(0,2)$). Therefore, $F^2_{(0,0)}(4) \geqslant 18$. In Fig. 2 we present a broadcasting scheme with $F^2_{(0,0)}(4) = 18$ which completes the proof. □

Proposition 2.2. *If* $A = \{(0,1),(0,2),(3,1),(3,2),(1,0),(2,0)(1,3),(3,2)\}$ *then*,

$$F^2_{\{(i,j)\in A\}}(4) = 16.$$

Proof. Since $|V| = 16$ then $F^2_{\{(i,j)\in A\}}(4) \geqslant 15$. Moreover, $F^2_{(i,j)\in A}(4) \geqslant 16$ since the originator has to transmit to 4 vertices and has only three adjacent vertices (degree 3). However, in Fig. 3 we present a broadcasting scheme with $F^2_{(0,0)}(4) = 16$ which completes the proof. □

Proposition 2.3. *Let* $B = \{(1,1),(1,2),(2,1),(2,2)\}$. *Then*,

$$F^2_{\{(i,j)\in B\}}(4) = 15.$$

Proof. W.l.o.g assume the originator is $(1,1)$.

Since $|V| = 16$, it follows that $F^2_{\{(i,j)\in B\}}(4) \geqslant 15$. However, in Fig. 4 we present a spanning tree of the two-dimensional grid which is an OBT for 16 vertices, rooted in $(1,1)$. Therefore, $F^2_{(1,1)}(4) = 15$. □

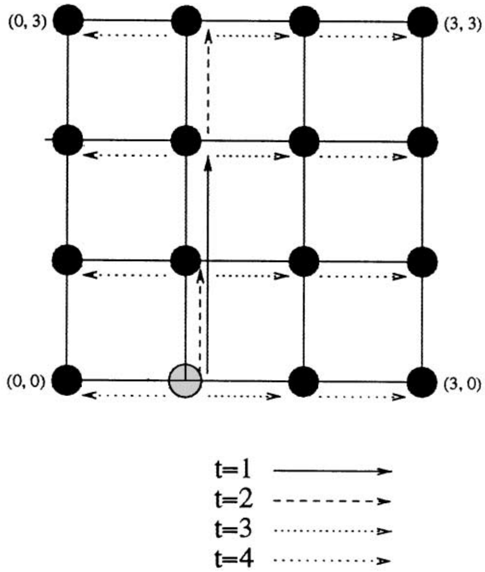

Fig. 3. Broadcasting scheme for $G^2(4)$ where the originator is a vertex of degree 3 ((0, 1)).

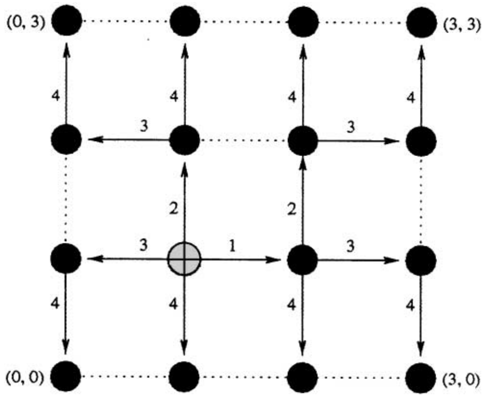

Fig. 4. Broadcasting scheme for $G^2(4)$ grid where the originator is a vertex of degree 4 ((1, 1)).

The results for the d-dimensional grid are divided into the following cases:
1. *The originator is a vertex of degree* $2d$:

Proof of Theorem 2.1. Since we have exactly 2^d vertices of degree $2d$, namely the internal vertices of the $G^d(4)$, we may assume that the originator is $O=(1,1,\ldots,1)$. $|V|=4^d$, therefore, $F^2_{\{(i_1,i_2,\ldots,i_d)\in D\}}(4)\geqslant 4^d-1$. To see that $F^d_{\{(i_1,i_2,\ldots,i_d)\in D\}}(4)\leqslant 4^d-1$ we show that $G^d(4)$ contains an OBT as a spanning tree with the originator O. Recall that the OBT for n vertices can be divided into four OBTs each contains $n/4$ vertices (see Fig. 1). We divide $G^d(4)$ into four sub-grids $G^{d-1}(4)$ denoted A_j, $j=0,1,2,3$

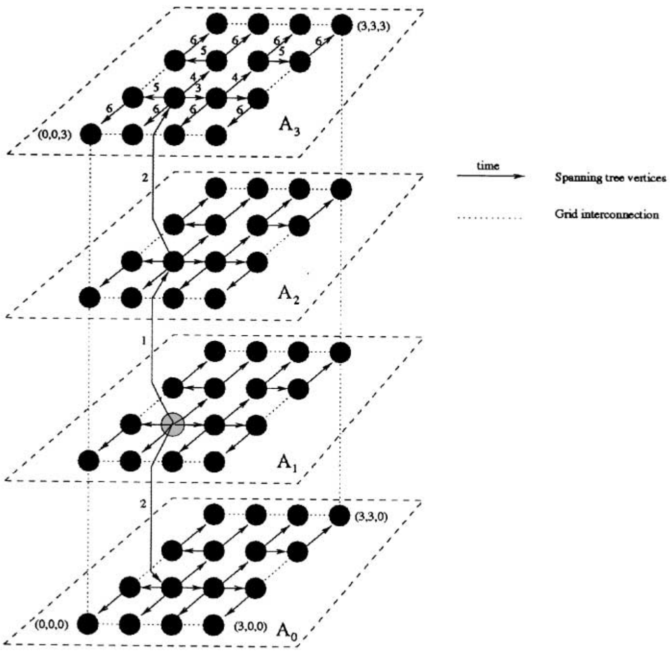

Fig. 5. Broadcasting scheme for $G^3(4)$ where the originator is a vertex of degree 6 ((1, 1, 1)).

as follows: $A_j = \{(i_1, i_2, \ldots, i_{d-1}, j) \mid i_s \in \{0, 1, 2, 3\},\ s = 1, \ldots, d-1\}$ (for example, see Fig. 5). Now to have the OBT, as described in Fig. 1, we choose:

$u_1 = (1, 1, \ldots, 1, 0) \in A_0$,
$u_2 = (1, 1, \ldots, 1, 1) \in A_1$,
$w_1 = (1, 1, \ldots, 1, 2) \in A_2$,
$w_2 = (1, 1, \ldots, 1, 3) \in A_3$

such that

$(u_1, u_2) = e_1$,
$(u_2, w_1) = e_2$,
$(w_2, w_1) = e_3$.

The vertices $u_1, u_2, w_1, , w_2$ of the OBT are the roots of the OBTs of the four $G^{d-1}(4)$. In order to build the OBT in each $G^{d-1}(4)$ we divide it into four $G^{d-2}(4)$ parts and proceed with the same algorithm as done in $G^d(4)$. This process is repeated until we left with $G^2(4)$ in which we build the OBT as presented in Proposition 2.3.

Now the proof is proceeded by induction on d which is the dimension of the grid that the spanning tree built is an OBT. For $G^3(4)$ we have an OBT since the above spanning tree composed of four spanning trees for each A_i, $i = 0, 1, 2, 3$ and three vertices combined those trees (see Fig. 1). By the induction hypothesis we assume that an OBT exists for $G^{d-1}(4)$. The $G^d(4)$ grid is divided into four $G^{d-1}(4)$, each contains the required OBT. The OBTs are connected via e_1, e_2 and e_3 to an OBT which is a spanning tree for the $G^d(4)$. □

2. *The originator is a vertex of degree* d:

Proof of Theorem 2.2. Since, we have exactly 2^d vertices of degree d, namely the corners of $G^d(4)$ we may assume that the originator is $O=(0,0,\dots,0)$.

First, we show the lower bound. We have a total time $2d$. So it follows that in order to accomplish broadcasting within that time, once a vertex receives the message it has to be busy to the reset of the time left [9]. Since each vertex must receive the message, the cost is at least, $4^d-1=n-1$. Also $deg(O)=d$. So that we must have at least d additional edges to pass at least twice. An edge that we pass, at least twice, is called a "double-passed" edge. Furthermore, in any optimal algorithm we may assume that the originator should transmits the message to its neighbors and to vertices at distance two. Otherwise, the cost would increase. If this is the case, we may assume that its neighbors receive the message at time: $t=d+1$ to $2d$ and at time $t=1$ to $t=d$, it transmits the message to d vertices at distance two from it. Otherwise, line-broadcasting is in action and the cost will increase.

Now, the d neighbors of O has to make

$$\sum_{i=0}^{d-1} i = \frac{d(d-1)}{2},$$

calls which costs at least $d(d-1)/2-d$ "double edges" (the subtraction of d is due to the possibly d vertices at distance 2 from O that did not get the message yet).

So that the lower bound is at least:

$$4^d-1+d+\frac{d(d-1)}{2}-d=4^d-1+\frac{d(d-1)}{2}.$$

To obtain the upper bound we present a broadcasting scheme with a cost of at most $4^d-1+d(d+1)/2$.

We divide $G^d(4)$ into four sub-grids $G^{d-1}(4)$ denoted A_j, $j=0,1,2,3$ as follows: $A_j=\{(i_1,i_2,\dots,i_{d-1},j)\,|\,i_s\in\{0,1,2,3\},\ s=1,\dots,d-1\}$ (for example, see Fig. 6). Put, $H=A_2\cup A_3$, and

$w_1=(1,1,\dots,1,2)\in A_2$,

$w_2=(1,1,\dots,1,3)\in A_3$.

Observe that H has an OBT as a spanning tree.

Our algorithm is:

(1) At $t=1$, O transmits the message to w_1, which is the closest root to O in the spanning OBT of H.

(2) From $t=2$ to $t=2d$, w_1 continues broadcasting in H using only local broadcasting. (Note that $|V(H)|=4^{d-1}=2^{2d-2}$).

(3) From $t=2$ to $2d-1$, O transmits in A_0 using the same algorithm, namely, the former steps.

(4) At $t=2d$ all vertices of A_0 transmit the message to the vertices of A_1 such that each vertex of A_0 transmits to the vertex in A_1 which differs only in the last coordinate, namely: $(i_1,i_2,\dots,i_{d-1},0)\to(i_1,i_2,\dots,i_{d-1},1)$.

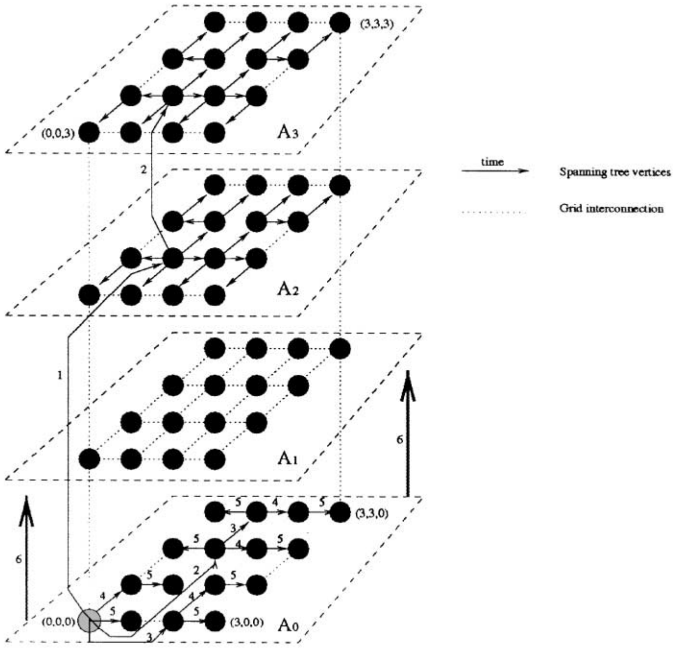

Fig. 6. Broadcasting scheme for $G^3(4)$ grid where the originator is a vertex of degree 3 ($(0,0,0)$).

For simplicity, we denote by $C(d)$ the cost function F of the theorem for the d-dimensional grid.

(1) The cost of the first step is: $d-1+2=d+1$.
(2) The cost of step 2 is: $2(4^{d-1}-1)+1=2\cdot 4^{d-1}-1$ (by Theorem 2.1 and the additional edge between w_1 and w_2).
(3) The cost of step 3 is at most $C(d-1)$ since A_0 is a $G^{d-1}(4)$ grid.
(4) The cost of step 4 is: 4^{d-1}.

Hence,

$$C(d)\leqslant C(d-1)+3\cdot 4^{d-1}+d,$$

with $C(2)=18$. Thus, the solution of the recurrence inequality yields

$$C(d)\leqslant 4^d+\frac{d(d+1)}{2}-1,$$

as required. □

2.1.1. Example

For $G^3(4)$ we have $66\leqslant F^3_{(0,0,0)}(4)\leqslant 69$. The upper bound algorithm is presented in Fig. 6 by dividing the $G^3(4)$ grid into four $G^2(4)$ grids, denoted A_0, A_1, A_2, A_3. The cost to broadcast from $(0,0,0)$ is the cost to transmit the message to $(1,1,2)\in A_2$, then the cost to transmit from it to $(1,1,3)\in A_3$ and then the cost in $A_2\cup A_3$ in dependently with the broadcast within A_0 so that in the last stage all vertices in A_0 transmit to all vertices in A_1. Hence, $F^3_{(0,0,0)} \leqslant 4 + 2 \times 15 + 1 + 18 + 16 = 69$.

2.2. Broadcasting in the d-dimensional, $d \geqslant 2$, grid with $n_i = 4^k$, $i = 1, \ldots, d$, and k a positive integer

In this section, we present a broadcast scheme for broadcasting one message from a corner vertex in a d-dimensional grid, $G^d(4^k)$.

Proof of Theorem 2.3. Again, the broadcast is done from a corner vertex. Since, we have exactly 2^d vertices of degree d, namely the corners of the $G^d(4^k)$, we may assume that the originator, O, is $(0, 0, \ldots, 0)$.

Proof of the lower bound. We have a total time $2dk$. Since $|V| = 4^{dk}$ it follows that in order to accomplish broadcasting within that time, once a vertex receives the message it has to be busy to the reset of the time left [9]. Since each vertex must receive the message the cost is at least $4^{dk} - 1 = n - 1$. Now, we compute the additional "double passed edges" obtained from the fact that each vertex that receives the message at time $t, 1 < t < 2kd - 2d$ must have at least $2dk - t - 2d - 1$ additional calls to make (assuming the degree of each vertex is $2d$), which are at least $2dk - t - 2d - 1$ additional "double passed edges". In addition, for each such t we have 2^t vertices that realized that amount of calls. Thus, we have at least

$$2kd - d + \sum_{t=0}^{2kd-2d-2} 2^t(2kd - t - 2d - 1) = 2^{2kd-2d} - 1 = \frac{n}{n^{1/k}} - 1.$$

The first term is due to the contribution of the originator to the additional edges. Hence, the lower bound is obtained.

Proof of the upper bound. In order to prove the upper bound, we present an algorithm realizing it.

We look at $G^d(4^k)$ as a grid $G^d(4)$ in which each vertex is a $G^d(4^{k-1})$. We denote the "new vertices" by $A^1_{a_1,a_2,\ldots,a_d}$, $a_i \in \{0, 1, 2, 3\}$ where, $|\{A^1_{a_1,a_2,\ldots,a_d}\}| = 4^d$. The same division is done with each $G^d(4^{k-1})$ where the "vertices" are $G^d(4^{k-2})$, denoted by $A^2_{a_1,a_2,\ldots,a_d}$ ($|\{A^2_{a_1,a_2,\ldots,a_d}\}| = 4^d \times 4^d$) and so on, until each "vertex" is a grid $G^d(4)$ ($|\{A^{k-1}_{a_1,a_2,\ldots,a_d}\}| = 4^{d(k-1)}$).

The broadcasting Algorithm

We start broadcasting from $O = (0, 0, \ldots, 0)$.

(1) The originator O transmits the message to the vertices:

$$\{S^1_{i_1,i_2,\dots,i_d},\ i_j \in \{a_j 4^{k-1} + 4^{k-2} + 4^{k-3} + \cdots + 4^0\},\ j \in \{b_j 4^{k-1} + 4^{k-2} + 4^{k-3} + \cdots + 4^0\},\ a_j, b_j \in \{0,1,2,3\},\quad j = 1,2,\dots d\}$$

(Observe that $|\{S^1_{i_1,i_2,\dots,i_d}\}| = 4^d - 1$ and that each vertex of $S^1_{i_1,i_2,\dots,i_d}$ is in different $A^1_{a_1,a_2,\dots,a_d}$).

(2) Now, the broadcast is done in each $A^1_{a_1,a_2,\dots,a_d}$, but $A^1_{0,0,\dots,0}$. W.l.o.g we shall describe the procedure in $A^1_{1,0,\dots,0}$ where the originator is $S^1_{1,0,\dots,0}$. Recall that $A^1_{1,0,\dots,0}$ is divided into 4^d d-dimensional grids such that each vertex is a $G^d(4^{k-2})$. The originator $S^1_{1,0,\dots,0}$ broadcast the message to

$$\{S^2_{i_1,i_2,\dots,i_d},\ i_j \in \{a_j 4^{k-2} + 4^{k-3} + \dots + 4^0\},\ j \in \{b_j 4^{k-2} + 4^{k-3} + \dots + 4^0\},\ a_j, b_j \in \{0,1,2,3\},\ j = 1,2,\dots d\}$$

which are the originators of each such $G^d(4^{k-2})$. This procedure is repeated until we get that each "vertex" is a grid $G^d(4)$. The broadcast within $G^d(4)$ is done as described in Theorem 2.1.

(3) The broadcasting scheme within $A^1_{0,0,\dots,0}$ is as described in step (1), where the grid $A^1_{0,0,\dots,0}$ is a $G^d(4^{k-1})$ grid and the originator is $O = (0,0,\dots,0)$.

(4) Broadcasting within $A^k_{0,0,\dots,0}$, which is a $G^d(4)$ grid, is done as for the upper bound in Theorem 2.2.

At the end of the above procedure all the 4^{dk} vertices receive the message, since this procedure is repeated $k-1$ times until we get $G^d(4)$. The duration of each step is $2d$ time units (multicast to 4^d vertices). The broadcast within $G^d(4)$ is done in $2d$ time units (Lemma 2.1). Therefore, the procedure is finished within $2dk$ time units as required.

Observe that the originator in each $A^i_{a_1,a_2,\dots,a_d}$ is of degree $2d$ while the originator in $A^i_{0,0,\dots,0}$ is of degree d. Denote by $C^i_{a_1,a_2,\dots,a_d}$ the cost to broadcast within $A^i_{a_1,a_2,\dots,a_d}$ (excluding $A^i_{0,0,\dots,0}$).

The cost to broadcast in the different stages is presented in the following Table 1. Therefore,

$$\begin{aligned}
&F^d_{\{(i_1,i_2,\dots,i_d)|(i_1,i_2,\dots,i_d)\in\{0,4^k-1\}\}}(4^k)\\
&\quad\leqslant \sum_{i=1}^{k-1}\left[(4^{d(i-1)}-1)\times(4^d-1)\times 4^{k-i} + \left(4^d + \frac{d(d+1)}{2} - 1\right)\right.\\
&\quad\quad\left.\times 4^{k-i} + 2d^2 \sum_{j=1}^{k-i} 4^{j-1}\right]\\
&\quad\quad + (4^{d(k-1)}-1)\times(4^d-1) + \left(4^d + \frac{d(d+1)}{2} - 1\right)\\
&\quad = 4^{dk} + \left(3 + \frac{1}{4^{d-1}-1}\right)4^{dk-d} + \left(\frac{2d^2+3d}{18} - \frac{4^d-1}{4^d-4}\right)4^k
\end{aligned}$$

Table 1

Cost	Location
$4^d - 1$	Number of transmissions within $A^i_{a_1,a_2,\dots,a_d}$
4^{k-i}	Length of each transmission within $A^i_{a_1,a_2,\dots,a_d}$
$4^{d(i-1)} - 1$	Number of grids of cost $C^i_{a_1,a_2,\dots,a_d}$ (number of grids $A^i_{a_1,a_2,\dots,a_d}$)
$4^{d(k-1)} - 1$	Number of $G^2(4)$ grids where their broadcasting cost is $C^k_{a_1,a_2,\dots,a_d}$
$4^d - 1$	Number of transmissions within $A^k_{a_1,a_2,\dots,a_d}$ ($C^k_{a_1,a_2,\dots,a_d}$)
$2d^2 \sum_{j=1}^{k-i} 4^{j-1}$	The cost to transmit the message from a corner vertex in $A^i_{a_1,a_2,\dots,a_d}$ to an internal vertex in $A^i_{a_1,a_2,\dots,a_d}$
$4^d + d(d+1)/2 - 1$	Number of transmissions within $A^i_{0,0,\dots,0}$
4^{k-i}	Length of each transmission within $A^i_{0,0,\dots,0}$
$4^d + d(d+1)/2 - 1$	The cost to transmit within $A^k_{0,0,\dots,0}$ which is $G^2(4)$

$$
\begin{aligned}
&-\frac{7d^2+3d}{9}-\frac{2d^2k}{3}\\
&\leqslant n+\left(3+\frac{1}{n^{1/k}-1}\right)\frac{n}{n^{1/k}}+d^2n^{1/d}. \qquad \square
\end{aligned}
$$

2.3. Final remark

A generalization for a d-dimensional grid where n is any positive integer is obtained as follows:

First, we find the minimal positive integer k such that $4^k \geqslant n$. If $4^k = n$ the broadcast is done as in Theorem 2.3. Otherwise, similar to the broadcasting scheme presented in Theorem 2.3 we divide the d-dimensional grid into sub-grids such that each vertex, excluding on the grid surface, resembles a grid $G^d(4^{k-2})$. A vertex resides on the grid surface resembles a grid $G^d(n-4^{k-1})$. The broadcasting within the vertices excluding on the grid surface is done as in Theorem 2.3. For vertices reside on the surface we continue the process until we get a grid $G^d(a)$, $a \geqslant 4$. The broadcast within $G^d(a)$ is done as described in Proposition 2.1.

Finally, concerning Theorem 2.3 we conjecture:

Conjecture. *The value of* $F^d_{\{(i_1,i_2,\dots,i_d)|(i_1,i_2,\dots,i_d)\in\{0,4^k-1\}\}}(4^k)$ *is about the upper bound of the theorem.*

3. Uncited references

[3,11,12]

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