Chapter 2

Vector Spaces

Vector spaces and their ancillary structures provide the common language of linear algebra, and, as such, are an essential prerequisite for understanding contemporary applied mathematics. The key concepts of vector space, subspace, linear independence, span, and basis will appear, not only in linear systems of equations and the geometry of $n$-dimensional Euclidean space, but also in the analysis of linear ordinary differential equations, linear partial differential equations, linear boundary value problems, all of Fourier analysis, numerical approximations like the finite element method, and many, many other fields. Therefore, in order to properly develop the wide variety of analytical methods used in mathematics and its manifold applications, the first order of business is to acquire a firm working knowledge of basic vector space constructions.

One of the great triumphs of modern mathematics was the recognition that many seemingly distinct constructions are, in fact, different manifestations of the same general mathematical structure. The abstract notion of a vector space serves to unify spaces of ordinary vectors, spaces of functions, such as polynomials, exponentials, trigonometric functions, as well as spaces of matrices, linear operators, etc., all in a common conceptual framework. Moreover, proofs that might look rather complicated in any particular context often turn out to be relatively transparent when recast in the abstract vector space framework. The price that one pays for the increased level of abstraction is that, while the underlying mathematics is not all that difficult, the student typically takes a long time to assimilate the material. In our opinion, the best way to approach the subject is to think in terms of concrete examples. First, make sure you understand what the concept or theorem says in the case of ordinary Euclidean space. Once this is grasped, the next important case to consider is an elementary function space, e.g., the space of continuous scalar functions. With these two examples firmly in hand, the leap to the general abstract version should not be too painful. Patience is essential; ultimately the only way to truly understand an abstract concept like a vector space is by working with it! And always keep in mind that the effort expended here will be amply rewarded later on.

Following an introduction to vector spaces and subspaces, we develop the notions of span and linear independence, first in the context of ordinary vectors, but then in more general vector spaces, particularly function spaces. These are combined into the all-important concept of a basis of a vector space, leading to a linear algebraic characterization of its dimension. Here is where the all-important distinction between finite-dimensional and infinite-dimensional vector spaces first becomes apparent, although the full ramifications of this dichotomy will take time to unfold. We will then study the four fundamental subspaces associated with a matrix — range, kernel, corange and cokernel — and explain how they help us understand the solution to linear algebraic systems. Of particular note is
the all-pervasive linear superposition principle that enables one to construct more general solutions to linear systems by combining known solutions. Superposition is the hallmark of linearity, and will apply not only to linear algebraic equations, but also linear ordinary differential equations, linear boundary value problems, linear partial differential equations, and so on. Some interesting applications in graph theory, to be used in our later study of electrical circuits, will form the final topic of this chapter.

2.1. Vector Spaces.

A vector space is the abstract formulation of the most basic underlying properties of $n$-dimensional Euclidean space $\mathbb{R}^n$, which is defined as the set of all real (column) vectors with $n$ entries. The basic laws of vector addition and scalar multiplication in $\mathbb{R}^n$ serve as the motivation for the general, abstract definition. In the beginning, we will refer to the individual elements of a vector space as “vectors”, even though, as we shall see, they might also be functions or matrices or even more general objects. Unless dealing with certain specific examples such as a space of functions or matrices, we will use bold face, lower case Latin letters to denote the elements of our vector space.

For most of this chapter we will deal with real vector spaces, in which the scalars are ordinary real numbers. Complex vector spaces, where complex scalars are allowed, will be introduced in Section 3.6. Vector spaces over other fields are studied in abstract algebra, [77]. We begin with the general definition.

**Definition 2.1.** A vector space is a set $V$ equipped with two operations:

(i) Addition: adding any pair of vectors $v, w \in V$ produces another vector $v + w \in V$;

(ii) Scalar Multiplication: multiplying a vector $v \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $cv \in V$.

that are required to satisfy the following axioms for all $u, v, w \in V$ and all scalars $c, d \in \mathbb{R}$:

(a) Commutativity of Addition: $v + w = w + v$.

(b) Associativity of Addition: $u + (v + w) = (u + v) + w$.

(c) Additive Identity: There is a zero element $0 \in V$ satisfying $v + 0 = v = 0 + v$.

(d) Additive Inverse: For each $v \in V$ there is an element $-v \in V$ such that $v + (-v) = 0 = (-v) + v$.

(e) Distributivity: $(c + d)v = (cv) + (dv)$, and $c(v + w) = (cv) + (cw)$.

(f) Associativity of Scalar Multiplication: $c(dv) = (cd)v$.

(g) Unit for Scalar Multiplication: the scalar $1 \in \mathbb{R}$ satisfies $1v = v$.

**Note:** We will use bold face $\mathbf{0}$ to denote the zero element of our vector space, while ordinary $0$ denotes the real number zero.

The following identities are elementary consequences of the vector space axioms:

(h) $0v = \mathbf{0}$.

(i) $(-1)v = -v$.

(j) $c0 = \mathbf{0}$.

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† The precise definition of dimension will appear later, in Theorem 2.28.
(k) If \(cv = 0\), then either \(c = 0\) or \(v = 0\).

Let us, as an example, prove \((h)\). Let \(z = 0v\). Then, by the distributive property,

\[
z + z = 0v + 0v = (0 + 0)v = 0v = z.
\]

Adding \(-z\) to both sides of this equation, and making use of axioms \((b)\), \((d)\), and then \((c)\), implies that \(z = 0\), which completes the proof. Verification of the other three properties is left as an exercise for the reader.

**Example 2.2.** As noted above, the prototypical example of a real vector space is the space \(\mathbb{R}^n\) consisting of column vectors or \(n\)-tuples of real numbers \(v = (v_1, v_2, \ldots, v_n)^T\). Vector addition and scalar multiplication are defined in the usual manner:

\[
v + w = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}, \quad cv = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{pmatrix}, \quad \text{whenever} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}.
\]

The zero vector is \(0 = (0, \ldots, 0)^T\). The fact that vectors in \(\mathbb{R}^n\) satisfy all of the vector space axioms is an immediate consequence of the laws of vector addition and scalar multiplication. Details are left to the reader.

**Example 2.3.** Let \(M_{m \times n}\) denote the space of all real matrices of size \(m \times n\). Then \(M_{m \times n}\) forms a vector space under the laws of matrix addition and scalar multiplication. The zero element is the zero matrix \(O\). Again, the vector space axioms are immediate consequences of the basic laws of matrix arithmetic. The preceding example of the vector space \(\mathbb{R}^n = M_{1 \times n}\) is a particular case when the matrices have only one column.

**Example 2.4.** Consider the space

\[
P^{(n)} = \left\{ p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \right\}
\]

consisting of all polynomials of degree \(\leq n\). Addition of polynomials is defined in the usual manner; for example,

\[
(x^2 - 3x) + (2x^2 - 5x + 4) = 3x^2 - 8x + 4.
\]

Note that the sum \(p(x) + q(x)\) of two polynomials of degree \(\leq n\) also has degree \(\leq n\). (However, it is not true that the sum of two polynomials of degree \(= n\) also has degree \(n\); for example \((x^2 + 1) + (-x^2 + x) = x + 1\) has degree 1 even though the two summands have degree 2. This means that the set of polynomials of degree \(= n\) is not a vector space.) The zero element of \(P^{(n)}\) is the zero polynomial. We can multiply polynomials by scalars — real constants — in the usual fashion; for example if \(p(x) = x^2 - 2x\), then \(3p(x) = 3x^2 - 6x\). The proof that \(P^{(n)}\) satisfies the vector space axioms is an easy consequence of the basic laws of polynomial algebra.
Remark: We are ignoring the fact that one can also multiply polynomials; this is not a vector space operation. Also, any scalar can be viewed as a constant polynomial, but one should really regard these as two completely different objects — one is a number, while the other is a constant function. To add to the confusion, one typically uses the same notation for these two objects; for instance, 1 could either mean the real number 1 or the constant function taking the value 1 everywhere. The reader needs to exercise due care when interpreting each occurrence.

For much of analysis, including differential equations, Fourier theory, numerical methods, and so on, the most important vector spaces consist of sets of functions with certain specified properties. The simplest such example is the following.

Example 2.5. Let \( I \subseteq \mathbb{R} \) be an interval. Consider the function space \( \mathcal{F} = \mathcal{F}(I) \) that consists of all real-valued functions \( f(x) \) defined for all \( x \in I \), which we also write as \( f: I \to \mathbb{R} \). The claim is that the function space \( \mathcal{F} \) has the structure of a vector space. Addition of functions in \( \mathcal{F} \) is defined in the usual manner: \( (f + g)(x) = f(x) + g(x) \) for all \( x \in I \). Multiplication by scalars \( c \in \mathbb{R} \) is the same as multiplication by constants, \( (cf)(x) = cf(x) \). The zero element is the constant function that is identically 0 for all \( x \in I \). The proof of the vector space axioms is straightforward, just as in the case of polynomials. Again, we ignore all additional operations — multiplication, division, inversion, composition, etc. — that affect functions; these are irrelevant as far as the vector space structure of \( \mathcal{F} \) goes.

Remark: An interval can be (a) closed, meaning that it includes its endpoints: \( I = [a, b] \), (b) open, which does not include either endpoint: \( I = (a, b) \), or (c) half open, which includes one but not the other endpoint, so \( I = [a, b) \) or \( (a, b] \). An open endpoint is allowed to be infinite; in particular, \( (-\infty, \infty) = \mathbb{R} \) is another way of writing the real line.

Example 2.6. The preceding examples are all, in fact, special cases of an even more general construction. A clue is to note that the last example of a function space does not make any use of the fact that the domain of definition of our functions is a real interval. Indeed, the construction produces a function space \( \mathcal{F}(I) \) corresponding to any subset \( I \subseteq \mathbb{R} \).

Even more generally, let \( S \) be any set. Let \( \mathcal{F} = \mathcal{F}(S) \) denote the space of all real-valued functions \( f:S \to \mathbb{R} \). Then we claim that \( V \) is a vector space under the operations of function addition and scalar multiplication. More precisely, given functions \( f \) and \( g \), we define their sum to be the function \( h = f + g \) such that \( h(x) = f(x) + g(x) \) for all \( x \in S \). Similarly, given a function \( f \) and a real scalar \( c \in \mathbb{R} \), we define the scalar multiple \( k = cf \) to be the function such that \( k(x) = cf(x) \) for all \( x \in S \). The validation of the vector space axioms is straightforward, and the reader should be able to fill in the necessary details.

In particular, if \( S \subseteq \mathbb{R} \) is an interval, then \( \mathcal{F}(S) \) coincides with the space of scalar functions described in the preceding example. If \( S \subseteq \mathbb{R}^n \) is a subset of Euclidean space, then the elements of \( \mathcal{F}(S) \) are functions \( f(x_1, \ldots, x_n) \) depending upon the \( n \) variables corresponding to the coordinates of points \( \mathbf{x} = (x_1, \ldots, x_n) \in S \) in the domain. In this fashion, the set of real-valued functions defined on any domain in \( \mathbb{R}^n \) is found to also form a vector space.
Another useful example is to let \( S = \{x_1, \ldots, x_n\} \subset \mathbb{R} \) be a finite set of real numbers. A real-valued function \( f: S \to \mathbb{R} \) is defined by its values \( f(x_1), f(x_2), \ldots, f(x_n) \) at the specified points. In applications, one can view such functions as indicating the sample values of a scalar function \( f(x) \in \mathcal{F}(\mathbb{R}) \) taken at the sample points \( x_1, \ldots, x_n \). For example, if \( f(x) = x^2 \) and the sample points are \( x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3 \), then the corresponding sample values are \( (f(x_1), f(x_2), f(x_3), f(x_4)) = (0, 1, 4, 9) \). When measuring a physical quantity, e.g., temperature, velocity, pressure, etc., one typically only measures a finite set of sample values. The intermediate, non-recorded values between the sample points are then reconstructed through some form of interpolation — a topic that we shall visit in depth later on. Interestingly, the sample values \( f(x_i) \) can be identified with the entries \( f_i \) of a vector

\[
\mathbf{f} = (f_1, f_2, \ldots, f_n)^T = (f(x_1), f(x_2), \ldots, f(x_n))^T \in \mathbb{R}^n,
\]

known as the vector sample. Every sampled function \( f \in \mathcal{F}(x_1, \ldots, x_n) \) corresponds to a unique vector \( \mathbf{f} \in \mathbb{R}^n \) and vice versa. (However, different scalar functions \( f: \mathbb{R} \to \mathbb{R} \) can have the same sample values.) Addition of sample functions corresponds to addition of their sample vectors, as does scalar multiplication. Thus, the vector space of sample functions \( \mathcal{F}(S) = \mathcal{F}(\{x_1, \ldots, x_n\}) \) is the same as the vector space \( \mathbb{R}^n \)! This connection between sampled functions and vectors is of fundamental importance in modern signal processing.

**Example 2.7.** The preceding construction admits yet a further generalization. We continue to let \( S \) be an arbitrary set. Let \( V \) be a vector space. The claim is that the space \( \mathcal{F}(S, V) \) consisting of all \( V \)-valued functions \( f: S \to V \) is a vector space. In other words, we replace the particular vector space \( \mathbb{R} \) in the preceding example by a general vector space, and the same conclusion holds. The operations of function addition and scalar multiplication are defined in the evident manner: \( (f + g)(x) = f(x) + g(x) \) and \( (cf)(x) = cf(x) \) for \( x \in S \), where we are using the vector addition and scalar multiplication operations on \( V \) to induce corresponding operations on \( V \)-valued functions. The proof that \( \mathcal{F}(S, V) \) satisfies all of the vector space axioms proceeds as before.

The most important example is when \( S \subset \mathbb{R}^n \) is a domain in Euclidean space and \( V = \mathbb{R}^m \) is itself a Euclidean space. In this case, the elements of \( \mathcal{F}(S, \mathbb{R}^m) \) consist of vector-valued functions \( f: S \to \mathbb{R}^m \), so that \( f(x) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))^T \) is a column vector consisting of \( m \) functions of \( n \) variables, all defined on a common domain \( S \). The general construction implies that addition and scalar multiplication of vector-valued functions is done componentwise; for example

\[
2 \begin{pmatrix} x^2 \\ e^x - 4 \end{pmatrix} - \begin{pmatrix} \cos x \\ x \end{pmatrix} = \begin{pmatrix} 2x^2 - \cos x \\ 2e^x - x - 8 \end{pmatrix}.
\]

Of particular importance are the vector fields arising in physics, including gravitational force fields, electromagnetic fields, fluid velocity fields, and so on; see Exercise [ ].

### 2.2. Subspaces.

In the preceding section, we were introduced to the most basic vector spaces that play a role in this text. Almost all of the important vector spaces arising in applications appear as particular subsets of these key examples.
**Definition 2.8.** A *subspace* of a vector space $V$ is a subset $W \subset V$ which is a vector space in its own right.

Since elements of $W$ also belong to $V$, the operations of vector addition and scalar multiplication for $W$ are induced by those of $V$. In particular, $W$ must contain the zero element of $V$ in order to satisfy axiom (c). The verification of the vector space axioms for a subspace is particularly easy: we only need check that addition and scalar multiplication keep us within the subspace.

**Proposition 2.9.** A subset $W \subset V$ of a vector space is a subspace if and only if

(a) for every $v, w \in W$, the sum $v + w \in W$, and

(b) for every $v \in W$ and every $c \in \mathbb{R}$, the scalar product $cv \in W$.

**Proof:** The proof is essentially trivial. For example, to show commutativity, given $v, w \in W$, we can regard them as elements of $V$, in which case $v + w = w + v$ because $V$ is a vector space. But the closure condition implies that the common sum also belongs to $W$, and so the commutativity axiom also holds for elements of $W$. The other axioms are equally easy to validate.  

*Q.E.D.*

**Remark:** Condition (a) says that a subspace must be closed under addition, while (b) says it must also be closed under scalar multiplication. It will sometimes be useful to combine the two closure conditions. Thus, to prove $W \subset V$ is a subspace it suffices to check that $cv + dw \in W$ for every $v, w \in W$ and $c, d \in \mathbb{R}$.

**Example 2.10.** Let us list some examples of subspaces of the three-dimensional Euclidean space $\mathbb{R}^3$. In each case, we must verify the closure conditions.

(a) The trivial subspace $W = \{0\}$. Demonstrating closure is easy: since there is only one element $0$ in $W$, we just need to check that $0 + 0 = 0 \in W$ and $c0 = 0 \in W$ for any scalar $c$.

(b) The entire space $W = \mathbb{R}^3$. Here closure is immediate because $\mathbb{R}^3$ is a vector space in its own right.

(c) The set of all vectors of the form $(x, y, 0)^T$, i.e., the $(x, y)$-coordinate plane. To prove closure, we check that the sum $(x, y, 0)^T + (\hat{x}, \hat{y}, 0)^T = (x + \hat{x}, y + \hat{y}, 0)^T$, and any scalar multiple $c(x, y, 0)^T = (cx, cy, 0)^T$, of vectors in the $(x, y)$-plane also lie in the plane.

(d) The set of solutions $(x, y, z)^T$ to the homogeneous linear equation

$$3x + 2y - z = 0.$$  

Indeed, if $x = (x, y, z)^T$ is a solution, then so is any scalar multiple $c x = (cx, cy, cz)^T$ since

$$3(c x) + 2(c y) - (c z) = c(3x + 2y - z) = 0.$$  

Moreover, if $\hat{x} = (\hat{x}, \hat{y}, \hat{z})$ is a second solution, the sum $x + \hat{x} = (x + \hat{x}, y + \hat{y}, z + \hat{z})^T$ is also a solution since

$$3(x + \hat{x}) + 2(y + \hat{y}) - (z + \hat{z}) = (3x + 2y - z) + (3 \hat{x} + 2 \hat{y} - \hat{z}) = 0.$$
Note that the solution space is a two-dimensional plane consisting of all vectors which are perpendicular (orthogonal) to the vector \((3, 2, -1)^T\).

\((e)\) The set of all vectors lying in the plane spanned by the vectors \(v_1 = (2, -3, 0)^T\) and \(v_2 = (1, 0, 3)^T\). In other words, we consider all vectors of the form

\[ v = a v_1 + b v_2 = a \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2a + b \\ -3a \\ 3b \end{pmatrix}, \]

where \(a, b \in \mathbb{R}\) are arbitrary scalars. If \(v = a v_1 + b v_2\) and \(w = \tilde{a} v_1 + \tilde{b} v_2\) are any two vectors of this form, so is

\[ c v + d w = c(a v_1 + b v_2) + d(\tilde{a} v_1 + \tilde{b} v_2) = (ac + \tilde{a} d)v_1 + (bc + \tilde{b} d)v_2 = \tilde{a} v_1 + \tilde{b} v_2, \]

where \(\tilde{a} = ac + \tilde{a} d, \tilde{b} = bc + \tilde{b} d\). This proves that the plane is a subspace of \(\mathbb{R}^3\). The reader might already have noticed that this subspace is the same plane that was considered in item \((d)\).

**Example 2.11.** The following subsets of \(\mathbb{R}^3\) are not subspaces.

\((a)\) The set \(P\) of all vectors of the form \((x, y, 1)^T\), i.e., the plane parallel to the \(xy\) coordinate plane passing through \((0, 0, 1)^T\). Indeed, \(0 \not\in P\), which is the most basic requirement for a subspace. In fact, neither of the closure axioms hold for this subset.

\((b)\) The positive orthant \(O^+ = \{x > 0, y > 0, z > 0\}\). While the sum of two vectors in \(O^+\) belongs to \(O^+\), multiplying by negative scalars takes us outside the orthant, violating closure under scalar multiplication.

\((c)\) The unit sphere \(S^2 = \{x^2 + y^2 + z^2 = 1\}\). Again, \(0 \not\in S^2\). More generally, curved surfaces, e.g., the paraboloid \(P = \{z = x^2 + y^2\}\), are not subspaces. Although \(0 \in P\), most scalar multiples of vectors in \(P\) do not belong to \(P\). For example, \((1, 1, 2)^T \in P\), but \(2(1, 1, 2)^T = (2, 2, 4)^T \not\in P\).

In fact, there are only four fundamentally different types of subspaces \(W \subset \mathbb{R}^3\) of three-dimensional Euclidean space:

\((i)\) The entire three-dimensional space \(W = \mathbb{R}^3\),

\((ii)\) a plane passing through the origin,

\((iii)\) a line passing through the origin,

\((iv)\) a point: the trivial subspace \(W = \{0\}\).

To verify this observation, we argue as follows. If \(W = \{0\}\) contains only the zero vector, then we are in case \((iv)\). Otherwise, \(W \subset \mathbb{R}^3\) contains a nonzero vector \(0 \neq v_1 \in W\). But since \(W\) must contain all scalar multiples \(cv_1\) of this element, it includes the entire line in the direction of \(v_1\). If \(W\) contains another vector \(v_2\) that does not lie in the line through \(v_1\), then it must contain the entire plane \(\{cv_1 + dv_2\}\) spanned by \(v_1, v_2\). Finally, if there is a third vector \(v_3\) not contained in this plane, then we claim that \(W = \mathbb{R}^3\). This final fact will be an immediate consequence of general results in this chapter, although the interested reader might try to prove it directly before proceeding.
Example 2.12. Let \( I \subseteq \mathbb{R} \) be an interval, and let \( \mathcal{F}(I) \) be the space of real-valued functions \( f : I \to \mathbb{R} \). Let us look at some of the most important examples of subspaces of \( \mathcal{F}(I) \). In each case, we need only verify the closure conditions to verify that the given subset is indeed a subspace.

(a) The space \( \mathcal{P}^{(n)} \) of polynomials of degree \( \leq n \), which we already encountered.

(b) The space \( \mathcal{P}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{P}^{(n)} \) consisting of all polynomials.

(c) The space \( C^0(I) \) of all continuous functions. Closure of this subspace relies on knowing that if \( f(x) \) and \( g(x) \) are continuous, then both \( f(x) + g(x) \) and \( cf(x) \) for any \( c \in \mathbb{R} \) are also continuous — two basic results from calculus.

(d) More restrictively, one can consider the subspace \( C^n(I) \) consisting of all functions \( f(x) \) that have \( n \) continuous derivatives \( f'(x), f''(x), \ldots, f^{(n)}(x) \) on \( I \). Again, we need to know that if \( f(x) \) and \( g(x) \) have \( n \) continuous derivatives, so do \( f(x) + g(x) \) and \( cf(x) \) for any \( c \in \mathbb{R} \).

(e) The space \( C^{\infty}(I) = \bigcap_{n \geq 0} C^n(I) \) of infinitely differentiable or smooth functions is also a subspace. (The fact that this intersection is a subspace follows directly from Exercise 1.)

(f) The space \( A(I) \) of analytic functions on the interval \( I \). Recall that a function \( f(x) \) is called analytic at a point \( a \) if it is smooth, and, moreover, its Taylor series

\[
f(a) + f'(a) (x - a) + \frac{1}{2} f''(a) (x - a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

converges to \( f(x) \) for all \( x \) sufficiently close to \( a \). (It does not have to converge on the entire interval \( I \).) Not every smooth function is analytic, and so \( A(I) \subseteq C^{\infty}(I) \). An explicit example is the function

\[
f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0. \end{cases}
\]

It can be shown that every derivative of this function at 0 exists and equals zero: \( f^{(n)}(0) = 0, \) \( n = 0, 1, 2, \ldots \), and so the function is smooth. However, its Taylor series at \( a = 0 \) is \( 0 + 0x + 0x^2 + \cdots \equiv 0 \), which converges to the zero function, not to \( f(x) \). Therefore \( f(x) \) is not analytic at \( a = 0 \).

(g) The set of all mean zero functions. The mean or average of an integrable function defined on a closed interval \( I = [a, b] \) is the real number

\[
\overline{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

In particular, \( f \) has mean zero if and only if \( \int_a^b f(x) \, dx = 0 \). Note that \( \overline{f + g} = \overline{f} + \overline{g} \), and so the sum of two mean zero functions also has mean zero. Similarly, \( c \overline{f} = c \overline{f} \), and any scalar multiple of a mean zero function also has mean zero.

\[\text{If } I = [a, b] \text{ is closed, we use the appropriate one-sided derivatives at its endpoints.}\]
(h) Let \( x_0 \in I \) be a given point. Then the set of all functions \( f(x) \) that vanish at the point, \( f(x_0) = 0 \), is a subspace. Indeed, if \( f(x_0) = 0 \) and \( g(x_0) = 0 \), then clearly \((f+g)(x_0) = 0\) and \( cf(x_0) = 0\), proving closure. This example can evidently be generalized to functions that vanish at several points, or even on an entire subset.

(i) The set of all solutions \( u = f(x) \) to the homogeneous linear differential equation

\[
u'' + 2u' - 3u = 0.
\]

Indeed, if \( u = f(x) \) and \( u = g(x) \) are solutions, so are \( u = f(x) + g(x) \) and \( u = cf(x) \) for any \( c \in \mathbb{R} \). Note that we do not need to actually solve the equation to verify these claims! They follow directly from linearity; for example

\[
(f + g)'' + 2(f + g)' - 3(f + g) = (f'' + 2f' - 3f) + (g'' + 2g' - 3g) = 0.
\]

**Warning:** In the last three examples, 0 is essential for the indicated set of functions to be a subspace. The set of functions such that \( f(x_0) = 1 \), say, is not a subspace. The set of functions with a fixed nonzero mean, say \( \bar{f} = 3 \), is also not a subspace. Nor is the set of solutions to an inhomogeneous ordinary differential equation, say \( u'' + 2u' - 3u = x - 3 \). None of these subsets contain the zero function, nor do they satisfy the closure conditions.

### 2.3. Span and Linear Independence.

The definition of the span of a collection of elements of a vector space generalizes, in a natural fashion, the geometric notion of two vectors spanning a plane in \( \mathbb{R}^3 \). As such, it forms the first of two important, general methods for constructing subspaces of vector spaces.

**Definition 2.13.** Let \( v_1, \ldots, v_k \) belong to a vector space \( V \). A sum of the form

\[
c_1v_1 + c_2v_2 + \cdots + c_kv_k = \sum_{i=1}^{k} c_i v_i,
\]

where the coefficients \( c_1, c_2, \ldots, c_k \) are any scalars, is known as a **linear combination** of the elements \( v_1, \ldots, v_k \). Their **span** is the subset \( W = \text{span} \{v_1, \ldots, v_k\} \subset V \) consisting of all possible linear combinations (2.5).

Thus, \( 3v_1 + v_2 - 2v_3, 8v_1 - \frac{1}{3}v_3, v_2 = 0v_1 + 1v_2 + 0v_3, \) and \( 0 = 0v_1 + 0v_2 + 0v_3 \) are four different linear combinations of the three vector space elements \( v_1, v_2, v_3 \in V \). The key observation is that the span always forms a subspace.

**Proposition 2.14.** The span \( W = \text{span} \{v_1, \ldots, v_k\} \) of a collection of vector space elements forms a subspace of the underlying vector space.

**Proof:** We need to show that if

\[
v = c_1v_1 + \cdots + c_kv_k \quad \text{and} \quad \tilde{v} = \tilde{c}_1v_1 + \cdots + \tilde{c}_kv_k
\]

are any two linear combinations, then their sum

\[
v + \tilde{v} = (c_1 + \tilde{c}_1)v_1 + \cdots + (c_k + \tilde{c}_k)v_k,
\]

is also a linear combination.
is also a linear combination, as is any scalar multiple

\[ a \mathbf{v} = (a c_1) \mathbf{v}_1 + \cdots + (a c_k) \mathbf{v}_k. \]  

Q.E.D.

**Example 2.15.** Examples of subspaces spanned by vectors in \(\mathbb{R}^3\):

(i) If \(\mathbf{v}_1 \neq \mathbf{0}\) is any non-zero vector in \(\mathbb{R}^3\), then its span is the line \(\{c \mathbf{v}_1 \mid c \in \mathbb{R}\}\) in the direction of \(\mathbf{v}_1\). If \(\mathbf{v}_1 = \mathbf{0}\), then its span just consists of the origin.

(ii) If \(\mathbf{v}_1\) and \(\mathbf{v}_2\) are any two vectors in \(\mathbb{R}^3\), then their span is the set of all vectors of the form \(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2\). Typically, such a span forms a plane passing through the origin. However, if \(\mathbf{v}_1\) and \(\mathbf{v}_2\) are parallel, then their span is just a line. The most degenerate case is when \(\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}\), where the span is just a point — the origin.

(iii) If we are given three non-coplanar vectors \(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\), then their span is all of \(\mathbb{R}^3\), as we shall prove below. However, if they all lie in a plane, then their span is the plane — unless they are all parallel, in which case their span is a line — or, when \(\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{0}\), a single point.

Thus, any subspace of \(\mathbb{R}^3\) can be realized as the span of some set of vectors. Note that we can also consider the span of four or more vectors, but the range of possible subspaces is limited, as we noted above, to either a point (the origin), a line, a plane, or the entire three-dimensional space. A crucial question, that we will return to shortly, is to determine when a given vector belongs to the span of a prescribed set of vectors.

**Remark:** It is entirely possible for different sets of vectors to span the same subspace. For instance, the pair of vectors \(\mathbf{e}_1 = (1, 0, 0)^T\) and \(\mathbf{e}_2 = (0, 1, 0)^T\) span the \(xy\) plane in \(\mathbb{R}^3\), as do the three coplanar vectors \(\mathbf{v}_1 = (1, -1, 0)^T, \mathbf{v}_2 = (-1, 2, 0)^T, \mathbf{v}_3 = (2, 1, 0)^T\).

**Example 2.16.** Let \(V = \mathcal{F}(\mathbb{R})\) denote the space of all scalar functions \(f(x)\).

(a) The span of the three monomials \(f_1(x) = 1, f_2(x) = x\) and \(f_3(x) = x^2\) is the set of all functions of the form

\[ f(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = c_1 + c_2 x + c_3 x^2, \]

where \(c_1, c_2, c_3\) are arbitrary scalars (constants). In other words, span \(\{1, x, x^2\} = \mathcal{P}^{(2)}\) is the subspace of all quadratic (degree \(\leq 2\)) polynomials. In a similar fashion, the space \(\mathcal{P}^{(n)}\) of polynomials of degree \(\leq n\) is spanned by the monomials \(1, x, x^2, \ldots, x^n\).

(b) The next example plays a key role in many applications. Let \(\omega \in \mathbb{R}\) be fixed. Consider the two basic trigonometric functions \(f_1(x) = \cos \omega x, f_2(x) = \sin \omega x\) of frequency \(\omega\), and hence period \(2\pi/\omega\). Their span consists of all functions of the form

\[ f(x) = c_1 f_1(x) + c_2 f_2(x) = c_1 \cos \omega x + c_2 \sin \omega x. \quad (2.6) \]

For example, the function \(\cos(\omega x + 2)\) lies in the span because, by the addition formula for the cosine,

\[ \cos(\omega x + 2) = \cos 2 \cos \omega x - \sin 2 \sin \omega x \]

is a linear combination of \(\cos \omega x\) and \(\sin \omega x\).

We can express a general function in the span in the alternative *phase-amplitude form*

\[ f(x) = c_1 \cos \omega x + c_2 \sin \omega x = r \cos(\omega x - \delta), \quad (2.7) \]
in which $r \geq 0$ is known as the *amplitude* and $0 \leq \delta < 2\pi$ the *phase shift*. Indeed, expanding the right hand side, we find

\[ r \cos(\omega x - \delta) = r \cos \delta \cos \omega x + r \sin \delta \sin \omega x, \quad \text{and hence} \quad c_1 = r \cos \delta, \quad c_2 = r \sin \delta. \]

Thus, $(r, \delta)$ are the polar coordinates of the point $c = (c_1, c_2) \in \mathbb{R}^2$ prescribed by the coefficients. Thus, any combination of $\sin \omega x$ and $\cos \omega x$ can be rewritten as a single cosine, with a phase lag. Figure 2.1 shows the particular function $3\cos(2x - 1)$ which has amplitude $r = 3$, frequency $\omega = 2$, and phase shift $\delta = 1$. The first peak appears at $x = \delta/\omega = \frac{1}{2}$.

(c) The space $T^{(2)}$ of *quadratic trigonometric polynomials* is spanned by the functions

\[ 1, \quad \cos x, \quad \sin x, \quad \cos^2 x, \quad \cos x \sin x, \quad \sin^2 x. \]

Its general element is a linear combination

\[ q(x) = c_0 + c_1 \cos x + c_2 \sin x + c_3 \cos^2 x + c_4 \cos x \sin x + c_5 \sin^2 x, \quad (2.8) \]

where $c_0, \ldots, c_5$ are arbitrary constants. A more useful spanning set for the same subspace is the trigonometric functions

\[ 1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x. \quad (2.9) \]

Indeed, by the double angle formulas, both

\[ \cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2 \sin x \cos x, \]

have the form of a quadratic trigonometric polynomial (2.8), and hence both belong to $T^{(2)}$. On the other hand, we can write

\[ \cos^2 x = \frac{1}{2} \cos 2x + \frac{1}{2}, \quad \cos x \sin x = \frac{1}{2} \sin 2x, \quad \sin^2 x = -\frac{1}{2} \cos 2x + \frac{1}{2}, \]

in terms of the functions (2.9). Therefore, the original linear combination (2.8) can be written in the alternative form

\[ q(x) = \left( c_0 + \frac{1}{2} c_3 + \frac{1}{2} c_5 \right) + c_1 \cos x + c_2 \sin x + \left( \frac{1}{2} c_3 - \frac{1}{2} c_5 \right) \cos 2x + \frac{1}{2} c_4 \sin 2x \]
and so the functions (2.9) do indeed span $T^{(2)}$. It is worth noting that we first characterized $T^{(2)}$ as the span of 6 functions, whereas the second characterization only required 5 functions. It turns out that 5 is the minimal number of functions needed to span $T^{(2)}$, but the proof of this fact will be deferred until Chapter 3.

(d) The homogeneous linear ordinary differential equation

$$u'' + 2u' - 3u = 0.$$  \hspace{1cm} (2.11)

considered in part (i) of Example 2.12 has two independent solutions: \( f_1(x) = e^x \) and \( f_2(x) = e^{-3x} \). (Now may be a good time for you to review the basic techniques for solving linear, constant coefficient ordinary differential equations.) The general solution to the differential equation is a linear combination

$$u = c_1 f_1(x) + c_2 f_2(x) = c_1 e^x + c_2 e^{-3x}.$$  

Thus, the vector space of solutions to (2.11) is described as the span of these two basic solutions. The fact that there are no other solutions is not obvious, but relies on the basic existence and uniqueness theorems for linear ordinary differential equations; see Theorem 7.33 for further details.

**Remark:** One can also define the span of an infinite collection of elements of a vector space. To avoid convergence issues, one should only consider finite linear combinations (2.5). For example, the span of the monomials \( 1, x, x^2, x^3, \ldots \) is the space \( P^{(\infty)} \) of all polynomials. (Not the space of convergent Taylor series.) Similarly, the span of the functions \( 1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots \) is the space of all trigonometric polynomials, to be discussed in great detail in Chapter 12.

### Linear Independence and Dependence

Most of the time, all of the vectors used to form a span are essential. For example, we cannot use fewer than two vectors to span a plane in \( \mathbb{R}^3 \) since the span of a single vector is at most a line. However, in degenerate situations, some of the spanning elements may be redundant. For instance, if the two vectors are parallel, then their span is a line, but only one of the vectors is really needed to define the line. Similarly, the subspace spanned by the polynomials

$$p_1(x) = x - 2, \quad p_2(x) = x^2 - 5x + 4, \quad p_3(x) = 3x^2 - 4x, \quad p_4(x) = x^2 - 1.$$  \hspace{1cm} (2.12)

is the vector space \( P^{(2)} \) of quadratic polynomials. But only three of the polynomials are really required to span \( P^{(2)} \). (The reason will become clear soon, but you may wish to see if you can demonstrate this on your own.) The elimination of such superfluous spanning elements is encapsulated in the following basic definition.

**Definition 2.17.** The vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \in V \) are called *linearly dependent* if there exists a collection of scalars \( c_1, \ldots, c_k, \) not all zero, such that

$$c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = \mathbf{0}.$$  \hspace{1cm} (2.13)

Vectors which are not linearly dependent are called *linearly independent.*
The restriction that the \(c_i\)'s not all simultaneously vanish is essential. Indeed, if \(c_1 = \cdots = c_k = 0\), then the linear combination (2.13) is automatically zero. To check linear independence, one needs to show that the only linear combination that produces the zero vector (2.13) is this trivial one. In other words, \(c_1 = \cdots = c_k = 0\) is the one and only solution to the vector equation (2.13).

**Example 2.18.** Some examples of linear independence and dependence:

(a) The vectors

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, & v_2 &= \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, & v_3 &= \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix},
\end{align*}
\]

are linearly dependent. Indeed,

\[
v_1 - 2v_2 + v_3 = 0.
\]

On the other hand, the first two vectors \(v_1, v_2\) are linearly independent. To see this, suppose that

\[
c_1 v_1 + c_2 v_2 = \begin{pmatrix} c_1 \\ 2c_1 + 3c_2 \\ -c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

For this to happen, the coefficients \(c_1, c_2\) must satisfy the homogeneous linear system

\[
c_1 = 0, \quad 2c_1 + 3c_2 = 0, \quad -c_1 + c_2 = 0,
\]

which, as you can check, has only the trivial solution \(c_1 = c_2 = 0\), proving linear independence.

(b) In general, any collection \(v_1, \ldots, v_k\) that includes the zero vector, say \(v_1 = 0\), is automatically linearly dependent, since \(1v_1 + 0v_2 + \cdots + 0v_k = 0\) is a nontrivial linear combination that adds up to 0.

(c) The polynomials (2.12) are linearly dependent; indeed,

\[
p_1(x) + p_2(x) - p_3(x) + 2p_4(x) \equiv 0
\]

is a nontrivial linear combination that vanishes identically. On the other hand, the first three polynomials,

\[
p_1(x) = x - 2, \quad p_2(x) = x^2 - 5x + 4, \quad p_3(x) = 3x^2 - 4x,
\]

are linearly independent. Indeed, if the linear combination

\[
c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = (c_2 + 3c_3)x^2 + (c_1 - 5c_2 - 4c_3)x - 2c_1 + 4c_2 \equiv 0
\]

is the zero polynomial, then its coefficients must vanish, and hence \(c_1, c_2, c_3\) are required to solve the homogeneous linear system

\[
c_2 + 3c_3 = 0, \quad c_1 - 5c_2 - 4c_3 = 0, \quad -2c_1 + 4c_2 = 0.
\]

But this has only the trivial solution \(c_1 = c_2 = c_3 = 0\), and so linear independence follows.
Remark: In the last example, we are using the basic fact that a polynomial is identically zero,

\[ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \equiv 0 \quad \text{for all} \quad x, \]
if and only if its coefficients all vanish: \( a_0 = a_1 = \cdots = a_n = 0 \). This is equivalent to the "self-evident" fact that the basic monomial functions \( 1, x, x^2, \ldots, x^n \) are linearly independent. Exercise \( \blacksquare \) asks for a bona fide proof.

**Example 2.19.** The set of quadratic trigonometric functions

\[ 1, \ \cos x, \ \sin x, \ \cos^2 x, \ \cos x \sin x, \ \sin^2 x, \]
that were used to define the vector space \( T^{(2)} \) of quadratic trigonometric polynomials, are, in fact, linearly dependent. This is a consequence of the basic trigonometric identity

\[ \cos^2 x + \sin^2 x \equiv 1, \]
which can be rewritten as a nontrivial linear combination

\[ 1 + 0 \cos x + 0 \sin x - \cos^2 x + 0 \cos x \sin x - \sin^2 x \equiv 0 \]
that sums to the zero function. On the other hand, the alternative spanning set

\[ 1, \ \cos x, \ \sin x, \ \cos 2x, \ \sin 2x, \]
is linearly independent, since the only identically zero linear combination

\[ c_0 + c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x \equiv 0 \]
turns out to be the trivial one \( c_0 = \ldots = c_4 = 0 \). However, the latter fact is not as obvious, and requires a bit of work to prove directly; see Exercise \( \blacksquare \). An easier proof, based on orthogonality, will appear in Chapter 5.

Let us now focus our attention on the linear independence or dependence of a set of vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n \) in Euclidean space. We begin by forming the \( n \times k \) matrix \( A = (\mathbf{v}_1 \ldots \mathbf{v}_k) \) whose columns are the given vectors. (The fact that we use column vectors is essential here.) The key is a very useful formula

\[ A \mathbf{c} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k, \quad \text{where} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}, \quad (2.14) \]
that expresses any linear combination in terms of matrix multiplication. For example,

\[
\begin{pmatrix}
 1 & 3 & 0 \\
-1 & 2 & 1 \\
4 & -1 & -2
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix}
= \begin{pmatrix}
c_1 + 3c_2 \\
-c_1 + 2c_2 + c_3 \\
4c_1 - c_2 - 2c_3
\end{pmatrix}
= \begin{pmatrix}
c_1 & -1 \\
4 & -2
\end{pmatrix}
\begin{pmatrix}
1 \\
3 \\
0
\end{pmatrix}
+ \begin{pmatrix}
c_2 \\
2 \\
1
\end{pmatrix}
\begin{pmatrix}
3 \\
-1 \\
-2
\end{pmatrix}.
\]
Formula (2.14) follows directly from the basic rules of matrix multiplication; see also Exercise \( \blacksquare \). It allows us to reformulate the notions of linear independence and span in terms of linear systems of equations. The key result is the following:
Theorem 2.20. Let \( v_1, \ldots, v_k \in \mathbb{R}^n \) and let \( A = (v_1 \ldots v_k) \) be the corresponding \( n \times k \) matrix.

(a) The vectors \( v_1, \ldots, v_k \in \mathbb{R}^n \) are linearly dependent if and only if there is a non-zero solution \( c \neq 0 \) to the homogeneous linear system \( A c = 0 \).

(b) The vectors are linearly independent if and only if the only solution to the homogeneous system \( A c = 0 \) is the trivial one \( c = 0 \).

(c) A vector \( b \) lies in the span of \( v_1, \ldots, v_k \) if and only if the linear system \( A c = b \) is compatible, i.e., it has at least one solution.

Proof: We prove the first statement, leaving the other two as exercises for the reader. The condition that \( v_1, \ldots, v_k \) be linearly dependent is that there is a nonzero vector

\[
\mathbf{c} = (c_1, c_2, \ldots, c_k)^T \neq 0
\]

such that the linear combination

\[
A \mathbf{c} = c_1 v_1 + \cdots + c_k v_k = 0.
\]

Therefore, linear dependence requires the existence of a nontrivial solution to the homogeneous linear system \( A \mathbf{c} = 0 \). Q.E.D.

Example 2.21. Let us determine whether the vectors

\[
\begin{align*}
v_1 & = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, & v_2 & = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, & v_3 & = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}, & v_4 & = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}, \\
\end{align*}
\]

are linearly independent or linearly dependent. We combine them as column vectors into a single matrix

\[
A = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 0 & -4 & 2 \\ -1 & 4 & 6 & 3 \end{pmatrix}.
\]

According to Theorem 2.20, we need to figure out whether there are any nontrivial solutions to the homogeneous equation \( A \mathbf{c} = 0 \); this can be done by reducing \( A \) to row echelon form

\[
U = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & -6 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

The general solution to the homogeneous system \( A \mathbf{c} = 0 \) is

\[
\mathbf{c} = (2c_3 - c_4, -c_3 - c_4, c_3, c_4)^T,
\]

where \( c_3, c_4 \) — the free variables — are arbitrary. Any nonzero choice of \( c_3, c_4 \) will produce a nontrivial linear combination

\[
(2c_3 - c_4) v_1 + (-c_3 - c_4) v_2 + c_3 v_3 + c_4 v_4 = 0
\]

that adds up to the zero vector. Therefore, the vectors (2.15) are linearly dependent.
In fact, Theorem 1.45 tells us that, in this particular case, we didn’t even need to do the row reduction if we only needed to answer the question of linear dependence or linear independence. Any coefficient matrix with more columns than rows automatically has a nontrivial solution to the associated homogeneous system. This implies the following:

**Lemma 2.22.** Any collection of \( k > n \) vectors in \( \mathbb{R}^n \) is linearly dependent.

*Warning:* The converse to this lemma is *not* true. For example, the two vectors \( \mathbf{v}_1 = (1,2,3)^T \) and \( \mathbf{v}_2 = (-2,-4,-6)^T \) in \( \mathbb{R}^3 \) are linearly dependent since \( 2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0} \). For a collection of \( n \) or fewer vectors in \( \mathbb{R}^n \), one does need to perform the elimination.

Lemma 2.22 is a particular case of the following general characterization of linearly independent vectors.

**Proposition 2.23.** A set of \( k \) vectors in \( \mathbb{R}^n \) is linearly independent if and only if the corresponding \( n \times k \) matrix \( A \) has rank \( k \). In particular, this requires \( k \leq n \).

Or, to state the result another way, the vectors are linearly independent if and only if the linear system \( A \mathbf{c} = \mathbf{0} \) has no free variables. The proposition is an immediate corollary of Propositions 2.20 and 1.45.

**Example 2.21. (continued)** Let us now see which vectors \( \mathbf{b} \in \mathbb{R}^3 \) lie in the span of the vectors (2.15). According to Theorem 2.20, this will be the case if and only if the linear system \( A \mathbf{x} = \mathbf{b} \) has a solution. Since the resulting row echelon form (2.16) has a row of all zeros, there will be a compatibility condition on the entries of \( \mathbf{b} \), and therefore not every vector lies in the span. To find the precise condition, we augment the coefficient matrix, and apply the same row operations, leading to the reduced augmented matrix

\[
\begin{pmatrix}
1 & 3 & 1 & 4 & b_1 \\
0 & -6 & -6 & -6 & b_2 - 2b_1 \\
0 & 0 & 0 & 0 & b_3 + \frac{7}{6}b_2 - \frac{4}{3}b_1
\end{pmatrix}.
\]

Therefore, \( \mathbf{b} = (b_1,b_2,b_3)^T \) lies in the span of these four vectors if and only if

\[-\frac{4}{3}b_1 + \frac{7}{6}b_2 + b_3 = 0.
\]

In other words, these four vectors only span a plane in \( \mathbb{R}^3 \).

The same method demonstrates that a collection of vectors will span all of \( \mathbb{R}^n \) if and only if the row echelon form of the associated matrix contains no all zero rows, or, equivalently, the rank is equal to \( n \), the number of rows in the matrix.

**Proposition 2.24.** A collection of \( k \) vectors will span \( \mathbb{R}^n \) if and only if their \( n \times k \) matrix has rank \( n \). In particular, this requires \( k \geq n \).

*Warning:* Not every collection of \( n \) or more vectors in \( \mathbb{R}^n \) will span all of \( \mathbb{R}^n \). A counterexample was already provided by the vectors (2.15).
2.4. Bases.

In order to span a vector space or subspace, we must use a sufficient number of distinct elements. On the other hand, including too many elements in the spanning set will violate linear independence, and cause redundancies. The optimal spanning sets are those that are also linearly independent. By combining the properties of span and linear independence, we arrive at the all-important concept of a “basis”.

**Definition 2.25.** A basis of a vector space $V$ is a finite collection of elements $v_1, \ldots, v_n \in V$ which (a) span $V$, and (b) are linearly independent.

Bases are absolutely fundamental in all areas of linear algebra and linear analysis, including matrix algebra, geometry of Euclidean space, solutions to linear differential equations, both ordinary and partial, linear boundary value problems, Fourier analysis, signal and image processing, data compression, control systems, and so on.

**Example 2.26.** The standard basis of $\mathbb{R}^n$ consists of the $n$ vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad (2.17)$$

so that $e_i$ is the vector with 1 in the $i^{th}$ slot and 0’s elsewhere. We already encountered these vectors — they are the columns of the $n \times n$ identity matrix (1.39). They clearly span $\mathbb{R}^n$ since we can write any vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n, \quad (2.18)$$

as a linear combination, whose coefficients are its entries. Moreover, the only linear combination that gives the zero vector $x = 0$ is the trivial one $x_1 = \cdots = x_n = 0$, which shows that $e_1, \ldots, e_n$ are linearly independent.

**Remark:** In the three-dimensional case $\mathbb{R}^3$, a common physical notation for the standard basis is

$$i = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad j = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad k = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.19)$$

But we will retain the general notation $e_i$ throughout this text. The standard basis is but one of many possible bases for $\mathbb{R}^3$. Indeed, any three non-coplanar vectors can be used to form a basis. This is a consequence of the following general characterization of bases in Euclidean space.
Theorem 2.27. Every basis of $\mathbb{R}^n$ contains exactly $n$ vectors. Furthermore, a set of $n$ vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ is a basis if and only if the $n \times n$ matrix $A = (v_1 \ldots v_n)$ is nonsingular.

Proof: This is a direct consequence of Theorem 2.20. Linear independence requires that the only solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ is the trivial one $\mathbf{x} = \mathbf{0}$. On the other hand, a vector $\mathbf{b} \in \mathbb{R}^n$ will lie in the span of $v_1, \ldots, v_n$ if and only if the linear system $A\mathbf{x} = \mathbf{b}$ has a solution. For $v_1, \ldots, v_n$ to span $\mathbb{R}^n$, this must hold for all possible right hand sides $\mathbf{b}$. Theorem 1.7 tells us that both results require that $A$ be nonsingular, i.e., have maximal rank $n$. Q.E.D.

Thus, every basis of $n$-dimensional Euclidean space $\mathbb{R}^n$ contains the same number of vectors, namely $n$. This is a general fact, and motivates a linear algebra characterization of dimension.

Theorem 2.28. Suppose the vector space $V$ has a basis $v_1, \ldots, v_n$. Then every other basis of $V$ has the same number of elements in it. This number is called the dimension of $V$, and written $\dim V = n$.

The proof of Theorem 2.28 rests on the following lemma.

Lemma 2.29. Suppose $v_1, \ldots, v_n$ span a vector space $V$. Then every set of $k > n$ elements $w_1, \ldots, w_k \in V$ is linearly dependent.

Proof: Let us write each element

$$w_j = \sum_{i=1}^{n} a_{ij} v_i, \quad j = 1, \ldots, k,$$

as a linear combination of the spanning set. Then

$$c_1 w_1 + \cdots + c_k w_k = \sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} c_j v_i.$$

This linear combination will be zero whenever $c = (c_1, c_2, \ldots, c_k)^T$ solves the homogeneous linear system

$$\sum_{j=1}^{k} a_{ij} c_j = 0, \quad i = 1, \ldots, n,$$

consisting of $n$ equations in $k > n$ unknowns. Theorem 1.45 guarantees that every homogeneous system with more unknowns than equations always has a non-trivial solution $c \neq \mathbf{0}$, and this immediately implies that $w_1, \ldots, w_k$ are linearly dependent. Q.E.D.

Proof of Theorem 2.28: Suppose we have two bases containing a different number of elements. By definition, the smaller basis spans the vector space. But then Lemma 2.29 tell us that the elements in the larger purported basis must be linearly dependent. This contradicts our assumption that the latter is a basis, and so proves the theorem. Q.E.D.
As a direct consequence, we can now give a precise meaning to the optimality of bases.

**Theorem 2.30.** Suppose $V$ is an $n$-dimensional vector space. Then

(a) Every set of more than $n$ elements of $V$ is linearly dependent.
(b) No set of less than $n$ elements spans $V$.
(c) A set of $n$ elements forms a basis if and only if it spans $V$.
(d) A set of $n$ elements forms a basis if and only if it is linearly independent.

In other words, once we determine the dimension of a vector space, to check that a given collection with the correct number of elements forms a basis, we only need check one of the two defining properties: span or linear independence. Thus, $n$ elements that span an $n$-dimensional vector space are automatically linearly independent and hence form a basis; vice versa, $n$ linearly independent elements of $n$-dimensional vector space automatically span the space and so form a basis.

**Example 2.31.** The standard basis of the space $\mathcal{P}^{(n)}$ of polynomials of degree $\leq n$ is given by the $n + 1$ monomials $1, x, x^2, \ldots, x^n$. (A formal proof of linear independence appears in Exercise [b] ) We conclude that the vector space $\mathcal{P}^{(n)}$ has dimension $n + 1$. Thus, any collection of $n + 2$ or more polynomials of degree $\leq n$ is automatically linearly dependent. Any other basis of $\mathcal{P}^{(n)}$ must contain precisely $n + 1$ polynomials. But, not every collection of $n+1$ polynomials in $\mathcal{P}^{(n)}$ is a basis — they must be linearly independent. See Exercise [b] for details.

**Remark:** By definition, every vector space of dimension $1 \leq n < \infty$ has a basis. If a vector space $V$ has no basis, it is either the trivial vector space $V = \{0\}$, which by convention has dimension 0, or, by definition, its dimension is infinite. An infinite-dimensional vector space necessarily contains an infinite collection of linearly independent vectors, and hence no (finite) basis. Examples of infinite-dimensional vector spaces include most spaces of functions, such as the spaces of continuous, differentiable, or mean zero functions, as well as the space of all polynomials, and the space of solutions to a linear homogeneous partial differential equation. On the other hand, the solution space for a homogeneous linear ordinary differential equation turns out to be a finite-dimensional vector space. The most important example of an infinite-dimensional vector space, “Hilbert space”, to be introduced in Chapter 12, underlies most of modern analysis and function theory, [122, 126], as well as providing the theoretical setting for all of quantum mechanics, [100, 104].

**Warning:** There is a well-developed concept of a “complete basis” of such infinite-dimensional function spaces, essential in Fourier analysis, [122, 126], but this requires additional analytical constructions that are beyond our present abilities. Thus, in this book the term “basis” always means a finite collection of vectors in a finite-dimensional vector space.

**Lemma 2.32.** The elements $v_1, \ldots, v_n$ form a basis of $V$ if and only if every $x \in V$ can be written uniquely as a linear combination thereof:

$$x = c_1 v_1 + \cdots + c_n v_n = \sum_{i=1}^{n} c_i v_i$$  \hfill (2.20)
Proof: The condition that the basis span $V$ implies every $x \in V$ can be written as some linear combination of the basis elements. Suppose we can write an element

$$x = c_1 v_1 + \cdots + c_n v_n = \hat{c}_1 v_1 + \cdots + \hat{c}_n v_n$$

as two different combinations. Subtracting one from the other, we find

$$(c_1 - \hat{c}_1) v_1 + \cdots + (c_n - \hat{c}_n) v_n = 0.$$ 

Linear independence of the basis elements implies that the coefficients $c_i - \hat{c}_i = 0$. We conclude that $c_i = \hat{c}_i$, and hence the linear combinations are the same. Q.E.D.

The coefficients $(c_1, \ldots, c_n)$ in (2.20) are called the coordinates of the vector $x$ with respect to the given basis. For the standard basis (2.17) of $\mathbb{R}^n$, the coordinates of a vector $x = (x_1, x_2, \ldots, x_n)^T$ are its entries — i.e., its usual Cartesian coordinates, cf. (2.18).

**Example 2.33. A Wavelet Basis.** The vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix},$$

form a basis of $\mathbb{R}^4$. This is verified by performing Gaussian elimination on the corresponding $4 \times 4$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix},$$

to check that it is nonsingular. This basis is a very simple example of a wavelet basis; the general case will be discussed in Section 13.2. Wavelets arise in modern applications to signal and digital image processing, [43, 128].

How do we find the coordinates of a vector, say $x = (4, -2, 1, 5)^T$, relative to the basis? We need to find the coefficients $c_1, c_2, c_3, c_4$ so that

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4.$$ 

We use (2.14) to rewrite this equation in matrix form

$$x = A c \quad \text{where} \quad c = (c_1, c_2, c_3, c_4)^T.$$ 

Solving the resulting linear system by Gaussian Elimination produces the unique solution $c_1 = 2, c_2 = -1, c_3 = 3, c_4 = -2$, which are the coordinates of

$$x = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 5 \end{pmatrix} = 2v_1 - v_2 + 3v_3 - 2v_4.$$ 

in the wavelet basis.
In general, to find the coordinates of a vector \( \mathbf{x} \) with respect to a new basis of \( \mathbb{R}^n \) requires the solution of a linear system of equations, namely

\[
A \mathbf{c} = \mathbf{x} \quad \text{for} \quad \mathbf{c} = A^{-1} \mathbf{x}. \tag{2.22}
\]

Here \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \) are the Cartesian coordinates of \( \mathbf{x} \), with respect to the standard basis \( \mathbf{e}_1, \ldots, \mathbf{e}_n \), while \( \mathbf{c} = (c_1, c_2, \ldots, c_n)^T \) denotes its coordinates with respect to the new basis \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) formed by the columns of the coefficient matrix \( A = (\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_n) \). In practice, one solves for the coordinates by Gaussian Elimination, \textit{not} matrix inversion.

Why would one want to change bases? The answer is simplification and speed — many computations and formulas become much easier, and hence faster, to perform in a basis that is adapted to the problem at hand. In signal processing, wavelet bases are particularly appropriate for denoising, compression, and efficient storage of signals, including audio, still images, videos, medical images, geophysical images, and so on. These processes would be quite time-consuming, if not impossible in the case of video processing, to accomplish in the standard basis. Many other examples will appear throughout the text.

2.5. The Fundamental Matrix Subspaces.

Let us now return to the general study of linear systems of equations, which we write in our usual matrix form

\[
A \mathbf{x} = \mathbf{b}. \tag{2.23}
\]

Here \( A \) is an \( m \times n \) matrix, where \( m \) is the number of equations, so \( \mathbf{b} \in \mathbb{R}^m \), and \( n \) the number of unknowns, i.e., the entries of \( \mathbf{x} \in \mathbb{R}^n \).

\textit{Kernel and Range}

There are four important vector subspaces associated with any matrix, which play a key role in the interpretation of our solution algorithm. The first two of these subspaces are defined as follows.

\textbf{Definition 2.34.} The \textit{range} of an \( m \times n \) matrix \( A \) is the subspace \( \text{rng} \, A \subset \mathbb{R}^m \) spanned by the columns of \( A \). The \textit{kernel} of \( A \) is the subspace \( \text{ker} \, A \subset \mathbb{R}^n \) consisting of all vectors which are annihilated by \( A \), so

\[
\text{ker} \, A = \{ \mathbf{z} \in \mathbb{R}^n \mid A \mathbf{z} = \mathbf{0} \} \subset \mathbb{R}^n. \tag{2.24}
\]

An alternative name for the range is the \textit{column space} of the matrix. By definition, a vector \( \mathbf{b} \in \mathbb{R}^m \) belongs to \( \text{rng} \, A \) if and only if it can be written as a linear combination,

\[
\mathbf{b} = x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n,
\]

of the columns of \( A = (\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_n) \). By our basic matrix multiplication formula (2.14), the right hand side of this equation equals the product \( A \mathbf{x} \) of the matrix \( A \) with the column vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \), and hence \( \mathbf{b} = A \mathbf{x} \) for some \( \mathbf{x} \in \mathbb{R}^n \). Thus,

\[
\text{rng} \, A = \{ A \mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \} \subset \mathbb{R}^m. \tag{2.25}
\]
Therefore, a vector $\mathbf{b}$ lies in the range of $A$ if and only if the linear system $A\mathbf{x} = \mathbf{b}$ has a solution. The compatibility conditions for linear systems can thereby be re-interpreted as the requirements for a vector to lie in the range of the coefficient matrix.

A common alternative name for the kernel is the null space of the matrix $A$. The kernel of $A$ is the set of solutions to the homogeneous system $A\mathbf{z} = \mathbf{0}$. The proof that ker $A$ is a subspace requires us to verify the usual closure conditions: Suppose that $\mathbf{z}, \mathbf{w} \in \text{ker} A$, so that $A\mathbf{z} = \mathbf{0} = A\mathbf{w}$. Then, for any scalars $c, d$,

$$A(c\mathbf{z} + d\mathbf{w}) = cA\mathbf{z} + dA\mathbf{w} = \mathbf{0},$$

which implies that $c\mathbf{z} + d\mathbf{w} \in \text{ker} A$, proving that ker $A$ is indeed a subspace. This fact can be re-expressed as the following important superposition principle for solutions to a homogeneous system of linear equations.

**Theorem 2.35.** If $\mathbf{z}_1, \ldots, \mathbf{z}_k$ are individual solutions to the same homogeneous linear system $A\mathbf{z} = \mathbf{0}$, then so is any linear combination $c_1\mathbf{z}_1 + \cdots + c_k\mathbf{z}_k$.

**Warning:** The set of solutions to an inhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \neq \mathbf{0}$ is **not** a subspace.

**Example 2.36.** Let us compute the kernel of the matrix $A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{pmatrix}$.

Our task is to solve the homogeneous system $A\mathbf{x} = \mathbf{0}$, we only need perform the elementary row operations on $A$ itself. The resulting row echelon form $U = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ corresponds to the equations $x - 2y + 3w = 0, \; y - z - 10w = 0$. The free variables are $z, w$, and the general solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2z + 17w \\ z + 10w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 17 \\ 10 \\ 0 \\ 1 \end{pmatrix}.$$ 

The result describes the most general vector in ker $A$, which is thus the two-dimensional subspace of $\mathbb{R}^4$ spanned by the linearly independent vectors $(2, 1, 1, 0)^T, (17, 10, 0, 1)^T$. This example is indicative of a general method for finding a basis for ker $A$ that will be developed in more detail in the following section.

Once we know the kernel of the coefficient matrix $A$, i.e., the space of solutions to the homogeneous system $A\mathbf{z} = \mathbf{0}$, we are in a position to completely characterize the solutions to the inhomogeneous linear system (2.23).

**Theorem 2.37.** The linear system $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}^*$ if and only if $\mathbf{b}$ lies in the range of $A$. If this occurs, then $\mathbf{x}$ is a solution to the linear system if and only if

$$\mathbf{x} = \mathbf{x}^* + \mathbf{z}, \quad (2.26)$$

where $\mathbf{z} \in \text{ker} A$ is an arbitrary element of the kernel of $A$. 

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Proof: We already demonstrated the first part of the theorem. If \( A \mathbf{x} = \mathbf{b} = A \mathbf{x}^* \) are any two solutions, then their difference \( \mathbf{z} = \mathbf{x} - \mathbf{x}^* \) satisfies
\[
A \mathbf{z} = A(\mathbf{x} - \mathbf{x}^*) = A \mathbf{x} - A \mathbf{x}^* = \mathbf{b} - \mathbf{b} = 0,
\]
and hence \( \mathbf{z} \) belongs to the kernel of \( A \). Therefore, \( \mathbf{x} \) and \( \mathbf{x}^* \) are related by formula (2.26), which proves the second part of the theorem. \( Q.E.D. \)

Therefore, to construct the most general solution to an inhomogeneous system, we need only know one particular solution \( \mathbf{x}^* \), along with the general solution \( \mathbf{z} \in \ker A \) to the homogeneous equation. This construction should remind the reader of the method of solution for inhomogeneous linear ordinary differential equations. Indeed, both linear algebraic systems and linear ordinary differential equations are but two particular instances of the general theory of linear systems, to be developed in Chapter 7. We can characterize the case when the linear system has a unique solution in any of the following equivalent ways.

**Proposition 2.38.** Let \( A \) be an \( m \times n \) matrix. Then the following conditions are equivalent:

(i) \( \ker A = \{0\} \), i.e., the homogeneous system \( A \mathbf{x} = \mathbf{0} \) has the unique solution \( \mathbf{x} = \mathbf{0} \).

(ii) \( \operatorname{rank} A = n \).

(iii) The linear system \( A \mathbf{x} = \mathbf{b} \) has no free variables.

(iv) The system \( A \mathbf{x} = \mathbf{b} \) has a unique solution for each \( \mathbf{b} \in \operatorname{rng} A \).

Thus, while existence of a solution may depend upon the right hand side \( \mathbf{b} \), uniqueness is universal: if one \( \mathbf{b} \), e.g., \( \mathbf{b} = \mathbf{0} \), has a unique solution, then all \( \mathbf{b} \in \operatorname{rng} A \) also have unique solutions. Specializing even further to square matrices, we can now characterize invertible matrices by looking either at their kernel or their range.

**Proposition 2.39.** If \( A \) is a square matrix, then the following three conditions are equivalent: (i) \( A \) is nonsingular. (ii) \( \ker A = \{0\} \). (iii) \( \operatorname{rng} A = \mathbb{R}^n \).

**Example 2.40.** Consider the system \( A \mathbf{x} = \mathbf{b} \), where
\[
A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},
\]
where the right hand side of the system will be left arbitrary. Applying our usual Gaussian Elimination procedure to the augmented matrix
\[
\begin{pmatrix} 1 & 0 & -1 & | & b_1 \\ 0 & 1 & -2 & | & b_2 \\ 1 & -2 & 3 & | & b_3 \end{pmatrix}
\]
leads to the row echelon form
\[
\begin{pmatrix} 1 & 0 & -1 & | & b_1 \\ 0 & 1 & -2 & | & b_2 \\ 0 & 0 & 0 & | & b_3 + 2b_2 - b_1 \end{pmatrix}.
\]
Therefore, the system has a solution if and only if the compatibility condition
\[
-b_1 + 2b_2 + b_3 = 0 \tag{2.27}
\]

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holds. This equation serves to characterize the vectors $b$ that belong to the range of the matrix $A$, which is therefore a plane in $\mathbb{R}^3$.

To characterize the kernel of $A$, we take $b = 0$, and solve the homogeneous system $A \mathbf{z} = 0$. The row echelon form corresponds to the reduced system

$$z_1 - z_3 = 0, \quad z_2 - 2z_3 = 0.$$  

The free variable is $z_3$, and the equations are solved to give

$$z_1 = c, \quad z_2 = 2c, \quad z_3 = c,$$

where $c$ is arbitrary. The general solution to the homogeneous system is $\mathbf{z} = \begin{pmatrix} c, 2c, c \end{pmatrix}^T = c \begin{pmatrix} 1, 2, 1 \end{pmatrix}^T$, and so the kernel is the line in the direction of the vector $\begin{pmatrix} 1, 2, 1 \end{pmatrix}^T$.

If we take $b = \begin{pmatrix} 3, 1, 1 \end{pmatrix}^T$ — which satisfies (2.27) and hence lies in the range of $A$ — then the general solution to the inhomogeneous system $A \mathbf{x} = b$ is

$$x_1 = 3 + c, \quad x_2 = 1 + 2c, \quad x_3 = c,$$

where $c$ is an arbitrary scalar. We can write the solution in the form (2.26), namely

$$\mathbf{x} = \begin{pmatrix} 3 + c \\ 1 + 2c \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \mathbf{x}^* + \mathbf{z},$$

where $\mathbf{x}^* = \begin{pmatrix} 3, 1, 0 \end{pmatrix}^T$ plays the role of the particular solution, and $\mathbf{z} = c \begin{pmatrix} 1, 2, 1 \end{pmatrix}^T$ is the general element of the kernel.

The Superposition Principle

The principle of superposition lies at the heart of linearity. For homogeneous systems, superposition allows one to generate new solutions by combining known solutions. For inhomogeneous systems, superposition combines the solutions corresponding to different inhomogeneities or forcing functions. Superposition is the reason why linear systems are so much easier to solve. Here, we shall explain the superposition principle in the context of inhomogeneous linear algebraic systems. In Chapter 7 we shall see that superposition applies to completely general linear systems, including linear differential equations, linear boundary value problems, linear integral equations, linear control systems, etc.

Suppose we know particular solutions $\mathbf{x}^*_1$ and $\mathbf{x}^*_2$ to two inhomogeneous linear systems

$$A \mathbf{x} = \mathbf{b}_1, \quad A \mathbf{x} = \mathbf{b}_2,$$

that have the same coefficient matrix $A$. Consider the system

$$A \mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2,$$

in which the right hand side is a linear combination or superposition of the previous two. Then a particular solution to the combined system is given by the same linear combination of the previous solutions:

$$\mathbf{x}^* = c_1 \mathbf{x}^*_1 + c_2 \mathbf{x}^*_2.$$
The proof is easy: we merely apply the rules of matrix arithmetic to compute

\[ A \mathbf{x}^* = A(c_1 \mathbf{x}_1^* + c_2 \mathbf{x}_2^*) = c_1 A \mathbf{x}_1^* + c_2 A \mathbf{x}_2^* = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2. \]

In many applications, the inhomogeneities \( \mathbf{b}_1, \mathbf{b}_2 \) represent external forces, and the solutions \( \mathbf{x}_1^*, \mathbf{x}_2^* \) represent the respective responses of the physical apparatus. The linear superposition principle says that if we know how the system responds to the individual forces, we immediately know its response to any combination. The precise details of the system are irrelevant — all that is required is linearity.

**Example 2.41.** For example, the system

\[
\begin{pmatrix}
4 & 1 \\
1 & 4
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
f \\
g
\end{pmatrix}
\]

models the mechanical response of a pair of masses connected by springs to an external force. The solution \( \mathbf{x} = (x, y)^T \) represent the respective displacements of the masses, while the components of the right hand side \( \mathbf{f} = (f, g)^T \) represent the respective forces applied to each mass. (See Chapter 6 for full details.) We can compute the response of the system \( \mathbf{x}_1^* = \left( \frac{4}{15}, -\frac{1}{15} \right)^T \) to a unit force \( \mathbf{e}_1 = (1, 0)^T \) on the first mass, and the response \( \mathbf{x}_2^* = \left( -\frac{1}{15}, \frac{4}{15} \right)^T \) to a unit force \( \mathbf{e}_2 = (0, 1)^T \) on the second mass. We then immediately deduce the response of the system to a general force, since we can write

\[
\mathbf{f} = \begin{pmatrix} f \\ g \end{pmatrix} = f \mathbf{e}_1 + g \mathbf{e}_2 = f \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

and hence the solution is

\[
\mathbf{x} = f \mathbf{x}_1^* + g \mathbf{x}_2^* = f \begin{pmatrix} \frac{4}{15} \\ -\frac{1}{15} \end{pmatrix} + g \begin{pmatrix} -\frac{1}{15} \\ \frac{4}{15} \end{pmatrix} = \begin{pmatrix} \frac{4}{15} f - \frac{1}{15} g \\ -\frac{1}{15} f + \frac{4}{15} g \end{pmatrix}.
\]

The preceding construction is easily extended to several inhomogeneities, and the result is a general *Superposition Principle* for inhomogeneous linear systems.

**Theorem 2.42.** Suppose that we know particular solutions \( \mathbf{x}_1^*, \ldots, \mathbf{x}_k^* \) to each of the inhomogeneous linear systems

\[ A \mathbf{x} = \mathbf{b}_1, \quad A \mathbf{x} = \mathbf{b}_2, \quad \ldots \quad A \mathbf{x} = \mathbf{b}_k, \quad (2.28) \]

where \( \mathbf{b}_1, \ldots, \mathbf{b}_k \in \text{rng} \ A \). Then, for any choice of scalars \( c_1, \ldots, c_k \), a particular solution to the combined system

\[ A \mathbf{x} = c_1 \mathbf{b}_1 + \cdots + c_k \mathbf{b}_k, \quad (2.29) \]

is the same superposition

\[ \mathbf{x}^* = c_1 \mathbf{x}_1^* + \cdots + c_k \mathbf{x}_k^* \quad (2.30) \]

of individual solutions. The general solution to (2.29) is

\[ \mathbf{u} = \mathbf{x}^* + \mathbf{z} = c_1 \mathbf{x}_1^* + \cdots + c_k \mathbf{x}_k^* + \mathbf{z}, \quad (2.31) \]

where \( \mathbf{z} \) is the general solution to the homogeneous equation \( A \mathbf{z} = \mathbf{0} \).
In particular, if we know particular solutions \( \mathbf{x}_1^*, \ldots, \mathbf{x}_m^* \) to
\[
A \mathbf{x} = \mathbf{e}_i, \quad \text{for each} \quad i = 1, \ldots, m, \quad (2.32)
\]
where \( \mathbf{e}_1, \ldots, \mathbf{e}_m \) are the standard basis vectors of \( \mathbb{R}^m \), cf. (2.17), then we can reconstruct a particular solution \( \mathbf{x}^* \) to the general linear system \( A \mathbf{x} = \mathbf{b} \) by first writing
\[
\mathbf{b} = b_1 \mathbf{e}_1 + \cdots + b_m \mathbf{e}_m
\]
as a linear combination of the basis vectors, and then using superposition to form
\[
\mathbf{x}^* = b_1 \mathbf{x}_1^* + \cdots + b_m \mathbf{x}_m^*. \quad (2.33)
\]

However, for linear algebraic systems, the practical value of this insight is rather limited. Indeed, in the case when \( A \) is square and nonsingular, the superposition formula (2.33) is merely a reformulation of the method of computing the inverse of the matrix. Indeed, the vectors \( \mathbf{x}_1^*, \ldots, \mathbf{x}_m^* \) which satisfy (2.32) are just the columns of \( A^{-1} \), cf. (1.39), and (2.33) is, using (2.14), precisely the solution formula \( \mathbf{x}^* = A^{-1} \mathbf{b} \) that we abandoned in practical computations, in favor of the more efficient Gaussian elimination method. Nevertheless, the implications of this result turn out to be of great importance in more general types of linear systems, such as linear boundary value problems.

**Adjoint Systems, Cokernel, and Corange**

A linear system of \( m \) equations in \( n \) unknowns requires an \( m \times n \) coefficient matrix \( A \). The transposed matrix \( A^T \) will be of size \( n \times m \), and forms the coefficient matrix of an associated linear system, consisting of \( n \) equations in \( m \) unknowns.

**Definition 2.43.** The adjoint† to a linear system \( A \mathbf{x} = \mathbf{b} \) of \( m \) equations in \( n \) unknowns is the linear system
\[
A^T \mathbf{y} = \mathbf{f} \quad (2.34)
\]
of \( n \) equations in \( m \) unknowns \( \mathbf{y} \in \mathbb{R}^m \) with right hand side \( \mathbf{f} \in \mathbb{R}^n \).

**Example 2.44.** Consider the linear system
\[
\begin{align*}
x_1 - 3x_2 - 7x_3 + 9x_4 &= b_1, \\
x_2 + 5x_3 - 3x_4 &= b_2, \\
x_1 - 2x_2 - 2x_3 + 6x_4 &= b_3,
\end{align*}
\]
(2.35)
of three equations in four unknowns. Its coefficient matrix
\[
A = \begin{pmatrix} 1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 1 & -2 & -2 & 6 \end{pmatrix}
\]
has transpose
\[
A^T = \begin{pmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ -7 & 5 & -2 \\ 9 & -3 & 6 \end{pmatrix}.
\]

† Warning: Many texts misuse the term “adjoint” to describe the classical adjugate or cofactor matrix. These are completely unrelated, and the latter will play no role in this book.
Thus, the adjoint system to (2.35) is the following system of four equations in three unknowns:

\[
\begin{align*}
    y_1 + y_3 &= f_1, \\
    -3y_1 + y_2 - 2y_3 &= f_2, \\
    -7y_1 + 5y_2 - 2y_3 &= f_3, \\
    9y_1 - 3y_2 + 6y_3 &= f_4.
\end{align*}
\] (2.36)

On the surface, there appears to be little direct connection between the solutions to a linear system and its adjoint. Nevertheless, as we shall soon see (and then in even greater depth in Sections 5.6 and 8.5) there are remarkable, but subtle interrelations between the two. To this end, we use the adjoint system to define the remaining two fundamental subspaces associated with a coefficient matrix \( A \).

**Definition 2.45.** The corange of an \( m \times n \) matrix \( A \) is the range of its transpose, 
\[
\text{corng} \ A = \text{rng} \ A^T = \{ \ A^T y \mid y \in \mathbb{R}^m \} \subset \mathbb{R}^n. \quad (2.37)
\]
The cokernel or left null space of \( A \) is the kernel of its transpose, 
\[
\text{coker} \ A = \ker \ A^T = \{ \ w \in \mathbb{R}^m \mid A^T w = 0 \} \subset \mathbb{R}^m, \quad (2.38)
\]
that is, the set of solutions to the homogeneous adjoint system.

The corange coincides with the subspace of \( \mathbb{R}^n \) spanned by the rows of \( A \), and is sometimes referred to as the row space. As a direct consequence of Theorem 2.37, the adjoint system \( A^T y = f \) has a solution if and only if \( f \in \text{rng} \ A^T = \text{corng} \ A \).

**Example 2.46.** To solve the linear system (2.35) appearing above, we perform Gaussian Elimination on its augmented matrix 
\[
\begin{pmatrix}
  1 & -3 & -7 & 9 & b_1 \\
  0 & 1 & 5 & -3 & b_2 \\
  1 & -2 & -2 & 6 & b_3
\end{pmatrix}
\]
that reduces it to the row echelon form 
\[
\begin{pmatrix}
  1 & -3 & -7 & 9 & b_1 \\
  0 & 1 & 5 & -3 & b_2 \\
  0 & 0 & 0 & 0 & b_3 - b_2 - b_1
\end{pmatrix}
\]. Thus, the system has a solution if and only if \( b \in \text{rng} \ A \) satisfies the compatibility condition \(-b_1 - b_2 + b_3 = 0\). For such vectors, the general solution is
\[
x = \begin{pmatrix}
  b_1 + 3b_2 - 8x_3 \\
  b_2 - 5x_3 + 3x_4 \\
  x_3 \\
  x_4
\end{pmatrix} = \begin{pmatrix}
  b_1 + 3b_2 \\
  b_2 \\
  0 \\
  0
\end{pmatrix} + x_3 \begin{pmatrix}
  -8 \\
  -5 \\
  1 \\
  0
\end{pmatrix} + x_4 \begin{pmatrix}
  0 \\
  3 \\
  0 \\
  1
\end{pmatrix}.
\]

In the second expression, the first vector is a particular solution and the remaining terms constitute the general element of the two-dimensional kernel of \( A \).

The solution to the adjoint system (2.36) is also obtained by Gaussian Elimination starting with its augmented matrix 
\[
\begin{pmatrix}
  1 & 0 & 1 & f_1 \\
  -3 & 1 & -2 & f_2 \\
  -7 & 5 & -2 & f_3 \\
  9 & -3 & 6 & f_4
\end{pmatrix}
\]. The resulting row echelon
form is
\[
\begin{pmatrix}
1 & 0 & 1 & f_1 \\
0 & 1 & 1 & f_2 \\
0 & 0 & 0 & f_3 - 5f_2 - 8f_1 \\
0 & 0 & 0 & f_4 + 3f_2
\end{pmatrix}.
\]
Thus, there are two compatibility constraints required for a solution to the adjoint system: 
\[-8f_1 - 5f_2 + f_3 = 0, \quad 3f_2 + f_4 = 0.\]
These are the conditions required for the right hand side to belong to the corange: \( f \in \text{rng} A^T = \text{corng} A \). If satisfied, the adjoint system has the general solution depending on the single free variable \( y_3 \):
\[
y = \begin{pmatrix}
    f_1 - y_3 \\
    3f_1 + f_2 - y_3 \\
y_3
\end{pmatrix} = \begin{pmatrix}
    f_1 \\
    3f_1 + f_2 \\
    0
\end{pmatrix} + y_3 \begin{pmatrix}
    -1 \\
    -1 \\
    1
\end{pmatrix}.
\]
In the latter formula, the first term represents a particular solution, while the second is the general element of \( \ker A^T = \text{coker} A \).

The Fundamental Theorem of Linear Algebra

The four fundamental subspaces associated with an \( m \times n \) matrix \( A \), then, are its range, corange, kernel and cokernel. The range and cokernel are subspaces of \( \mathbb{R}^m \), while the kernel and corange are subspaces of \( \mathbb{R}^n \). Moreover, these subspaces are not completely arbitrary, but are, in fact, profoundly related through both their numeric and geometric properties.

The Fundamental Theorem of Linear Algebra\(^\dagger\) states that their dimensions are entirely prescribed by the rank (and size) of the matrix.

**Theorem 2.47.** Let \( A \) be an \( m \times n \) matrix of rank \( r \). Then
\[
\dim \text{corng} A = \dim \text{rng} A = \rank A = \rank A^T = r, \\
\dim \ker A = n - r, \quad \dim \text{coker} A = m - r.
\]

**Remark:** Thus, the rank of a matrix, i.e., the number of pivots, indicates the number of linearly independent columns, which, remarkably, is always the same as the number of linearly independent rows. A matrix and its transpose inevitably have the same rank, i.e., the same number of pivots, despite the fact that their row echelon forms are quite different, and rarely transposes of each other. Theorem 2.47 also proves our earlier contention that the rank of a matrix is an *intrinsic* quantity, and does not depend on which specific elementary row operations are employed during the reduction process, nor on the final row echelon form.

**Proof:** Since the dimension of a subspace is prescribed by the number of vectors in any basis, we need to relate bases of the fundamental subspaces to the rank of the matrix. Rather than try to present a completely general argument, we will show how to construct

\(^{\dagger}\) Not to be confused with the Fundamental Theorem of Algebra, that states that every polynomial has a complex root; see Theorem 16.62.
bases for each of the subspaces in a particular instance, and thereby illustrate the method of proof. Consider the matrix

\[
A = \begin{pmatrix} 2 & -1 & 1 & 2 \\ -8 & 4 & -6 & -4 \\ 4 & -2 & 3 & 2 \end{pmatrix}.
\]

Its row echelon form

\[
U = \begin{pmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
is obtained in the usual manner. There are two pivots, and thus the rank of \( A \) is \( r = 2 \).

**Kernel:** We need to find the solutions to the homogeneous system \( A \mathbf{x} = \mathbf{0} \). In our example, the pivots are in columns 1 and 3, and so the free variables are \( x_2, x_4 \). Using back substitution on the reduced homogeneous system \( U \mathbf{x} = \mathbf{0} \), we find the general solution

\[
\mathbf{x} = \begin{pmatrix} \frac{1}{2} x_2 - 2 x_4 \\ x_2 \\ 2 x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \end{pmatrix}
\]

written as a linear combination of the vectors

\[
\mathbf{z}_1 = \left( \frac{1}{2} \ 1 \ 0 \ 0 \right)^T, \quad \mathbf{z}_2 = (-2 \ 0 \ 2 \ 1)^T.
\]

We claim that \( \mathbf{z}_1, \mathbf{z}_2 \) form a basis of \( \text{ker} \ A \). By construction, they span the kernel, and linear independence follows easily since the only way in which the linear combination \( \mathbf{2.40} \) could vanish, \( \mathbf{x} = \mathbf{0} \), is if both free variables vanish: \( x_2 = x_4 = 0 \). In general, there are \( n - r \) free variables, each corresponding to one of the basis elements of the kernel, which thus implies the dimension formula for \( \text{ker} \ A \).

**Corange:** The corange is the subspace of \( \mathbb{R}^n \) spanned by the rows of \( A \). We claim that applying an elementary row operation does not alter the corange. To see this for row operations of the first type, suppose, for instance, that \( \widehat{A} \) is obtained adding \( a \) times the first row of \( A \) to the second row. If \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_m \) are the rows of \( A \), then the rows of \( \widehat{A} \) are \( \mathbf{r}_1, \widehat{\mathbf{r}}_2 = \mathbf{r}_2 + a \mathbf{r}_1, \mathbf{r}_3, \ldots, \mathbf{r}_m \). If

\[
\mathbf{v} = c_1 \mathbf{r}_1 + c_2 \mathbf{r}_2 + c_3 \mathbf{r}_3 + \cdots + c_m \mathbf{r}_m
\]
is any vector belonging to \( \text{corng} \ A \), then

\[
\mathbf{v} = \widehat{c}_1 \mathbf{r}_1 + c_2 \widehat{\mathbf{r}}_2 + c_3 \mathbf{r}_3 + \cdots + c_m \mathbf{r}_m,
\]
where \( \widehat{c}_1 = c_1 - a c_2 \), is also a linear combination of the rows of the new matrix, and hence lies in \( \text{corng} \ \widehat{A} \). The converse is also valid — \( \mathbf{v} \in \text{corng} \ \widehat{A} \) implies \( \mathbf{v} \in \text{corng} \ A \) — and we conclude that elementary row operations of Type \#1 do not change \( \text{corng} \ A \). The proof for the other two types of elementary row operations is even easier, and left to the reader.

Since the row echelon form \( U \) is obtained from \( A \) by a sequence of elementary row operations, we conclude that \( \text{corng} \ A = \text{corng} \ U \). Moreover, because each nonzero row in \( U \) contains a pivot, it is not hard to see that the nonzero rows of \( \text{corng} \ U \) are linearly independent, and hence form a basis of both \( \text{corng} \ U \) and \( \text{corng} \ A \). Since there is one row
per pivot, corng $U = \text{corng } A$ has dimension $r$, the number of pivots. In our example, then, a basis for corng $A$ consists of the nonzero row vectors

$$s_1 = (2\; -1\; 1\; 2), \quad s_2 = (0\; 0\; -2\; 4),$$

in $U$. The reader may wish to verify their linear independence, as well as the fact that every row of $A$ lies in their span.

**Range:** There are two methods for computing a basis of the range or column space. The first proves that it has dimension equal to the rank. This has the important, and remarkable consequence that the space spanned by the rows of a matrix and the space spanned by its columns always have the same dimension, even though they are, in general, subspaces of different vector spaces.

Now the range of $A$ and the range of $U$ are, in general, different subspaces, so we cannot directly use a basis for rng $U$ as a basis for rng $A$. However, the linear dependencies among the columns of $A$ and $U$ are the same. It is not hard to see that the columns of $U$ that contain the pivots form a basis for rng $U$; they are easily seen to be linearly independent and every other column can be written as a linear combination of them. This implies that the same columns of $A$ form a basis for $\text{rng } A$. In particular, this implies that $\dim \text{rng } A = \dim \text{rng } U = r$.

In our example, the pivots lie in the first and third columns of $U$, and hence the first and third columns of $A$, namely

$$v_1 = \begin{pmatrix} 2 \\ -8 \\ 4 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix},$$

form a basis for $\text{rng } A$. This implies that every column of $A$ can be written uniquely as a linear combination of the first and third column, as you can validate directly.

In more detail, using our matrix multiplication formula (2.14), we see that a linear combination of columns of $A$ is trivial,

$$c_1 v_1 + \cdots + c_n v_n = A c = 0,$$

if and only if $c \in \ker A$. But we know $\ker A = \ker U$, and so the same linear combination of columns of $U$, namely

$$U c = c_1 u_1 + \cdots + c_n u_n = 0,$$

is also trivial. In particular, the linear independence of the pivot columns of $U$, labeled $u_{j_1}, \ldots, u_{j_r}$, implies the linear independence of the same collection, $v_{j_1}, \ldots, v_{j_r}$, of columns of $A$. Moreover, the fact that any other column of $U$ can be written as a linear combination

$$u_k = d_1 u_{j_1} + \cdots + d_r u_{j_r},$$

of the pivot columns implies that the same holds for the corresponding column of $A$, so

$$v_k = d_1 v_{j_1} + \cdots + d_r v_{j_r}.$$ 

We conclude that the pivot columns of $A$ form a basis for its range or column space.
An alternative method to find a basis for the range is to note that \( \text{rng } A = \text{corng } A^T \). Thus, we can employ our previous algorithm to compute \( \text{corng } A^T \). In our example, applying Gaussian elimination to

\[
A^T = \begin{pmatrix}
2 & -8 & 4 \\
-1 & 4 & -2 \\
1 & -6 & 3 \\
2 & -4 & 2
\end{pmatrix}
\]

leads to the row echelon form

\[
\tilde{U} = \begin{pmatrix}
2 & -8 & 4 \\
0 & -2 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Observe that the row echelon form of \( A^T \) is not the transpose of the row echelon form of \( A \)! However, they do have the same number of pivots since both \( A \) and \( A^T \) have the same rank. We conclude that

\[
y_1 = (2 \ -8 \ 4)^T, \quad y_2 = (0 \ -2 \ 1)^T,
\]

forms an alternative basis for \( \text{rng } A \).

**Cokernel:** Finally, to determine a basis for the cokernel of the matrix, we apply the preceding algorithm for finding a basis for \( \text{ker } A^T = \text{coker } A \). Since the ranks of \( A \) and \( A^T \) coincide, there are now \( m - r \) free variables, which is the same as the dimension of \( \text{ker } A^T \).

In our particular example, using the reduced form (2.41), the only free variable is \( y_3 \), and the general solution to the homogeneous adjoint system \( A^T y = 0 \) is

\[
y = \begin{pmatrix}
0 \\
\frac{1}{2} y_3 \\
y_3
\end{pmatrix} = y_3 \begin{pmatrix}
0 \\
\frac{1}{2} \\
1
\end{pmatrix}.
\]

We conclude that \( \text{coker } A \) is one-dimensional, with basis \( (0 \ \frac{1}{2} \ 1)^T \).

Summarizing, given an \( m \times n \) matrix \( A \) with row echelon form \( U \), to find a basis for

- \( \text{rng } A \): choose the \( n - r \) columns of \( A \) where the pivots appear in \( U \);
- \( \text{ker } A \): write the general solution to \( A x = 0 \) as a linear combination of the basis vectors whose coefficients are the free variables.
- \( \text{corng } A \): choose the nonzero rows of \( U \);
- \( \text{coker } A \): write the general solution to the adjoint system \( A^T y = 0 \) as a linear combination of the basis vectors whose coefficients are the free variables.

### 2.6. Graphs and Incidence Matrices.

We now present an intriguing application of linear algebra to graph theory. A **graph** consists of one or more points, called **vertices**, and lines or curves connecting them, called **edges**. Edge edge connects exactly two vertices, which, for simplicity, are assumed to always be distinct, so that no edge forms a **loop** that connects a vertex to itself. However, we do permit two vertices to be connected by multiple edges. Some examples of graphs appear in Figure 2.2; the vertices are the black dots. In a planar representation of the graph, the edges may cross over each other at non-nodal points, but do not actually meet — think of a circuit where the (insulated) wires lie on top of each other, but do not touch. Thus, the first graph in Figure 2.2 has 5 vertices and 8 edges; the second has 4 vertices...
and 6 edges — the two central edges do not meet; the final graph has 5 vertices and 10 edges.

Graphs arise in a multitude of applications. A particular case that will be considered in depth is electrical networks, where the edges represent wires, and the vertices represent the nodes where the wires are connected. Another example is the framework for a building — the edges represent the beams and the vertices the joints where the beams are connected. In each case, the graph encodes the topology — meaning interconnectedness — of the system, but not its geometry — lengths of edges, angles, etc.

Two graphs are considered to be the same if one can identify all their edges and vertices, so that they have the same connectivity properties. A good way to visualize this is to think of the graph as a collection of wires joined at the vertices. Moving the vertices and wires around without cutting or rejoining them will have no effect on the underlying graph. Consequently, there are many ways to draw a given graph; three versions of one graph appear in Figure 2.3.

Two vertices in a graph are adjacent if there is an edge connecting them. Two edges are adjacent if they meet at a common vertex. For instance, in the graph in Figure 2.4, all vertices are adjacent; edge 1 is adjacent to all edges except edge 5. A path is a sequence of distinct, i.e., non-repeated, edges, with each edge adjacent to its successor. For example, in Figure 2.4, one path starts at vertex #1, then goes in order along the edges labeled as 1, 4, 3, 2, thereby passing through vertices 1, 2, 4, 1, 3. Note that while an edge cannot be
repeated in a path, a vertex may be. A circuit is a path that ends up where it began. For example, the circuit consisting of edges $1, 4, 5, 2$ starts at vertex 1, then goes to vertices $2, 4, 3$ in order, and finally returns to vertex 1. The starting vertex for a circuit is not important. For example, edges $4, 5, 2, 1$ also represent the same circuit we just described. A graph is called connected if you can get from any vertex to any other vertex by a path, which is by far the most important case for applications. We note that every graph can be decomposed into a disconnected collection of connected subgraphs.

In electrical circuits, one is interested in measuring currents and voltage drops along the wires in the network represented by the graph. Both of these quantities have a direction, and therefore we need to specify an orientation on each edge in order to quantify how the current moves along the wire. The orientation will be fixed by specifying the vertex the edge “starts” at, and the vertex it “ends” at. Once we assign a direction to an edge, a current along that wire will be positive if it moves in the same direction, i.e., goes from the starting vertex to the ending one, and negative if it moves in the opposite direction. The direction of the edge does not dictate the direction of the current — it just fixes what directions positive and negative values of current represent. A graph with directed edges is known as a directed graph or digraph for short. The edge directions are represented by arrows; examples of digraphs can be seen in Figure 2.5.
Consider a digraph $D$ consisting of $n$ vertices connected by $m$ edges. The \textit{incidence matrix} associated with $D$ is an $m \times n$ matrix $A$ whose rows are indexed by the edges and whose columns are indexed by the vertices. If edge $k$ starts at vertex $i$ and ends at vertex $j$, then row $k$ of the incidence matrix will have a $+1$ in its $(k,i)$ entry and $-1$ in its $(k,j)$ entry; all other entries of the row are zero. Thus, our convention is that a $+1$ entry represents the vertex at which the edge starts and a $-1$ entry the vertex at which it ends.

A simple example is the digraph in Figure 2.6, which consists of five edges joined at four different vertices. Its $5 \times 4$ incidence matrix is

$$A = \begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1 \\
\end{pmatrix}.$$  

(2.42)

Thus the first row of $A$ tells us that the first edge starts at vertex 1 and ends at vertex 2. Similarly, row 2 says that the second edge goes from vertex 1 to vertex 3. Clearly one can completely reconstruct any digraph from its incidence matrix.

\textbf{Example 2.48.} The matrix

$$A = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 \\
\end{pmatrix}.$$  

(2.43)

qualifies as an incidence matrix because each row contains a single $+1$, a single $-1$, and the other entries are 0. Let us construct the digraph corresponding to $A$. Since $A$ has five columns, there are five vertices in the digraph, which we label by the numbers 1, 2, 3, 4, 5. Since it has seven rows, there are 7 edges. The first row has its $+1$ in column 1 and its $-1$ in column 2 and so the first edge goes from vertex 1 to vertex 2. Similarly, the second
Figure 2.7. Another Digraph.

edge corresponds to the second row of $A$ and so goes from vertex 3 to vertex 1. The third row of $A$ gives an edge from vertex 3 to vertex 2; and so on. In this manner we construct the digraph drawn in Figure 2.7.

The incidence matrix has important geometric and quantitative consequences for the graph it represents. In particular, its kernel and cokernel have topological significance. For example, the kernel of the incidence matrix (2.43) is spanned by the single vector $z = (1, 1, 1, 1)^T$, and represents the fact that the sum of the entries in any given row of $A$ is zero. This observation holds in general for connected digraphs.

**Proposition 2.49.** If $A$ is the incidence matrix for a connected digraph, then $\ker A$ is one-dimensional, with basis $z = (1 \ 1 \ldots \ 1)^T$.

**Proof:** If edge $k$ connects vertices $i$ and $j$, then the $k$th equation in $Az = 0$ is $z_i = z_j$. The same equality holds, by a simple induction, if the vertices $i$ and $j$ are connected by a path. Therefore, if $D$ is connected, all the entries of $z$ are equal, and the result follows. $Q.E.D.$

**Corollary 2.50.** If $A$ is the incidence matrix for a connected digraph with $n$ vertices, then $\text{rank } A = n - 1$.

**Proof:** This is an immediate consequence of Theorem 2.47. $Q.E.D.$

Next, let us look at the cokernel of an incidence matrix. Consider the particular example (2.42) corresponding to the digraph in Figure 2.6. We need to compute the kernel of the transposed incidence matrix

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}. \quad (2.44)$$
Solving the homogeneous system $A^T y = 0$ by Gaussian elimination, we discover that $\text{coker } A = \ker A^T$ is spanned by the two vectors

\[
y_1 = (1 \ 0 \ -1 \ 1 \ 0)^T, \quad y_2 = (0 \ 1 \ -1 \ 0 \ 1)^T.
\]

Each of these vectors represents a circuit in the digraph, the nonzero entries representing the direction in which the edges are traversed. For example, $y_1$ corresponds to the circuit that starts out along edge #1, then traverses edge #4 and finishes by going along edge #3 in the reverse direction, which is indicated by the minus sign in its third entry. Similarly, $y_2$ represents the circuit consisting of edge #2, followed by edge #5, and then edge #3, backwards. The fact that $y_1$ and $y_2$ are linearly independent vectors says that the two circuits are "independent".

The general element of $\text{coker } A$ is a linear combination $c_1 y_1 + c_2 y_2$. Certain values of the constants lead to other types of circuits; for example $-y_1$ represents the same circuit as $y_1$, but traversed in the opposite direction. Another example is

\[
y_1 - y_2 = (1 \ -1 \ 0 \ 1 \ -1)^T,
\]

which represents the square circuit going around the outside of the digraph, along edges 1, 4, 5, 2, the fourth and second being in the reverse direction. We can view this circuit as a combination of the two triangular circuits; when we add them together the middle edge #3 is traversed once in each direction, which effectively "cancels" its contribution. (A similar cancellation occurs in the theory of line integrals; see Section A.5.) Other combinations represent "virtual" circuits; for instance, one can interpret $2y_1 - \frac{1}{2}y_2$ as two times around the first triangular circuit plus one half of the other triangular circuit, in the opposite direction—whatever that might mean.

Let us summarize the preceding discussion.

**Theorem 2.51.** Each circuit in a digraph $D$ is represented by a vector in the cokernel of its incidence matrix, whose entries are $+1$ if the edge is traversed in the correct direction, $-1$ if in the opposite direction, and $0$ if the edge is not in the circuit. The dimension of the cokernel of $A$ equals the number of independent circuits in $D$.

The preceding two theorems have an important and remarkable consequence. Suppose $D$ is a connected digraph with $m$ edges and $n$ vertices and $A$ its $m \times n$ incidence matrix. Corollary 2.50 implies that $A$ has rank $r = n - 1 = n - \dim \ker A$. On the other hand, Theorem 2.51 tells us that $\dim \text{coker } A = l$ equals the number of independent circuits in $D$. The Fundamental Theorem 2.47 says that $r = m - l$. Equating these two formulas for the rank, we find $r = n - 1 = m - l$, or $n + l = m + 1$. This celebrated result is known as Euler's formula for graphs, first discovered by the extraordinarily prolific eighteenth century Swiss mathematician Leonhard Euler.

**Theorem 2.52.** If $G$ is a connected graph, then

\[
\# \text{ vertices } + \# \text{ independent circuits } = \# \text{ edges } + 1.
\] (2.45)
Remark: If the graph is **planar**, meaning that it can be graphed in the plane without any edges crossing over each other, then the number of independent circuits is equal to the number of “holes” in the graph, i.e., the number of distinct polygonal regions bounded by the edges of the graph. For example, the pentagonal digraph in Figure 2.7 bounds three triangles, and so has three independent circuits. For non-planar graphs, (2.45) gives a possible definition of the number of independent circuits, but one that is not entirely standard. A more detailed discussion relies on further developments in the topological properties of graphs.

**Example 2.53.** Consider the graph corresponding to the edges of a cube, as illustrated in Figure 2.8, where the second figure represents the same graph squashed down onto a plane. The graph has 8 vertices and 12 edges. Euler’s formula (3.78) tells us that there are 5 independent circuits. These correspond to the interior square and four trapezoids in the planar version of the digraph, and hence to circuits around 5 of the 6 faces of the cube. The “missing” face does indeed define a circuit, but it can be represented as the sum of the other five circuits, and so is not independent. In Exercise [ ], the reader is asked to write out the incidence matrix for the cubical digraph and explicitly identify the basis of its kernel with the circuits.

We do not have the space to further develop the remarkable connections between graph theory and linear algebra. The interested reader is encouraged to consult a more specialized text in graph theory, e.g., [33].