Interpolatory frames in signal space*

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Abstract

We present a new family of frames, which are generated by perfect reconstruction filter banks of linear phased filters. The filter banks are based on discrete interpolatory splines and are related to Butterworth filters. Each filter bank contains one interpolatory symmetric low-pass filter and two high-pass filters, one of which is also interpolatory and symmetric. The second high-pass filter is either symmetric or antisymmetric. These filter banks generate the analysis and synthesis scaling functions and pairs of framelets. We introduce the concept of *semi-tight* frame. All the analysis waveforms in a tight frame coincide with their synthesis counterparts. In the semi-tight frame we can trade properties of smoothness and number of vanishing moments between the synthesis and the analysis framelets. We construct dual pairs of frames, where all the waveforms are symmetric and all the framelets have the same number of vanishing moments. Although most of the designed filters are IIR, they allow fast implementation via recursive procedures. The waveforms are well localized in time domain despite their infinite support. The frequency response of the designed filters is flat.

Introduction

Recently frames or redundant expansions of signals have attracted considerable interest from researchers working in signal processing although one particular class of frames, the Gabor systems, is being applied and investigated since 1946 [12]. As the requirement of one-to-one correspondence between the signal and its transform coefficients is dropped, there is more freedom to design and implement frame transforms. The frame expansions of signals demonstrate resilience to quantization noise and to coefficients losses [13, 14, 16]. Thus, frames may serve as a tool for error correction of signals that are transmitted through lossy/noisy channels. Recently, overcomplete representation signals benefits image reconstruction ([5, 19]).

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Frames generated by filter banks [4, 8, 16] are important. Actually, under some relaxed conditions, a perfect reconstruction oversampled filter bank produces frame expansion. In this paper we use filter banks as an engine that constructs a new family of frames that have properties, which are attractive for signal processing: symmetry, interpolation, flat spectra, combined with fine time-domain localization, efficient implementation to name some.

Infinite iteration of the frame filter banks results in limit functions that are called framelets. The framelets, which are derived in the paper, are smooth, symmetric, interpolatory and may have any number of vanishing moments. Non-compactness of their support is compensated by exponential decay as time tends to infinity.

We consider in the paper 3-channel analysis and synthesis filter banks where each contains one low-pass and two high-pass filters. The downsampling factor $N = 2$ and the transfer functions of all filters are rational functions. The low-pass filter and one of the high-pass filters in each filter bank are interpolatory, whereas the even polyphase components of the other high-pass filters are zero. Our approach to the design of interpolatory perfect reconstruction filter banks is, to some extent, similar to the approach, which we used in the construction of the biorthogonal wavelet transforms [1]. For example, the output of the low-pass component of the analysis filter bank is the sum of the even polyphase component of the input signal and the approximation of the even component by discrete spline of order $2r$, which interpolates the odd samples of the signal. This procedure is equivalent to the application to the signal of the causal followed by the application of the anticausal half-band low-pass Butterworth filters of order r. By using this approach, we construct a diverse family of tight, semi-tight and bi-frames. Note that the causal Butterworth filters were used by Herley and Vetterli [15] to construct orthogonal non-symmetric wavelets. In our previous papers [1, 2] we presented a family of biorthogonal symmetric wavelets related to the Butterworth filters and their application to image compression.

The paper is organized as follows. In the introductory Section 1 we recall some facts about filter banks and frames, which are necessary for the rest of the presentation. In Section 2 we describe how to construct a tight frame and a bundle of semi-tight frames starting from arbitrary interpolatory low-pass filter. From a pair of interpolatory low-pass filters we construct a set of bi-frames. In Section 3 we derive interpolatory filters from discrete splines and explain the relation between the designed filters and the Butterworth filters. In addition, we establish some properties of these filters and their corresponding waveforms. Section 4 is devoted to the construction of tight, semi-tight and bi-frames using the designed filters. We call these frames the Butterworth frames. We provide numerous examples that are accompanied by graphical illustrations. In the concluding section we compare a frame transform with the related biorthogonal wavelet transform and provide a concrete example that shows how the constructed frames can be used to correct errors in a transmitted image.

1 Preliminaries: filter banks and frames

1.1 Filter banks

We call the sequences $\mathbf{x} \stackrel{\Delta}{=} \{x_k\},\ k \in \mathbb{Z}$, which belong to the space l_1, (and, consequently, to l_2) discrete-time signals. The z-transform of a signal $\mathbf{x}$ is defined as $X(z) \stackrel{\Delta}{=} \sum_{k\in\mathbb{Z}} z^{-k}\, x_k$. Throughout the paper we assume that $z = e^{j\omega}$.

The input x_n and the output y_n of a linear discrete time shift-invariant system are linked by $y_n = \sum_{k\in\mathbb{Z}} f_{n-k} x_k$. This processing of the signal $\mathbf{x}$ is called digital filtering and the sequence $\{f_n\}$ is called the impulse response of the filter $\mathbf{f}$. Its z-transform $F(z) \stackrel{\Delta}{=} \sum_{n\in\mathbb{Z}} z^{-n} f_n$ is called the transfer function of the filter. Usually, a filter is designated by its transfer function $F(z)$. The function $\widehat{F}(\omega) = F(e^{j\omega n})$ is called the frequency response of the digital filter.

The set of filters $\left\{\tilde{F}^k(z) = \sum_{n\in\mathbb{Z}} z^{-n} \tilde{f}_n^k\right\}_{k=1}^{K}$, which, being time-reversed and applied to the input signal $\mathbf{x}$, produce the set of decimated output signals $\{\tilde{\mathbf{y}}^k\}_{k=1}^K$

$$\tilde{y}_l^k = \sum_{n\in\mathbb{Z}} \tilde{f}_{n-Nl}^k\, x_n, \quad k = 1, \ldots, K,$$

is called the $K-$channel analysis filter bank. Here $N \in \mathbb{N}$ is the downsampling factor. The set of filters $\left\{F^k(z) = \sum_{n\in\mathbb{Z}} z^{-n} f_n^k\right\}_{k=1}^{K}$, which, being applied to the set of input signals $\{\mathbf{y}^k\}_{k=1}^K$ that are upsampled by factor N, produces the output signal $\hat{\mathbf{x}}$

$$\hat{x}_l = \sum_{k=1}^{K} \sum_{n\in\mathbb{Z}} f_{l-Nn}^k\, y_n^k, \quad l = 1, \ldots, K,$$

is called the $K-$channel synthesis filter bank. If the number of channels K equals to the downsampling factor N then the filter bank is said to be critically sampled. If $K > N$ then the filter bank is oversampled.

In this paper we consider only 3-channel filter banks where each contains one low-pass and two high-pass filters. Their transfer functions are rational functions and where their downsampling factor $N = 2$. We denote the analysis and synthesis low-pass filters by $\tilde{H}(z)$ and $H(z)$, respectively, and the high-pass filters are denoted by $\tilde{G}^r(z)$ and $G^r(z),\ r = 1, 2$. We denote by $\mathbf{s}$, $\mathbf{d}^r,\ r = 1, 2$ the output signals from the analysis filter bank. These signals are the input for the synthesis filter bank. Then, the analysis and synthesis formulas are:

$$s_l = 2 \sum_{n\in\mathbb{Z}} \tilde{h}_{n-2l}\, x_n \Leftrightarrow S(z^2) = \tilde{H}(1/z)X(z) + \tilde{H}(-1/z)X(-z), \tag{1.1}$$

$$d_l^r = 2 \sum_{n\in\mathbb{Z}} \tilde{g}_{n-2l}^r\, x_n \Leftrightarrow D^r(z^2) = \tilde{G}^r(1/z)X(z) + \tilde{G}^r(-1/z)X(-z), \quad r = 1, 2, \tag{1.2}$$

$$\hat{x}_l = \sum_{n\in\mathbb{Z}} h_{l-2n}\, s_n + \sum_{r=1}^{2} \sum_{n\in\mathbb{Z}} g_{l-2n}^r\, d_n^r \Leftrightarrow \hat{X}(z) = H(z)S(z^2) + \sum_{r=1}^{2} G^r(z) d^r(z^2),\ r = 1, 2. \tag{1.3}$$

1.1.1 Polyphase representation of filtering:

Denote

$$F_e(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} f_{2k}, \quad F_o(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} f_{2k+1},$$
$$E(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} x_{2k}, \quad O(z) \triangleq \sum_{k\in\mathbb{Z}} z^{-k} x_{2k+1}. \tag{1.4}$$

We have

$$F(z) = F_e(z^2) + z^{-1}F_o(z^2), \quad X(z) = E(z^2) + z^{-1}O(z^2) \Rightarrow Y(z) = F(z)X(z)$$
$$= \left(F_e(z^2)E(z^2) + z^{-2}F_o(z^2))O(z^2)\right) + z^{-1}\left(F_o(z^2)E(z^2) + F_e(z^2)O(z^2)\right).$$

Hence, the $z-$transforms of the even and odd subarrays of the array $\mathbf{y} = \{y_k\}$ are

$$Y_e(z) = F_e(z)E(z) + z^{-1}F_o(z)O(z), \;\; Y_o(z) = F_o(z)E(z) + F_e(z)O(z),$$

respectively.

The analysis $\tilde{\mathbf{P}}(z)$ and the synthesis $\mathbf{P}(z)$ *polyphase matrices* , respectively, are:

$$\tilde{\mathbf{P}}(z) \triangleq \begin{pmatrix} \tilde{H}_e(z) & \tilde{H}_o(z) \\ \tilde{G}_e^1(z) & \tilde{G}_o^1(z) \\ \tilde{G}_e^2(z) & \tilde{G}_o^2(z) \end{pmatrix}, \quad \mathbf{P}(z) \triangleq \begin{pmatrix} H_e(z) & G_e^1(z) & G_e^2(z) \\ H_o(z) & G_o^1(z) & G_o^2(z) \end{pmatrix}.$$

Then,

$$\begin{pmatrix} S(z) \\ D^1(z) \\ D^2(z) \end{pmatrix} = 2\tilde{\mathbf{P}}(1/z) \cdot \begin{pmatrix} E(z) \\ O(z) \end{pmatrix}, \quad \begin{pmatrix} \hat{E}(z) \\ \hat{O}(z) \end{pmatrix} = \mathbf{P}(\mathbf{z}) \cdot \begin{pmatrix} S(z) \\ D^1(z) \\ D^2(z) \end{pmatrix}.$$

Here $\hat{E}(z)$ and $\hat{O}(z)$ are the z-transforms of the even and odd components of the output signal $\hat{\mathbf{x}}$, respectively. If the signal $\hat{\mathbf{x}} = \mathbf{x}$ then the analysis and synthesis filter banks form a perfect reconstruction filter bank. Analytically, this property is expressed via the polyphase matrices as:

$$\mathbf{P}(z) \cdot \tilde{\mathbf{P}}(1/z) = \frac{1}{2}\mathbf{I}, \tag{1.5}$$

where $\mathbf{I}$ is the 2×2 identity matrix. Thus, the synthesis polyphase matrix must be left inverse of the analysis matrix (up to factor 1/2). Obviously, if such a matrix exists, it is not unique.

1.2 Frames

Definition 1.1 *A system* $\tilde{\Phi} \triangleq \{\tilde{\phi}_j\}_{j\in\mathbb{Z}}$ *of signals forms a frame of the signal space if there exist positive constants* A *and* B *such that for any signal* $\boldsymbol{x} = \{x_l\}_{l\in\mathbb{Z}}$

$$A\|\boldsymbol{x}\|^2 \le \sum_{j\in\mathbb{Z}} |\langle \boldsymbol{x}, \tilde{\phi}_j\rangle|^2 \le B\|\boldsymbol{x}\|^2.$$

If the frame bounds A *and* B *are equal to each other then the frame is said to be tight.*

If the system $\tilde{\Phi}$ is a frame then there exists another frame $\Phi \stackrel{\Delta}{=} \{\phi_i\}_{i\in\mathbb{Z}}$ of the signals space such that any signal $\mathbf{x}$ can be expanded into the sum $\mathbf{x} = \sum_{i\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\phi}_i\rangle\, \phi_i$. The frames $\tilde{\Phi}$ and Φ can be interchanged. Together they form the so-called bi-frame. If the frame is tight then Φ can be chosen as $\Phi = c\tilde{\Phi}$.

Assume we have the analysis $\tilde{H}(z),\ \tilde{G}^1(z)\ \tilde{G}^2(z)$ and the synthesis $H(z),\ G^1(z)\ G^2(z)$ filter banks. We denote:

$$\tilde{\varphi}^1 \stackrel{\Delta}{=} \{\tilde{\varphi}^1(n) \stackrel{\Delta}{=} 2\tilde{h}(n)\},\ \tilde{\psi}^{r,1} \stackrel{\Delta}{=} \{\tilde{\psi}^{r,1}(n) \stackrel{\Delta}{=} 2\tilde{g}^r(n)\},$$
$$\varphi^1 \stackrel{\Delta}{=} \{\varphi^1(n) \stackrel{\Delta}{=} 2h(n)\},\ \psi^{r,1} \stackrel{\Delta}{=} \{\psi^{r,1}(n) \stackrel{\Delta}{=} 2g^r(n)\},\quad r = 1,2\ n \in \mathbb{Z}.$$

Then, the analysis and synthesis Eqs. (1.1) and (1.2) can be presented in the following way:

$$s_l^1 = \langle \mathbf{x}, \tilde{\varphi}^1(\cdot - 2l)\rangle, \qquad d_l^{r,1} = \langle \mathbf{x}, \tilde{\psi}^{r,1}(\cdot - 2l)\rangle, \qquad r = 1,2,\ l \in \mathbb{Z},$$
$$\hat{\mathbf{x}} = \frac{1}{2}\sum_{l\in\mathbb{Z}} s_l^1 \varphi^1(\cdot - 2l) + \frac{1}{2}\sum_{r=1}^{2}\sum_{l\in\mathbb{Z}} d_l^{r,1}\psi^{r,1}(\cdot - 2l).$$

If the given set of filters forms a perfect reconstruction filter bank then we have

$$\mathbf{x} = \frac{1}{2}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\varphi}^1(\cdot - 2l)\rangle \varphi^1(\cdot - 2l) + \frac{1}{2}\sum_{r=1}^{2}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\psi}^{r,1}(\cdot - 2l)\rangle \psi^{r,1}(\cdot - 2l). \tag{1.6}$$

Propositon 1.1 formulates the condition for the analysis filter bank to provide the frame expansion of the signal $\mathbf{x}$.

Proposition 1.1 ([4]) *Assume the impulse response $\{\tilde{h}(n)\},\ \{\tilde{g}^r(n)\},\ r = 1,2$ of the analysis filter bank $\tilde{H},\ \tilde{G}^r,\ r = 1,2$ belong to l_1. Then, the filter bank provides a frame expansion of signals $\boldsymbol{x} \in l_2$ if and only if the polyphase analysis matrix $\tilde{P}(z)$ has a full rank 2 on the unit circle $|z| = 1$.*

A similar condition for FIR filter banks was established in [8]. Obviously, if the polyphase matrix $P(z)$ of the filter bank $H(z),\ G^1(z)\ G^2(z)$ satisfies the condition (1.5) then the matrix $\tilde{P}(z)$ has a full rank.

Corollary 1.1 *Assume the impulse response $\{\tilde{h}(n)\},\ \{\tilde{g}^r(n)\},\ r = 1,2$ of the analysis filters $\tilde{H}$, $\tilde{G}^r,\ r = 1,2$ belong to l_1. Then, the perfect reconstruction filter bank $\tilde{H}(z),\ \tilde{G}^1(z)\ \tilde{G}^2(z)$ and $H(z),\ G^1(z)\ G^2(z)$ provides a frame expansion of signals $\boldsymbol{x} \in l_2$ and the set of two-sample shifts of the signals $\tilde{\varphi}^1,\ \tilde{\psi}^{r,1},\ \varphi^1,\ \psi^{r,1},\ r = 1,2$ forms a bi-frame of the signal space.*

One solution to (1.5) is the parapseudoinverse of $\tilde{\mathbf{P}}$:

$$\mathbf{P}(z) = \tilde{\mathbf{P}}^{+}(z) \stackrel{\Delta}{=} \frac{1}{2}\left(\tilde{\mathbf{P}}^T(z)\cdot\tilde{\mathbf{P}}(1/z)\right)^{-1}\cdot\tilde{\mathbf{P}}^T(z). \tag{1.7}$$

The synthesis frame that corresponds to the polyphase matrix $\tilde{\mathbf{P}}^{+}(z)$ is dual to the analysis frame. If $\mathbf{P}(z) =$ then the signals $\tilde{\varphi}^1$ and $\tilde{\psi}^{r,1},\ r = 1,2$ generate a tight frame.

1.3 Multiscale frame transforms

The iterated application of the analysis filter bank to the signal $\mathbf{s}^1 = \{s_k\}$ produces the following three signals:

$$
\begin{aligned}
s_l^2 &= \sum_{n\in\mathbb{Z}} \tilde{h}_{n-2l}\, s_n^1, = \sum_{n\in\mathbb{Z}} \tilde{h}_{n-2l} \sum_{m\in\mathbb{Z}} \tilde{h}_{m-2n}\, x_m = \langle \mathbf{x}, \tilde{\varphi}^2(\cdot - 4l)\rangle,\\
d_l^{r,2} &= \sum_{n\in\mathbb{Z}} \tilde{g}_{n-2l}^r\, s_n^1, = \sum_{n\in\mathbb{Z}} \tilde{g}_{n-2l}^r \sum_{m\in\mathbb{Z}} \tilde{h}_{m-2n}\, x_m = \langle \mathbf{x}, \tilde{\psi}^{r,2}(\cdot - 4l)\rangle,\\
&\text{where } \tilde{\varphi}^2(l) \triangleq 2\sum_{n\in\mathbb{Z}} \tilde{h}_n\, \tilde{\varphi}^1(n-2l), \quad \tilde{\psi}^{r,2}(l) \triangleq 2\sum_{n\in\mathbb{Z}} \tilde{g}_n^r\, \tilde{\varphi}^1(n-2l), \quad r = 1,2.
\end{aligned}
$$

Then, the signal $\mathbf{s}^1$ is restored as

$$
s_l^1 = \frac{1}{2}\sum_{n\in\mathbb{Z}} h_{l-2n}\, s_n^2 + \frac{1}{2}\sum_{r=1}^{2}\sum_{n\in\mathbb{Z}} g_{l-2n}^r\, d_n^{r,1}, \; r = 1,2,
$$

and the signal $\mathbf{x}$ is expanded into the following sums:

$$
\begin{aligned}
\mathbf{x} &= \frac{1}{4}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\varphi}^2(\cdot - 4l)\rangle \varphi^2(\cdot - 4l) + \frac{1}{4}\sum_{r=1}^{2}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\psi}^{r,2}(\cdot - 4l)\rangle \psi^{r,2}(\cdot - 4l)\\
&+\frac{1}{2}\sum_{r=1}^{2}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\psi}_r^1(\cdot - 2l)\rangle \psi^{r,1}(\cdot - 2l),\\
&\text{where } \varphi^2(l) \triangleq 2\sum_{n\in\mathbb{Z}} h_n\, \varphi^1(n-2l), \quad \psi^{r,2}(l) \triangleq 2\sum_{n\in\mathbb{Z}} g_n^r\, \varphi^1(n-2l), \quad r = 1,2.
\end{aligned}
$$

Thus, if the condition (1.5) is satisfied then the sets of four-sample shifts of the signals $\tilde{\varphi}^2$, $\tilde{\psi}^{r,2}$, φ^2, $\psi^{r,2}$, $r = 1,2$ and of two-sample shifts of signals $\tilde{\psi}^{r,1}$, $\psi^{r,1}$, $r = 1,2$ form a new bi-frame of the signal space. If $\mathbf{P}(z) = c\tilde{\mathbf{P}}^T(z)$ then the signals φ^2, $\psi^{r,2}$ and $\psi^{r,1}$, $r = 1,2$ generate a tight frame.

Successive iterations lead to the following expansion of the signal $\mathbf{x}$:

$$
\begin{aligned}
\mathbf{x} &= 2^{-N}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\varphi}^N(\cdot - 2^N l)\rangle \varphi^N(\cdot - 2^N l) + \sum_{r=1}^{2}\sum_{\nu=1}^{N} 2^{-\nu}\sum_{l\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\psi}^{r,\nu}(\cdot - 2^\nu l)\rangle \psi^{r,\nu}(\cdot - 2^\nu l),\\
&\text{where } \tilde{\varphi}^N(l) \triangleq 2\sum_{n\in\mathbb{Z}} h_n\, \tilde{\varphi}^{N-1}(n-2l), \quad \tilde{\psi}^{r,\nu}(l) \triangleq 2\sum_{n\in\mathbb{Z}} g_n^r\, \tilde{\varphi}^{\nu-1}(n-2l),\\
&\varphi^N(l) \triangleq 2\sum_{n\in\mathbb{Z}} h_n\, \varphi^{N-1}(n-2l), \quad \psi^{r,\nu}(l) \triangleq 2\sum_{n\in\mathbb{Z}} g_n^r\, \varphi^{\nu-1}(n-2l), \quad r = 1,2.
\end{aligned}
$$

Thus, we have a new bi-frame consisting of shifts of the signals $\tilde{\varphi}^N$, $\{\tilde{\psi}^{r,\nu}\}$ and φ^N, $\{\psi^{r,\nu}\}$, $r = 1,2$, $\nu = 1,\ldots,N$.

1.4 Scaling functions and framelets

It is well known ([9]) that under certain conditions the low-pass filter $H(z)$ such that $H(1) = 1$ generates a continuous scaling function $\varphi(t)$. To be specific, if the infinite product

$$\lim_{N\to\infty} \prod_{\nu=1}^{N} H(e^{j2^{-\nu}\omega}) \tag{1.8}$$

converges to $\Phi(\omega) \in L^2(\mathbb{R})$ then the inverse Fourier transform of $\Phi(\omega)$ is the scaling function $\varphi(t) \in L^2(\mathbb{R})$, which is a solution to the refinement equation $\varphi(t) = 2\sum_{k\in\mathbb{Z}} h_k\, \varphi(2t-k)$.

A simple sufficient condition for the existence of a smooth scaling function was established in [9].

Proposition 1.2 ([9]) *Let the transfer function $H(z)$ be factorized as $H(z) = \left(\frac{1+z^{-1}}{2}\right)^p K(z)$, where $K(z)$ is a rational function such that $K(1) = 1$. If the condition $\kappa \stackrel{\Delta}{=} \sup_{|z|=1} |K(z)| < 2^{p-1-m}$ is satisfied then there exists a scaling function $\varphi(t) \in L^2(\mathbb{R})$, which is continuous together with its derivatives up to order m.*

The relation (1.8) in time domain corresponds to infinite iteration of subdivision scheme, whose symbol is equal to $2H(z)$ and the initial data is the delta sequence. The limit function of this scheme is the scaling function $\varphi(t)$. This method is called the cascade algorithm [9]. Therefore, methods from the subdivision theory can be used for convergence analysis of the cascade algorithm and for the analysis of regularity of the scaling functions. Sometimes these methods can provide more accurate estimation of the regularity than the Fourier transform method in Proposition 1.2. The following proposition is a direct consequence of [10].

Proposition 1.3 *Let the transfer function $H(z)$ be factorized as $H(z) = \left(\frac{1+z^{-1}}{2}\right)^p K(z)$, where $K(z)$ is a rational function such that $K(1) = 1$. If subdivision scheme whose symbol is $2K(z)$, converges to a continuous function then there exists a scaling function $\varphi(t) \in L^2(\mathbb{R})$, which is continuous together with its derivatives up to order p.*

Under certain relaxed conditions on the low-pass filter, whose transfer function $H(z)$ is a rational function, the generated scaling function $\varphi(t)$ decays exponentially. We cite the following sufficient conditions.

Proposition 1.4 ([25]) *Let the transfer function $H(z)$ be factorized as $H(z) = \left(\frac{1+z^{-1}}{2}\right) M(z)$, where $M(z) = \sum_{i\in\mathbb{Z}} z^{-i} m_i$ is a rational function, which has no poles on the unit circle $|z| = 1$. If the inequality $\max\{\sum_{i\in\mathbb{Z}} |m_{2i}|, \sum_{i\in\mathbb{Z}} |m_{2i+1}|\} < 1$ holds then there exists a continuous scaling function $\varphi(t)$ and positive numbers A and g such that $|\varphi(t)| \le Ae^{-g|t|}$.*

Definition 1.2 *The set of functions $\{\psi^k(t)\}_{k=1}^n$ such that $\left\{\{2^{\nu/2}\psi^k(2^j t - l)\}_{\nu,l\in\mathbb{Z}}\right\}_{k=1}^n$ form a frame for $L^2(\mathbb{R})$ is called a wavelet frame. The functions $\{\psi^k(t)\}$ are called framelets.*

The Mixed Extension Principle ([22]) implies the following statement.

Proposition 1.5 *Let $\tilde{H}$, $\tilde{G}^1$ $\tilde{G}^2$ and H, G^1 G^2 be a perfect reconstruction filter bank and the impulse response $\{\tilde{h}(n)\}$, $\{\tilde{g}^r(n)\}$, $r = 1, 2$ and $\{h(n)\}$, $\{g^r(n)\}$, $r = 1, 2$ decay exponentially. If the low-pass filters $\tilde{H}$ and H generate square integrable scaling functions $\tilde{\varphi}(t)$, and $\varphi(t)$, respectively, then the functions*

$$\tilde{\psi}^r(t) \triangleq 2\sum_{k\in\mathbb{Z}} \tilde{g}_k^r\, \tilde{\varphi}(2t-k), \quad \psi^r(t) \triangleq 2\sum_{k\in\mathbb{Z}} g_k^r\, \varphi(2t-k)\ , r = 1, 2, \tag{1.9}$$

generate the dual wavelet frames of $L^2(\mathbb{R})$ i.e. they are the dual framelets.

If the scaling functions $\tilde{\varphi}(t)$ and $\varphi(t)$ decay exponentially and the rational functions $\tilde{G}^r(z)$, $G^r(z)$, $r = 1, 2$, have no poles on the unit circle $|z| = 1$, then, their impulse response g_i^r, $r = 1, 2$, decay exponentially. Thus, the framelets $\tilde{\psi}^r(t)$ and $\psi^r(t)$, defined in (1.9), also decay exponentially.

A framelet $\psi^r(t)$ has p vanishing moments if $\int_{-\infty}^{\infty} t^s \psi^r(t)\, dt = 0, \quad s = 0, \ldots p-1$. The number of vanishing moments of the framelet $\psi^r(t)$ is equal to the multiplicity of zero of the filter $G^r(z)$ at $z = 1$ [23].

2 Interpolatory frames

2.1 Bi-frames

Assume that the even polyphase component $F_e(z)$ of a filter $F(z)$ is 1/2. Then, the filter is called interpolatory. In the rest of the paper we deal exclusively with filter banks, whose low-pass filters are interpolatory:

$$H(z) = \frac{1 + z^{-1}U(z^2)}{2}, \qquad \tilde{H}(z) = \frac{1 + z^{-1}\tilde{U}(z^2)}{2}. \tag{2.1}$$

We assume that $U(z)$ and $\tilde{U}(z)$ are rational functions that have no poles on the unit circle $|z| = 1$, $U(1) = \tilde{U}(1) = 1$ and the following symmetry conditions hold

$$z^{-1}U(z^2) = zU(z^{-2}), \qquad z^{-1}\tilde{U}(z^2) = z\tilde{U}(z^{-2}). \tag{2.2}$$

If an interpolatory low-pass filter generates the scaling function $\varphi(t)$ then this scaling function is interpolatory. In other words, $\varphi(n) = \delta_n,\ n \in \mathbb{Z}$.

The polyphase matrices for a filter bank that use the interpolatory low-pass filters $H(z)$ and $\tilde{H}(z)$ are

$$\tilde{\mathbf{P}}(z) \triangleq \begin{pmatrix} 1/2 & \tilde{U}(z)/2 \\ \tilde{G}_e^1(z) & \tilde{G}_o^1(z) \\ \tilde{G}_e^2(z) & \tilde{G}_o^2(z) \end{pmatrix}, \quad \mathbf{P}(z) \triangleq \begin{pmatrix} 1/2 & G_e^1(z) & G_e^2(z) \\ U(z)/2 & G_o^1(z) & G_o^2(z) \end{pmatrix}.$$

Then, the perfect reconstruction condition (1.5) leads to

$$\mathbf{P}_g(z) \cdot \ \tilde{\mathbf{P}}_g(1/z) = \mathbf{Q}(z), \tag{2.3}$$

where

$$\tilde{\mathbf{P}}_g(z) \triangleq \begin{pmatrix} \tilde{G}_e^1(z) & \tilde{G}_o^1(z) \\ \tilde{G}_e^2(z) & \tilde{G}_o^2(z) \end{pmatrix}, \quad \mathbf{P}_g(z) \triangleq \begin{pmatrix} G_e^1(z) & G_e^2(z) \\ G_o^1(z) & G_o^2(z) \end{pmatrix},$$
$$\mathbf{Q}(z) \triangleq \begin{pmatrix} 1/4 & -\tilde{U}(z^{-1})/4 \\ -U(z)/4 & (2 - U(z)\tilde{U}(z^{-1}))/4 \end{pmatrix}.$$

We can immediately obtain a solution to (2.3) with the interpolatory filters $G^1(z)$ and $\tilde{G}^1(z)$:

$$G_e^1(z) = \tilde{G}_e^1(z) = \frac{1}{2}, \quad \tilde{G}_o^1(z) = -\frac{\tilde{U}(z)}{2}, \quad G_o^1(z) = -\frac{U(z)}{2}, \quad G_e^2(z) = \tilde{G}_e^2(z) = 0.$$

The odd components of the filters $G^2(z)$ and $\tilde{G}^2(z)$ are derived from the factorization

$$v(z)\tilde{v}(z^{-1}) = V(z), \text{ where } V(z) \triangleq \frac{1 - U(z)\tilde{U}(z^{-1})}{2} \tag{2.4}$$

and the filters

$$G^2(z) = z^{-1}v(z^2), \qquad \tilde{G}^2(z) = z^{-1}\tilde{v}(z^2). \tag{2.5}$$

Note that the filters

$$G^1(z) = \frac{1 - z^{-1}U(z^2)}{2} = H(-z), \qquad \tilde{G}^1 = \frac{1 - z^{-1}\tilde{U}(z^2)}{2} = \tilde{H}(-z) \tag{2.6}$$

are interpolatory. They are high-pass filters because $U(1) = \tilde{U}(1) = 1$. The transfer functions $G^1(z)$ and $\tilde{G}^1(z)$ are invariant about inversion $z \longrightarrow z^{-1}$ due to Eq. (2.2).

Proposition 2.1 *Let the rational functions $\tilde{U}(z)$ and $U(z)$ have no poles on the unit circle and $\tilde{U}(1) = U(1) = 1$. Then, the perfect reconstruction filter bank $\tilde{H}(z)$, $\tilde{G}^1(z)$ $\tilde{G}^2(z)$ and $H(z)$, $G^1(z)$ $G^2(z)$, defined in Eqs. (2.1), (2.5) and (2.6), implements a frame expansion of signal $\boldsymbol{x} \in l_2$.*

Proof: Since the functions $\tilde{U}(z)$ and $U(z)$ have no poles on the unit circle, the impulse response $\{\tilde{h}(n)\}$, $\{\tilde{g}^r(n)\}$, $r = 1, 2$ of the analysis filters $\tilde{H}(z)$ $\tilde{G}^r(z)$, $r = 1, 2$ belong to l_1. The minors of the analysis polyphase matrix

$$\tilde{\mathbf{P}}(z) \triangleq \begin{pmatrix} 1/2 & \tilde{U}(z)/2 \\ 1/2 & -\tilde{U}(z)/2 \\ 0 & \tilde{v}(z) \end{pmatrix}$$

are $m_1(z) = \tilde{U}(z)/2$, $m_2(z) = \tilde{v}(z)/2$. They can not vanish simultaneously due to Eq. (2.4) . Thus the matrix $\tilde{\mathbf{P}}(z)$ has full rank and the assertion follows from Proposition 1.1. ■

The rational function $V(z^2)$ can be written as $V(z^2) = (1 - z^{-1}U(z^2) \cdot z\tilde{U}(z^{-2}))/2 = V(z^{-2})$. Thus, a rational symmetric or antisymmetric factorization is possible. The trivial rational symmetric factorizations are $v(z) = 1$, $\tilde{v}(z) = V(z)$ or $\tilde{v}(z) = 1$, $v(z) = V(z)$. Since $V(1) = 0$, at least one of the filters $G^2(z)$ and $\tilde{G}^2$ is high-pass and the corresponding framelet has vanishing moments.

2.2 Tight and semi-tight frames

If filter $U(z) = \tilde{U}(z)$ then we get $H(z) = \tilde{H}(z)$, $G^1(z) = \tilde{G}^1(z)$ and

$$V(z) = (1 - |U(z)|^2)/2, \; V(z^2) = 2H(z)H(-z). \tag{2.7}$$

If the inequality

$$|U(z)| \leq 1 \;\text{ as } |z| = 1 \tag{2.8}$$

holds, then, the function $V(z)$ can be factored as $V(z) = v(z)\, v(1/z)$. This factorization is not unique. Due to Riesz's lemma [9], a rational factorization is possible. Then, we have $G^2(z) = \tilde{G}^2(z)$. Thus, the synthesis filter bank coincides with the analysis filter bank and generates a tight frame. Note that, due to (2.7), the (anti)symmetric rational factorization is possible if and only if all roots and poles of the function $H(z)$ have even multiplicity. If $H(z)$ has a root of multiplicity $2m$ at $z = 1$, the filter $G^2(z)$ has roots of multiplicity m at $z = 1$ and $z = -1$. The corresponding framelet $\psi^2(t)$ has m vanishing moments. A similar construction for the tight frame based on a family of interpolatory symmetric FIR filters was presented in [6]. However, the filter $G^2(z)$ in [6] lacks symmetry.

If the condition (2.8) is not satisfied we are still able to generate frames, which are very close to a tight frame. Namely,

$$\begin{aligned} &H(z) = \tilde{H}(z) = (1 + z^{-1}U(z^2))/2, \; G^1(z) = \tilde{G}^1(z) = (1 - z^{-1}U(z^2))/2, \\ &G^2(z) = z^{-1}v(z^2), \quad \tilde{G}^2(z) = z^{-1}\tilde{v}(z^2), \quad v(z)\, \tilde{v}(1/z) = V(z) = (1 - |U(z)|^2)/2. \end{aligned} \tag{2.9}$$

It is natural to refer to such a frame as a *semi-tight* frame. Due to the symmetry of $V(z)$, an (anti)symmetric factorization of type (2.9) is always possible. Therefore, even when (2.8) holds, sometimes it is preferable to construct a semi-tight rather than a tight frame. For example, it was proved in [18] that a compactly supported interpolatory symmetric tight frame with two framelets is possible only with the low-pass filter $H(z) = 1/2 + (z + 1/z)/4$. In this case the scaling function and the framelets are piece-wise linear. The framelets $\psi^1(t)$ and $\psi^2(t)$ have two and one vanishing moments, respectively. However, it is possible to construct a variety of compactly supported interpolatory symmetric semi-tight frames with smooth framelets. The construction of compactly supported interpolatory symmetric tight frame with three framelets is always possible [6].

2.3 Dual frame

Let

$$\tilde{\mathbf{P}}(z) \triangleq \begin{pmatrix} 1/2 & \tilde{U}(z)/2 \\ 1/2 & -\tilde{U}(z)/2 \\ 0 & \tilde{v}(z) \end{pmatrix}$$

be the polyphase matrix of an interpolatory filter bank, which generates an analysis frame. The dual synthesis frame is generated by a filter bank, whose polyphase matrix is the parapseudoinverse $\tilde{\mathbf{P}}^{+}(z)$ of $\tilde{\mathbf{P}}(z)$ (see (1.7)). Denote

$$U(z) \triangleq \frac{\tilde{U}(z)}{|\tilde{U}(z)|^2 + 2|\tilde{v}(z)|^2}, \quad v(z) \triangleq \frac{\tilde{v}(z)}{|\tilde{U}(z)|^2 + 2|\tilde{v}(z)|^2}. \tag{2.10}$$

Since $\tilde{v}(1) = 0$, we have $U(1) = 1$. Obviously, $U(z)$ has no poles on the unit circle and $z^{-1}U(z^2)$ is symmetric about inversion $z \to 1/z$. The function $v(z)$ has zero of the same multiplicity as $\tilde{v}(z)$ at $z = 1$. It is readily verified that

$$\tilde{\mathbf{P}}^{+}(z) = \begin{pmatrix} 1/2 & 1/2 & 0 \\ U(z)/2 & -U(z)/2 & v(z) \end{pmatrix}.$$

The matrix $\tilde{\mathbf{P}}^{+}(z)$ has the same structure as $\tilde{\mathbf{P}}^{T}(z)$. The filters

$$H(z) \triangleq 1/2 + z^{-1}U(z^2)/2 \text{ and } G^1(z) \triangleq 1/2 - z^{-1}U(z^2)/2 \tag{2.11}$$

are interpolatory. The product $v(z)\tilde{v}(1/z) = (1 - U(z)\tilde{U}(1/z))/2$. Since $U(1) = 1$, we have $G^1(1) = 0$. Thus, the framelet $\psi^1(t)$ has vanishing moments.

Theorem 2.1 *Assume that an interpolatory low-pass symmetric filter $\tilde{H}(z) = 1/2 + z^{-1}\tilde{U}(z^2)/2$ has zero of multiplicity m at $z = -1$ and no poles on the unit circle $|z| = 1$. Then it generates an invertible analysis filter bank $\tilde{H}(z)$, $\tilde{G}^1(z)$, $\tilde{G}^2(z)$ such that the high-pass filters $\tilde{G}^1(z)$ and $\tilde{G}^2(z)$ are symmetric and have zero of multiplicity m at $z = 1$. In addition, the filter $\tilde{G}^1(z)$ is interpolatory and the even polyphase component of the filter $\tilde{G}^2(z)$ is zero. The filter $\tilde{G}^2(z)$ has zero of multiplicity m at $z = -1$. The dual synthesis filter bank $H(z)$, $G^1(z)$, $G^2(z)$ has the same properties.*

Proof: Obviously, the high-pass filter $\tilde{G}^1(z) = 1/2 - z^{-1}\tilde{U}(z^2)/2$ is interpolatory, symmetric and has zero of multiplicity m at $z = 1$. To obtain the filter $\tilde{G}^2(z)$ we choose $\tilde{v}(z) \triangleq V(z) = (1 - |\tilde{U}(z)|^2)/2$. Then, the filter $\tilde{G}^2(z) \triangleq z^{-1}\tilde{v}(z^2) = 2z^{-1}\tilde{G}^1(z)\tilde{H}(z)$ is symmetric and has zero of multiplicity m at $z = 1$ and also at $z = -1$. The dual filter $G^2(z) \triangleq z^{-1}v(z^2)$, where $v(z)$ is defined in (2.10), has the same properties. The dual high-pass filter $G^1(z)$ is defined in (2.11). We have $|\tilde{U}(z)|^2 + 2|\tilde{v}(z)|^2 =$

$\left(1+|\tilde{U}(z)|^4\right)/2$. Then, the symmetric interpolatory filter

$$\begin{aligned}
G^1(z) &\triangleq \frac{1}{2}\left(1-z^{-1}U(z^2)\right) = \frac{1}{2}\left(1-z^{-1}\frac{\tilde{U}(z^2)}{|\tilde{U}(z^2)|^2+2|\tilde{v}(z^2)|^2}\right)\\
&= \frac{1+|\tilde{U}(z^2)|^4-2z^{-1}\tilde{U}(z^2)}{2(1+|\tilde{U}(z^2)|^4)} = \frac{1+|\tilde{U}(z^2)|^4-2z^{-1}\tilde{U}(z^2)}{2(1+|\tilde{U}(z^2)|^4)}\\
&= \frac{\left(1-z^{-1}\tilde{U}(z^2)\right)+\left(z^{-2}\tilde{U}^2(z^2)\, z^2\tilde{U}^2(z^{-2})-z^{-1}\tilde{U}(z^2)\right)}{2(1+|\tilde{U}(z^2)|^4)}\\
&= \frac{\left(1-z^{-1}\tilde{U}(z^2)\right)+z^{-1}\tilde{U}(z^2)\left(z^{-3}\tilde{U}^3(z^2)-1\right)}{2(1+|\tilde{U}(z^2)|^4)}\\
&= \left(1-z^{-1}\tilde{U}(z^2)\right)\frac{1-z^{-1}\tilde{U}(z^2)-z^{-2}\tilde{U}^2(z^2)-z^{-3}\tilde{U}^3(z^2)}{2(1+|\tilde{U}(z^2)|^4)}.
\end{aligned}$$

Finally, we get

$$G^1(z) = 2\tilde{G}^1(z)\frac{\tilde{G}^1(z)-z^{-2}\tilde{U}^2(z^2)\tilde{H}(z)}{2(1+|\tilde{U}(z^2)|^4)}.$$

Hence, $G^1(z)$ has zero of multiplicity m at $z=1$. By similar calculations we derive

$$H(z) = 2\tilde{H}(z)\frac{\tilde{H}(z)-z^{-2}\tilde{U}^2(z^2)\tilde{G}^1(z)}{2(1+|\tilde{U}(z^2)|^4)}.$$

Hence, $H(z)$ has zero of multiplicity m at $z=-1$. ■

Remark. All the four framelets $\tilde{\psi}^1(t)$, $\tilde{\psi}^2(t)$, $\psi^1(t)$ and $\psi^2(t)$ that were generated by the above filter bank $\tilde{H}(z)$, $\tilde{G}^1(z)$, $H(z)$, $G^1(z)$ and $G^2(z)$ have m vanishing moments.

3 Design of interpolatory filters

In order to generate a continous scaling function, the interpolatory rational transfer function $\tilde{H}(z) = (1+z^{-1}\tilde{U}(z^2))/2$ must have zero of multiplicity $m>0$ at $z=-1$. Then $\tilde{G}^1(z) = (1-z^{-1}\tilde{U}(z^2))/2$ has zero of multiplicity m at $z=1$. The filter with a transfer function $\tilde{G}^1(z)$ eliminates sampled polynomials up to degree $m-1$. The filter with the transfer function $\tilde{H}(z)$ restores polynomials up to degree $m-1$. To achieve it, the filter $\tilde{U}(z)$, being applied to the even subarray of a sampled polynomial $\mathbf{p}$, has to produce exactly the odd subarray of $\mathbf{p}$ and the filter $z^{-1}\tilde{U}(z)$, being applied to the odd subarray of $\mathbf{p}$, must produce the even subarray. In other words, the filtered half-array of $\mathbf{p}$ has to exactly predict another half-array. As a source for the design of these filters we use the so-called discrete splines. We will show that the derived filters are related to the Butterworth filters, which are commonly used in signal processing [17].

3.1 Discrete splines

We briefly outline the properties of discrete splines, which will be needed later. For a detailed description of the subject - see [20, 21]. The discrete splines are defined on the grid $\{k\}$ and they are the counterparts of the continuous polynomial splines.

The signal

$$\mathbf{b}^{1,n} = \{b_k^{1,n}\} \stackrel{\Delta}{=} \begin{cases} 1, & \text{as } k = 0, \dots, 2n-1 \\ 0, & \text{otherwise} \end{cases} \Longleftrightarrow B^{1,n}(z) = \frac{1 - z^{2n}}{1 - z},$$

is called the discrete B-spline of first order.

We define by recurrence the higher order B-splines via discrete convolutions:

$$\mathbf{b}^{p,n} = \mathbf{b}^{1,n} * \mathbf{b}^{p-1,n} \Longleftrightarrow B^{p,n}(z) = \left(\frac{1 - z^{2n}}{1 - z}\right)^p.$$

In this paper we are interested only in the case when $p = 2r,\ r \in \mathbb{N}$ and $n = 1$. In this case we have $B^{2r,1}(z) = (1 + z^{-1})^{2r}$. The B-spline $\mathbf{b}^{2r,1}$ is symmetric about the point $k = r$ where it attains its maximal value. We define the centered B-spline $\mathbf{q}^{2r}$ of order $2r$ as the shift of the B-spline: $\mathbf{q}^{2r} \stackrel{\Delta}{=} \{q_k^{2r} = b_{k+r}^{2r,1}\},\quad Q^{2r}(z) = z^r B^{2r,1}(z) = z^r(1 + z^{-1})^{2r}$. The discrete spline $\mathbf{a}^{2r} = \{a_k^{2r}\}_{k \in \mathbb{Z}}$ of order $2r$ on the grid $\{2k\}$ is defined as a linear combination with real-valued coefficients of shifts of the centered B-spline:

$$\begin{aligned} a_k^{2r} &\stackrel{\Delta}{=} \sum_{l=-\infty}^{\infty} c_l\, q_{k-2l}^{2r} \Longleftrightarrow A^{2r}(z) = C(z^2)Q^{2r}(z) = C(z^2)\left(\upsilon^{2r}(z^2) + z^{-1}\theta^{2r}(z^2)\right), \\ \upsilon^{2r}(z^2) &\stackrel{\Delta}{=} Q_e^{2r}(z^2) = \frac{1}{2}\left(z^r\left(1 + z^{-1}\right)^{2r} + (-z)^r\left(1 - z^{-1}\right)^{2r}\right), \\ \theta^{2r}(z^2) &\stackrel{\Delta}{=} Q_o^{2r}(z^2) = \frac{z}{2}\left(z^r\left(1 + z^{-1}\right)^{2r} - (-z)^r\left(1 - z^{-1}\right)^{2r}\right). \end{aligned}$$

Our scheme to design prediction filters that use the discrete splines consists of the following. We construct the discrete spline $\mathbf{a}^{2r}$, which interpolates even samples $\{e_k = x_{2k}\}$ of a signal $\mathbf{x} \stackrel{\Delta}{=} \{x_k\}_{k \in \mathbb{Z}}$, that is $a_{2k}^{2r} = e_k,\ k \in \mathbb{Z}$. Then, we use the values a_{2k+1}^{2r} for the prediction of odd samples $\{o_k = x_{2k+1}\}$.

The z-transform of the even component of the spline $\mathbf{a}^{2r}$ is

$$A_e^{2r}(z) = C(z)\upsilon^{2r}(z) = E(z) \Longrightarrow C(z) = E(z)/\upsilon^{2r}(z).$$

Then, the z-transform of the odd component of the spline $\mathbf{a}^{2r}$

$$A_o^{2r}(z) = C(z)\theta^{2r}(z) = U^{2r}(z)E(z) \text{ where } U^{2r}(z) \stackrel{\Delta}{=} \frac{\theta^{2r}(z)}{\upsilon^{2r}(z)}. \tag{3.1}$$

Thus, in order to predict the odd samples of the signal $\mathbf{x}$, we filter the even subarray of $\mathbf{x}$ with the filter $U^{2r}(z)$.

3.2 Properties of the designed filters

In this section we prove that the designed filters can serve as a source for frame constructions. Denote

$$\chi^{2r}(z) \triangleq \frac{1}{2}\left(1 + z^{-1}U^{2r}(z^2)\right), \quad \gamma^{2r}(z) \triangleq \frac{1}{2}\left(1 - z^{-1}U^{2r}(z^2)\right).$$

Proposition 3.1 *The rational functions $U^{2r}(z)$, defined in (3.1), have the following properties:*

P1. *No poles exist on the unit circle $|z| = 1$.*

P2. $U^{2r}(1) = 1$.

P3. *Symmetry:* $z^{-1}U^{2r}(z^2) = zU^{2r}(z^{-2})$.

P4. $|U^{2r}(z)| \leq 1$.

P5. *The function $\chi^{2r}(z)$ has a root of multiplicity $2r$ at $z = -1$ and the function $\gamma^{2r}(z)$ has a root of multiplicity $2r$ at $z = 1$.*

Proof: We substitute $z = e^{j\omega}$ into $z^{-1}U^{2r}(z^2)$. We have

$$z^{-1}U^{2r}(z^2) = \frac{e^{ir\omega}(1 - e^{-j\omega})^{2r} + (-1)^r e^{ir\omega}(1 - e^{-j\omega})^{2r}}{e^{ir\omega}(1 + e^{-j\omega})^{2r} + (-1)^r e^{ir\omega}(1 - e^{-j\omega})^{2r}} = \frac{\left(\cos\frac{\omega}{2}\right)^{2r} - \left(\sin\frac{\omega}{2}\right)^{2r}}{\left(\cos\frac{\omega}{2}\right)^{2r} + \left(\sin\frac{\omega}{2}\right)^{2r}}.$$

Hence **P1** – **P4** follow. The function

$$\chi^{2r}(z) = \frac{\left(\cos\frac{\omega}{2}\right)^{2r}}{\left(\cos\frac{\omega}{2}\right)^{2r} + \left(\sin\frac{\omega}{2}\right)^{2r}} = \frac{\left(1 + z^{-1}\right)^{2r}}{\left(1 + z^{-1}\right)^{2r} + (-1)^r\left(1 - z^{-1}\right)^{2r}}, \tag{3.2}$$

$$\gamma^{2r}(z) = \frac{\left(\sin\frac{\omega}{2}\right)^{2r}}{\left(\cos\frac{\omega}{2}\right)^{2r} + \left(\sin\frac{\omega}{2}\right)^{2r}} = \frac{(-1)^r\left(1 - z^{-1}\right)^{2r}}{\left(1 + z^{-1}\right)^{2r} + (-1)^r\left(1 - z^{-1}\right)^{2r}}. \tag{3.3}$$

Equations (3.2) and (3.3) imply **P5**. ■

Remark. From (3.2) and (3.3) we can see that the functions $\chi^{2r}(z)$ and $\gamma^{2r}(z)$ coincide with the squared magnitudes of the frequency response of the low- and high-pass digital Butterworth filters of order r, respectively. For details see [17].

Proposition 3.2 *The filter $\chi^{2r}(z)$ generates the scaling function $\Phi^{2r}(t) \in L^2(\mathbb{R})$ such that*

$$\hat{\Phi}^{2r}(\omega) = \lim_{N\to\infty}\prod_{\nu=1}^{N}\chi^{2r}(e^{j2^{-\nu}\omega}), \quad \Phi^{2r}(t) = 2\sum_{k\in\mathbb{Z}}\chi_k^{2r}\Phi^{2r}(2t-k).$$

The scaling function $\Phi^{2r}(t)$ is continuous together with its derivatives up to the order $r-1$ (belongs to C^{r-1}). The filter $\gamma^{2r}(z)$ generates the framelet $\Psi^{2r}(t) \in L^2(\mathbb{R})$ with the same smoothness as $\Phi^{2r}(t)$, such that

$$\Psi^{2r}(t) = 2\sum_{k\in\mathbb{Z}}\gamma_k^{2r}\Phi^{2r}(2t-k). \tag{3.4}$$

The framelet $\Psi^{2r}(t)$ has $2r$ vanishing moments.

Proof: From (3.2) we have that the function $\chi^{2r}(z)$ can be factorized as

$$\chi^{2r}(z) = \left(\frac{1+z^{-1}}{2}\right)^p K(z), \quad K(e^{j\omega}) = \frac{e^{-jr\omega}}{\left(\cos\frac{\omega}{2}\right)^{2r} + \left(\sin\frac{\omega}{2}\right)^{2r}}.$$

$K(1) = 1$ and the following estimate is true: $\kappa \stackrel{\Delta}{=} \sup_{|z|=1} |K(z)| = 2^{r-1} < 2^{2r-1-(r-1)}$. Then, Proposition 1.2 implies that there exists a scaling function $\Phi^{2r}(t) \in L^2(\mathbb{R})$, which belongs to C^{r-1}.

The rational function $\gamma^{2r}(z)$ has no poles on the unit circle $|z| = 1$. Therefore, its impulse response $\{\gamma_k^{2r}\}_{k\in\mathbb{Z}}$ decays exponentially as $k \to \infty$. Therefore, the function $\Psi^{2r}(t) \in L^2(\mathbb{R})$, defined in (3.4), exists and has the same smoothness as $\Phi^{2r}(t)$. The multiplicity of zero of the filter $\gamma^{2r}(z)$ at $z = 1$ is $2r$. Therefore, the framelet $\Psi^{2r}(t)$ has $2r$ vanishing moments.■

Using Propositions 1.3 and 1.4 we established in [25] improved evaluations of the smoothness for a few scaling functions and framelets.

Proposition 3.3 *The filters $\chi^{2r}(z)$, $r = 2, 3, 4$ generate the scaling functions $\Phi^{2r}(t)$, which decay exponentially as $t \to \infty$. In addition, $\Phi^4(t) \in C^2$, $\Phi^6(t) \in C^4$ and $\Phi^8(t) \in C^5$.*

4 Butterworth frames

The above considerations suggest that the filters $U^{2r}(z)$, $\chi^{2r}(z)$ and $\gamma^{2r}(z)$, which originate from discrete splines, can be useful for the construction of frames in the signal space. To be specific, we choose $U(z) = U^{2r}(z)$, $H(z) = \chi^{2r}(z)$, $\tilde{U}(z) = U^{2p}(z)$ and $\tilde{H}(z) = \chi^{2p}(z)$ where r and p are some natural numbers that may be equal to each other. Since there is a relation between the filters and the Butterworth filters we call the corresponding frames the Butterworth frames. We denote $\rho(z) \stackrel{\Delta}{=} z + 2 + z^{-1}$. Thus $\rho(-z) = -z + 2 - z^{-1}$.

4.1 Tight frames

We define the filters

$$H(z) = \tilde{H}(z) \stackrel{\Delta}{=} \chi^{2r}(z) = \frac{1 + z^{-1}U^{2r}(z^2)}{2} = \frac{\rho^r(z)}{\rho^r(z) + \rho^r(-z)},$$
$$G^1(z) = \tilde{G}^1(z) = \gamma^{2r}(z) = \frac{\rho^r(-z)}{\rho^r(z) + \rho^r(-z)}.$$

Due to **P4** in Proposition 3.1 we get a tight frame when we factorize $V^{2r}(z)$ to be

$$V^{2r}(z) = \frac{1}{2}\left(1 - |U^{2r}(z)|^2\right) = v^r(z)v^r(1/z).$$

From (2.7) we have

$$V^{2r}(z^2) = 2H(z)\,H(-z) = \frac{2(-1)^r z^{-2r}\left(1-z^2\right)^{2r}}{\left(z^r\left(1+z^{-1}\right)^{2r} + (-z)^r\left(1-z^{-1}\right)^{2r}\right)^2}$$

$$= v^r(z^2)v^r(z^{-2}), \quad v^r(z^2) \triangleq \frac{\sqrt{2}\left(1-z^2\right)^r}{\rho^r(z)+\rho^r(-z)}. \tag{4.1}$$

If $r = 2n$ then we can define $v^r(z^2)$ differently:

$$v^r(z^2) \triangleq \frac{\sqrt{2}\left(z-z^{-1}\right)^{2n}}{\rho^{2n}(z)+\rho^{2n}(-z)}.$$

Hence, the three filters $H(z) = \chi^{2r}(z)$, $G^1(z) = H(-z) = \gamma^{2r}(z)$ and $G^2(z) \triangleq z^{-1}v^r(z^2)$ generate a tight frame in the signal space. The scaling function $\varphi(t)$ and the framelet $\psi^1(t)$ are symmetric, whereas the framelet $\psi^2(t)$ is symmetric when r is even and antisymmetric when r is odd. The framelet $\psi^1(t)$ has $2r$ vanishing moments and the framelet $\psi^2(t)$ has r vanishing moments. The frequency response of the filter $H(z)$ is maximally flat. The frequency response of the filter $G^1(z)$ is a mirrored version of $H(z)$. The frequency response of the filter $G^2(z)$ is symmetric about $\omega = \pi/2$ and it vanishes at the points $\omega = 0$ and $\omega = \pi$.

Examples:

The simplest case, $r = 1$:

$$U^2(z) = \frac{1+z}{2}, \quad H(z) = \frac{1+z^{-1}U^2(z^2)}{2} = \frac{z^{-1}+2+z}{4} \tag{4.2}$$

$$G^1(z) = H(-z) = \frac{-z^{-1}+2-z}{4}, \quad G^2(z) = \frac{\sqrt{2}(1-z^2)}{4z}.$$

The filter $U^2(z)$ is FIR and, therefore, the scaling function $\varphi(t)$ and the framelets $\psi^1(t)$ and $\psi^2(t)$ are compactly supported. The framelet $\psi^1(t)$ has two vanishing moments. The framelet $\psi^2(t)$ is antisymmetric and has one vanishing moment.

Cubic discrete spline, $r = 2$:

$$U^4(z) = 4\frac{1+z}{z+6+z^{-1}}, \quad H(z) = \frac{(z+2+z^{-1})^2}{2\left(z^{-2}+6+z^2\right)} \tag{4.3}$$

$$G^1(z) = \frac{(z-2+z^{-1})^2}{2\left(z^{-2}+6+z^2\right)}, \quad G^2(z) = \frac{\sqrt{2}z^{-1}(z-z^{-1})^2}{2\left(z^{-2}+6+z^2\right)}.$$

The framelet $\psi^1(t)$ has four vanishing moments. The framelet $\psi^2(t)$ is symmetric and has two vanishing moments.

Discrete spline of sixth order, $r = 3$**:**

$$U^6(z) = \frac{(z + 14 + z^{-1})(1 + z)}{6z^{-1} + 20 + 6z}, \quad H(z) = \frac{(z^{-1} + 2 + z)^3}{2\,(6z^2 + 20 + 6z^{-2})} \tag{4.4}$$
$$G^1(z) = \frac{(-z^{-1} + 2 - z)^3}{2\,(6z^2 + 20 + 6z^{-2})}, \quad G^2(z) = \frac{\sqrt{2}z^{-1}(1 - z^2)^3}{2\,(6z^2 + 20 + 6z^{-2})}.$$

The framelet $\psi^1(t)$ has six vanishing moments. The framelet $\psi^2(t)$ is antisymmetric and has three vanishing moments.

Discrete spline of eighth order, $r = 4$**:**

$$U_d^8(z) = \frac{8(1 + z)(z^{-1} + 6 + z)}{z^{-2} + 28z^{-1} + 70 + 28z + z^2} \quad H(z) = \frac{(z^{-1} + 2 + z)^4}{2\,(z^{-4} + 28z^{-2} + 70 + 28z^2 + z^4)} \tag{4.5}$$
$$G^1(z) = \frac{(z^{-1} - 2 + z)^4}{2\,(z^{-4} + 28z^{-2} + 70 + 28z^2 + z^4)} \quad G^2(z) = \frac{\sqrt{2}z^{-1}(z - z^{-1})^4}{2\,(z^{-4} + 28z^{-2} + 70 + 28z^2 + z^4)}.$$

The framelet $\psi^1(t)$ has eight vanishing moments. The framelet $\psi^2(t)$ is symmetric and has four vanishing moments.

4.2 Semi-tight frames

Unlike tight frames, a symmetric factorization of type (2.9) of the function $V^{2r}(z)$ is possible for either of even and odd values of r:

$$V^{2r}(z^2) = \frac{2\left(2 - z^{-2} - z^2\right)^r}{\left(\rho^r(z) + \rho^r(-z)\right)^2} = v^{2p,s}(z^2)\tilde{v}^{2(r-p),2-s}(z^{-2}),$$
$$v^{2p,s}(z^2) \triangleq \frac{\sqrt{2}\left(2 - z^{-2} - z^2\right)^p}{\left(\rho^r(z) + \rho^r(-z)\right)^s}, \quad \tilde{v}^{2(r-p),2-s}(z^2) \triangleq \frac{\sqrt{2}\left(2 - z^{-2} - z^2\right)^{r-p}}{\left(\rho^r(z) + \rho^r(-z)\right)^{2-s}}. \tag{4.6}$$

We can get an antisymmetric factorization by choosing an odd p:

$$v^{2p,s}(z^2) \triangleq -\frac{\sqrt{2}(-z^2)^{-r}\left(1 - z^2\right)^p}{\left(\rho^r(z) + \rho^r(-z)\right)^s} \tag{4.7}$$
$$\tilde{v}^{2(r-p),2-s}(z^2) \triangleq \frac{\sqrt{2}(-z^2)^{p-2r}\left(1 - z^2\right)^{2r-p}}{\left(\rho^r(z) + \rho^r(-z)\right)^{2-s}}, \quad s \in \mathbb{Z}.$$

With this factorization we can change the number of vanishing moments in the framelets $\psi^2(t)$ and $\tilde{\psi}^2(t)$. One option is that one of the filters $G^2(z) = z^{-1}v^{p,s}(z^2)$ or $\tilde{G}^2(z) = z^{-1}\tilde{v}_2^{2r-p,2-s}(z^2)$ has a finite impulse response. It is achieved if $s \leq 0$ or $s \geq 2$.

Examples:

The simplest case, $r = 1$**:**

$$U^2(z) = \frac{1 + z}{2}, \quad H(z) = \frac{z^{-1} + 2 + z}{4} \quad G^1(z) = \frac{-z^{-1} + 2 - z}{4}.$$

Increase of the number of vanishing moments in the analysis framelet $\tilde{\psi}_2^1$ to two on the expense of the synthesis "framelet" ψ_2^1 [1] which does not have vanishing moments.

$$G^2(z) = 2z^{-1} \quad \tilde{G}^2(z) = \frac{z^{-1}(-z^2 + 2 - z^{-2})}{4}. \tag{4.8}$$

The framelets are symmetric. The synthesis framelet is $\tilde{\psi}^2(t) = 4\varphi(2t)$.

Cubic discrete spline, $r = 2$:

$$U^4(z) = 4\frac{1+z}{z+6+z^{-1}}, \quad H(z) = \frac{(z+2+z^{-1})^2}{2\,(z^{-2}+6+z^2)}, \quad G^1(z) = \frac{(z-2+z^{-1})^2}{2\,(z^{-2}+6+z^2)}.$$

1. Increase of the number of vanishing moments in the analysis framelet $\tilde{\psi}_2^1$ to four on the expense of the synthesis "framelet" ψ_2^1 which does not have vanishing moments

$$G^2(z) = \frac{\sqrt{2}z^{-1}}{2\,(z^{-2}+6+z^2)} \quad \tilde{G}^2(z) = \frac{\sqrt{2}z^{-1}(z-z^{-1})^4}{2\,(z^{-2}+6+z^2)}. \tag{4.9}$$

2. The synthesis filter $G^2(z)$ is FIR. Both the synthesis and analysis framelets are symmetric and have two vanishing moments.

$$G^2(z) = \frac{\sqrt{2}z^{-1}(z-z^{-1})^2}{2} \quad \tilde{G}^2(z) = \frac{\sqrt{2}z^{-1}(z-z^{-1})^2}{2\,(z^{-2}+6+z^2)^2}. \tag{4.10}$$

3. Antisymmetric factorization. Increase the number of vanishing moments in the analysis framelet $\tilde{\psi}^2$ to three at the cost of reducing the number of vanishing moments in the synthesis framelet ψ^2 to one.

$$G^2(z) = -\frac{\sqrt{2}z^{-1}z^{-4}(1-z^2)}{2\,(z^{-2}+6+z^2)} \quad \tilde{G}^2(z) = \frac{\sqrt{2}z^{-1}(-z^2)^{-3}(1-z^2)^3}{2\,(z^{-2}+6+z^2)}. \tag{4.11}$$

Discrete spline of sixth order, $r = 3$:

$$U^6(z) = \frac{(z+14+z^{-1})(1+z)}{6z^{-1}+20+6z}, \quad H(z) = \frac{(z^{-1}+2+z)^3}{2\,(6z^2+20+6z^{-2})}, \quad G^1(z) = \frac{(-z^{-1}+2-z)^3}{2\,(6z^2+20+6z^{-2})},$$

1. Symmetric factorization. Increase the number of vanishing moments in the analysis framelet $\tilde{\psi}^2$ to four whereas the synthesis framelet ψ^2 has two vanishing moments.

$$G^2(z) = \frac{\sqrt{2}z^{-1}(2-z^2-z^2)}{2\,(6z^2+20+6z^{-2})} \quad \tilde{G}^2(z) = \frac{\sqrt{2}z^{-1}(2-z^2-z^2)^2}{2\,(6z^2+20+6z^{-2})}. \tag{4.12}$$

2. Antisymmetric factorization. Increase the number of vanishing moments in the analysis framelet $\tilde{\psi}^2$ to five whereas the synthesis framelet ψ^2 has only one vanishing moment.

$$G^2(z) = -\frac{\sqrt{2}z^{-1}(-z^2)^{-3}(1-z^2)}{2\,(6z^2+20+6z^{-2})} \quad \tilde{G}^2(z) = -\frac{\sqrt{2}z^{-1}(-z^2)^{-5}(1-z^2)^5}{2\,(6z^2+20+6z^{-2})}. \tag{4.13}$$

3. The synthesis filter $G^2(z)$ is FIR. Both the synthesis and analysis framelets are antisymmetric and have three vanishing moments.

$$G^2(z) = \frac{\sqrt{2}}{2}z^{-1}\left(1-z^2\right)^3 \quad \tilde{G}^2(z) = \frac{\sqrt{2}z^{-1}(1-z^2)^3}{2\,(6z^2+20+6z^{-2})^2}. \tag{4.14}$$

[1] Here the word framelet is in quotation because the function ψ_2^1 has no vanishing moments.

4.3 Bi-frames

Let $U(z)=U^{2r}(z)$, $\tilde{U}(z)=U^{2p}(z),\ p,r\in\mathbb{N}$. Then, we have

$$\begin{aligned}
H(z) &\triangleq \tfrac{\rho^r(z)}{\rho^r(z)+\rho^r(-z)}, & \tilde{H}(z) &\triangleq \frac{\rho^p(z)}{\rho^p(z)+\rho^p(-z)},\\
G^1(z) &\triangleq \tfrac{\rho^r(-z)}{\rho^r(z)+\rho^r(-z)}, & \tilde{G}^1(z) &\triangleq \frac{\rho^p(-z)}{\rho^p(z)+\rho^p(-z)},\\
G^2(z) &\triangleq z^{-1}v(z^2), & \tilde{G}^2(z) &\triangleq z^{-1}\tilde{v}(z^2).
\end{aligned}$$

where

$$\begin{aligned}
&2v(z^2)\tilde{v}(z^{-2}) = 1-U^{2r}(z^2)U^{2p}(z^{-2}) = 1-z^{-1}U^{2r}(z^2)z^{-1}U^{2p}(z^2)\\
&= 1-\frac{\left(\rho^r(z)-\rho^r(-z)\right)\left(\rho^p(z)-\rho^p(-z)\right)}{\left(\rho^r(z)+\rho^r(-z)\right)\left(\rho^p(z)+\rho^p(-z)\right)} = 2\frac{\rho^r(z)\rho^p(-z)+\rho^p(z)\rho^r(-z)}{\left(\rho^r(z)+\rho^r(-z)\right)\left(\rho^p(z)+\rho^p(-z)\right)}.
\end{aligned}$$

Assume $p<r$. Then, we have

$$v(z^2)\tilde{v}(z^{-2}) = \frac{(-1)^p\left(z-z^{-1}\right)^{2p}\left(\rho^{r-p}(z)+\rho^{r-p}(-z)\right)}{\left(\rho^r(z)+\rho^r(-z)\right)\left(\rho^p(z)+\rho^p(-z)\right)}.$$

One way to (anti)symmetrically factorize this function is

$$v(z^2) = \frac{\left(1-z^2\right)^p\left(\rho^{r-p}(z)+\rho^{r-p}(-z)\right)}{\rho^p(z)+\rho^p(-z)},\quad \tilde{v}(z^2) = \frac{\left(1-z^2\right)^p}{\rho^r(z)+\rho^r(-z)}.$$

If $p=2n$ then we have a symmetric factorization is possible

$$v(z^2) = \frac{\left(z-z^{-1}\right)^{2n}\left(\rho^{r-p}(z)+\rho^{r-p}(-z)\right)}{\rho^p(z)+\rho^p(-z)},\quad \tilde{v}(z^2) = \frac{\left(z-z^{-1}\right)^{2n}}{\rho^r(z)+\rho^r(-z)}.$$

Examples:

$p=2$, $r=1$:

$$\begin{aligned}
H(z) &= \tfrac{z^{-1}+2+z}{4}, & \tilde{H}(z) &= \frac{(z+2+z^{-1})^2}{2\left(z^{-2}+6+z^2\right)},\\
G^1(z) &= \tfrac{-z^{-1}+2-z}{4}, & \tilde{G}^1(z) &= \frac{(z-2+z^{-1})^2}{2\left(z^{-2}+6+z^2\right)},\\
G^2(z) &\triangleq z^{-1}v(z^2), & \tilde{G}^2(z) &\triangleq z^{-1}\tilde{v}(z^2).
\end{aligned} \tag{4.15}$$

where

$$v(z^2)\tilde{v}(z^{-2}) = \frac{-\left(z-z^{-1}\right)^2}{2\left(z^{-2}+6+z^2\right)}.$$

The following options for factorization are available:

1. Antisymmetric factorization:

$$v(z^2) = \frac{1-z^2}{2},\quad \tilde{v}(z^2) = \frac{1-z^2}{z^{-2}+6+z^2}. \tag{4.16}$$

All the synthesis filters are FIR. Consequently, the synthesis scaling function $\varphi(t)$ and the framelets $\psi^1(t)$ and $\psi^2(t)$ are compactly supported. The analysis framelet $\tilde{\psi}^1(t)$ has four vanishing moments and the synthesis framelet $\psi^1(t)$ has two vanishing moments. The synthesis and the analysis framelets $\psi^2(t)$ and $\tilde{\psi}^2(t)$ are antisymmetric and have one vanishing moment.

2. Symmetric factorization:

$$v(z^2) = \frac{1}{z^{-2}+6+z^2}, \quad \tilde{v}(z^2) = \frac{-\left(z-z^{-1}\right)^2}{2}. \tag{4.17}$$

The analysis filter $\tilde{G}^2(z)$ is FIR, the analysis framelet $\tilde{\psi}^2$ is symmetric and have two vanishing moments. The synthesis "framelet" ψ^2 is symmetric and does not have vanishing moments.

3. Trivial factorization:

$$v(z^2) = 1, \quad \tilde{v}(z^2) = \frac{-\left(z-z^{-1}\right)^2}{2(z^{-2}+6+z^2)}. \tag{4.18}$$

The analysis framelet $\tilde{\psi}^2$ is symmetric and has two vanishing moments. The synthesis framelet $\psi^2(t) = 2\varphi(2t)$ is compactly supported.

$p = 2$, $r = 3$:

$$\begin{aligned} H(z) &= \frac{(z^{-1}+2+z)^3}{2\left(6z^2+20+6z^{-2}\right)}, \quad \tilde{H}(z) = \frac{(z+2+z^{-1})^2}{2\left(z^{-2}+6+z^2\right)}, \\ G^1(z) &= \frac{(z-2+z^{-1})^2}{2\left(z^{-2}+6+z^2\right)}, \quad \tilde{G}^1(z) = \frac{(-z^{-1}+2-z)^3}{2\left(6z^2+20+6z^{-2}\right)}, \\ G^2(z) &\triangleq z^{-1}v(z^2), \qquad \tilde{G}^2 \triangleq z^{-1}\tilde{v}(z^2) \end{aligned} \tag{4.19}$$

where

$$v(z^2)\tilde{v}(z^{-2}) = \frac{\left(z-z^{-1}\right)^4}{(z^{-2}+6+z^2)\left(6z^2+20+6z^{-2}\right)}.$$

The analysis framelet $\tilde{\psi}^1(t)$ has four vanishing moments and the synthesis framelet $\psi^1(t)$ has six vanishing moments.

We have the following factorization options:

1. Symmetric factorization

$$v(z^2) = \frac{\left(z-z^{-1}\right)^2}{z^{-2}+6+z^2} \quad \tilde{v}(z^2) = \frac{\left(z-z^{-1}\right)^2}{6z^2+20+6z^{-2}}. \tag{4.20}$$

Both the synthesis and the analysis framelets are symmetric and have two vanishing moments.

2. Another symmetric factorization with maximal number of vanishing moments in the analysis framelet is

$$v(z^2) = \frac{1}{z^{-2}+6+z^2} \quad \tilde{v}(z^2) = \frac{\left(z-z^{-1}\right)^4}{6z^2+20+6z^{-2}}. \tag{4.21}$$

3. Antisymmetric factorization

$$v(z^2) = \frac{1-z^2}{z^{-2}+6+z^2} \quad \tilde{v}(z^2) = \frac{\left(-z^2\right)^3\left(1-z^2\right)^3}{6z^2+20+6z^{-2}}. \tag{4.22}$$

Both the synthesis and the analysis framelets are antisymmetric. The synthesis framelet has one vanishing moment, whereas the analysis framelet one has three vanishing moments.

4.4 Dual frames

Let $\tilde{U}(z) = U^{2r}(z), \quad \tilde{v}(z) = (1-|U^{2r}(z)|^2)/2$. Then, due to (2.10),

$$z^{-1}U(z^2) = \frac{2z^{-1}U^{2r}(z^2)}{1+|U^{2r}(z^2)|^4}, \quad v(z^2) = \frac{1-|U^{2r}(z^2)|^2}{1+|U^{2r}(z^2)|^4}.$$

Then,

$$1+|U^{2r}(z^2)|^4 = 1+\left(\frac{\rho^r(z)-\rho^r(-z)}{\rho^r(z)+\rho^r(-z)}\right)^4 = 2\frac{\rho^{4r}(z)+6\rho^{2r}(-z)\rho^{2r}(z)+\rho^{4r}(-z)}{\left(\rho^r(z)+\rho^r(-z)\right)^4},$$

$$1-|U^{2r}(z^2)|^2 = 1-\left(\frac{\rho^r(z)-\rho^r(-z)}{\rho^r(z)+\rho^r(-z)}\right)^2 = 4\frac{\rho^r(z)\rho^r(-z)}{\left(\rho^r(z)+\rho^r(-z)\right)^2}.$$

Thus

$$z^{-1}U(z^2) = \frac{\left(\rho^{2r}(z)-\rho^{2r}(-z)\right)\left(\rho^r(z)+\rho^r(-z)\right)^2}{\rho^{4r}(z)+6\rho^{2r}(-z)\rho^{2r}(z)+\rho^{4r}(-z)}, \tag{4.23}$$

$$v(z^2) = 2\frac{\rho^r(z)\rho^r(-z)\left(\rho^r(z)+\rho^r(-z)\right)^2}{\rho^{4r}(z)+6\rho^{2r}(-z)\rho^{2r}(z)+\rho^{4r}(-z)}. \tag{4.24}$$

Theorem 2.1 implies that the filter bank $H(z) = 1/2 + z^{-1}U(z^2)/2$, $G^1(z) = 1/2 - z^{-1}U(z^2)/2$, $G^2(z) = z^{-1}v(z^2)/2$ is dual to the filter bank $\tilde{H}(z) = 1/2 + z^{-1}\tilde{U}(z^2)/2$, $\tilde{G}^1(z) = 1/2 - z^{-1}\tilde{U}(z^2)/2$, $\tilde{G}^2(z) = z^{-1}\tilde{v}(z^2)/2$. All the framelets have $2r$ vanishing moments.

Example:

$r=1$**:**

$$\tilde{U}^2(z) = \frac{1+z}{2}, \quad \tilde{H}(z) = \frac{z^{-1}+2+z}{4}, \quad \tilde{G}^1(z) = \frac{-z^{-1}+2-z}{4}, \tag{4.25}$$
$$\tilde{v}(z) = (1-|\tilde{U}(z)|^2)/2 = \frac{-z^{-1}+2-z}{8}.$$

By using Eqs. (4.23) and (4.24), we get

$$v(z) = \frac{4\left(-z+2-z^{-1}\right)}{z^2+4z+22+4z^{-1}+z^{-2}}, \quad U(z) = 16\frac{z+1}{z^2+4z+22+4z^{-1}+z^{-2}} \tag{4.26}$$
$$H(z) = \frac{1+z^{-1}U(z^2)}{2} = \left(z+2+z^{-1}\right)\frac{z^3-2z^2+7z+4+7z^{-1}-2z^{-2}+z^{-3}}{2\left(z^4+4z^2+22+4z^{-2}+z^{-4}\right)}$$
$$G^1(z) = \frac{1-z^{-1}U(z^2)}{2} = \left(z-2+z^{-1}\right)\frac{z^3+2z^2+7z-4+7z^{-1}+2z^{-2}+z^{-3}}{2\left(z^4+4z^2+22+4z^{-2}+z^{-4}\right)}$$
$$G^2(z) = z^{-1}v(z^2).$$

The analysis filters are FIR. The analysis scaling function and framelets are compactly supported. All framelets have two vanishing moments.

4.5 Graphic illustrations of the frames

In this section we display the frequency response of the filters and the framelets generated by these filters. These are based on the equations in Sections 4.1–4.4.

Figure 1: Tight frames, which originate from the discrete splines of second and eight order, are displayed. The plots in the first column from bottom up display the scaling function $\varphi(t)$ and the framelets $\psi^1(t)$ and $\psi^2(t)$, which are generated by the $H(z)$, $G^1(z)$ and $G^2(z)$ filters, respectively. The filters are defined in Eq. (4.2). They are FIR and the waveforms are compactly supported. The framelet $\psi^1(t)$ is symmetric and has two vanishing moments. $\psi^2(t)$ is antisymmetric and has one vanishing moment. The frequency responses of the filters are displayed in the second column from the left. The waveforms and the filters (Eq. (4.5)), which result from the eight order discrete splines, are displayed in a similar way in the third and fourth columns from the left. In this case, the framelet $\psi^1(t)$ has eight vanishing moments and $\psi^2(t)$ has four. Both framelets are symmetric. We observe that the frequency response of the filters $H(z)$ and $G^1(z)$ have near-rectangle shape. They are mirrored versions of each other.

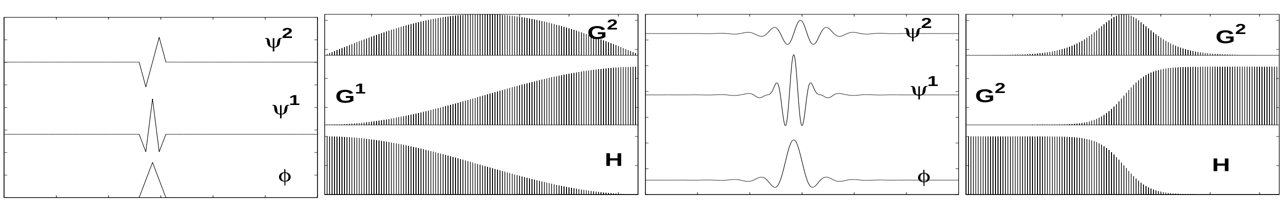

Figure 1: Bottom up in the left column: the scaling function $\varphi(t)$, and the framelets $\psi^1(t)$ and $\psi^2(t)$ for the tight frame originated from the second order discrete spline. Second column from left: corresponding filters. Third and fourth columns from left: the same as in the two columns on the left while the frame is related to the eight order discrete spline.

Figure 2: We present tight and semi-tight frames that originate from the fourth order (cubic) discrete spline. Picture is divided into three pairs - left, middle and right. Each pair contains two columns. The left columns displays the waveforms and the right column displays the frequency response of the generating filters. The left pair displays the waveforms and filters related to the tight frame (Eq. (4.3)). The framelet $\psi^1(t)$ has four vanishing moments and $\psi^2(t)$ has two. Both framelets are symmetric. The middle and the rights pairs illustrate three ways to factorize $V(z)$, which are given in Eqs. (4.9), (4.10) and (4.11). The bottom row in the four rightmost columns is related to Eq. (4.9). The synthesis framelet $\psi^2(t)$, synthesis filter $G^2(z)$, analysis framelet $\tilde{\psi}^2(t)$ and the analysis filter $\tilde{G}^2(z)$ are displayed in this row. All four vanishing moments are assigned to the analysis framelet. Both $\psi^2(t)$ and $\tilde{\psi}^2(t)$ are symmetric. The filter $G^2(z)$ is all-pass. The middle row in these columns depicts similar objects that are related to Eq. (4.10). The analysis and synthesis framelets have two vanishing moments and they are symmetric. The filter $G^2(z)$ is FIR. The upper row is related to Eq. (4.11). The framelets here are antisymmetric, $\tilde{\psi}^2(t)$ has three vanishing moments and $\psi^2(t)$ has only one. The analysis framelets and filters can be interchanged with the synthesis filters.

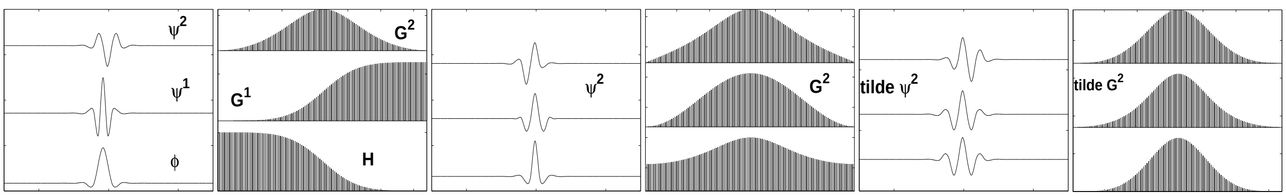

Figure 2: Tight and semi-tight frames originated from the fourth order discrete spline. Left pair of pictures: dislay the waveforms and filters of the tight frame. Middle pair: synthesis framelets $\psi^2(t)$ and filters $G^2(z)$ for different factorizations of $V(z)$. Right pair: the corresponding analysis framelets $\tilde{\psi}^2(t)$ and filters $\tilde{G}^2(z)$.

Figure 3: Presentation of the tight and semi-tight frames originated from the sixth order discrete spline. The left pair of columns display the waveforms and filters related to the tight frame (Eq. (4.4)). The framelet $\psi^1(t)$ is symmetric and has six vanishing moments, the framelet $\psi^2(t)$ is antisymmetric and has three vanishing moments. The middle and the right pairs illustrate three ways to factorize the $V(z)$, which are given in Eqs. (4.12), (4.13) and (4.14). The bottom row in middle and right pair is related to Eq. (4.12). The synthesis framelet $\psi^2(t)$, synthesis filter $G^2(z)$, analysis framelet $\tilde{\psi}^2(t)$ and analysis filter $\tilde{G}^2(z)$ are displayed. $\psi^2(t)$ and $\tilde{\psi}^2(t)$ are symmetric. The synthesis framelet $\psi^2(t)$ has two vanishing moments and the analysis framelet $\tilde{\psi}^2(t)$ has four. The middle row depicts similar objects related to Eq. (4.13). The analysis and synthesis framelets are antisymmetric. The analysis framelet $\tilde{\psi}^2(t)$ has five vanishing moments

and only one is left for the synthesis framelet. The upper row is related to Eq. (4.14). The framelets are antisymmetric and have three vanishing moments. The filter $G^2(z)$ is FIR.

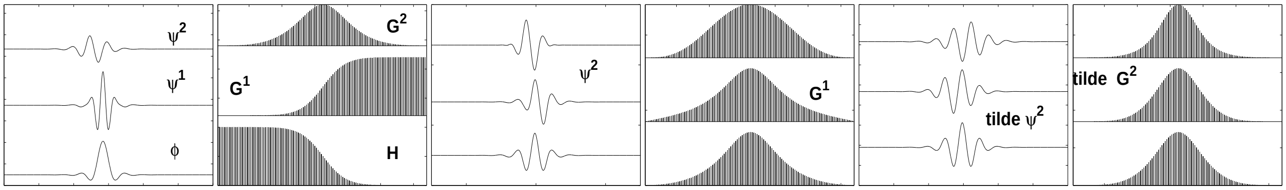

Figure 3: Tight and semi-tight frames that are originated from the sixth order discrete spline. Left pair of columns: waveforms and filters for the tight frame. Middle pair: synthesis framelets $\psi^2(t)$ and filters $G^2(z)$ for the different factorizations of $V(z)$. Right pair: analysis framelets $\tilde{\psi}^2(t)$ and filters $\tilde{G}^2(z)$.

Figure 4: This figure corresponds to bi-frames that are generated by a pair of low-pass filters: synthesis $H(z)$, which stems from the second order discrete spline and analysis $\tilde{H}(z)$, which stems from the fourth order discrete spline. These filters and the related high-pass filters $G^1(z)$ and $\tilde{G}^1(z)$ are defined in Eq. (4.15). The frequency response of the filters $H(z)$ and $G^1(z)$ and the generated waveforms $\varphi(t)$ and $\psi^1(t)$ are displayed in Fig. 1. The filters $\tilde{H}(z)$ and $\tilde{G}^1(z)$ and the waveforms $\tilde{\varphi}(t)$ and $\tilde{\psi}^1(t)$ are displayed in Fig. 2. We present in Fig. 4 the filters $G^2(z)$ and $\tilde{G}^2(z)$ and the framelets $\psi^2(t)$ and $\tilde{\psi}^2(t)$, which result from different ways of factorization of the function $V(z)$ (Eqs. (4.16), (4.17) and (4.18)). The bottom row is related to Eq. (4.16). Both $\psi^2(t)$ and $\tilde{\psi}^2(t)$ are antisymmetric and have one vanishing moment. The synthesis filter $G^2(z)$ is FIR and $\psi^2(t)$ is compactly supported. The middle row illustrates the symmetric factorization of Eq. (4.17). The analysis framelet $\tilde{\psi}^2(t)$ has two vanishing moments at the expense of $\psi^2(t)$, which has none. In the trivial factorization of Eq. (4.18), which is illustrated in the upper row, $\tilde{\psi}^2$ is symmetric and has two vanishing moments. The synthesis framelet $\psi^2(t) = 2\varphi(2t)$ is compactly supported.

Figure 5: This figure corresponds to bi-frames that are generated by the pair of low-pass filters: synthesis $H(z)$, which stems from the sixth order discrete spline and analysis $\tilde{H}(z)$, which stems from the fourth order discrete spline. These filters and the related high-pass filters $G^1(z)$ and $\tilde{G}^1(z)$ are defined in Eq. (4.19). The frequency response of the filters $H(z)$ and $G^1(z)$ and the generated waveforms $\varphi(t)$ and $\psi^1(t)$ are displayed in Fig. 3. The filters $\tilde{H}(z)$ and $\tilde{G}^1(z)$ and the waveforms $\tilde{\varphi}(t)$ and $\tilde{\psi}^1(t)$ are displayed in Fig. 2. We present in Figure 5 the filters $G^2(z)$ and $\tilde{G}^2(z)$ and the framelets $\psi^2(t)$ and $\tilde{\psi}^2(t)$, which result from different factorizations of the function $V(z)$ (Eqs. (4.20), (4.21) and (4.22)). The bottom row is related to Eq. (4.20). Both

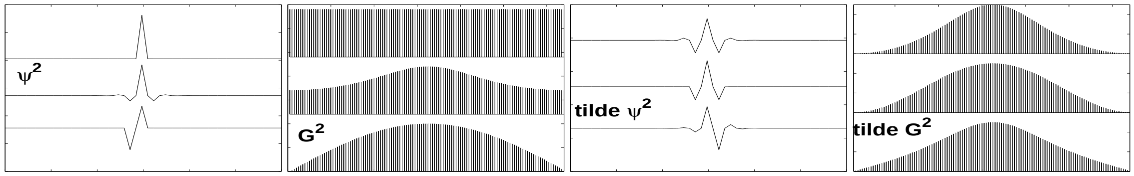

Figure 4: Filters and framelets for the bi-frames resulting from a pair of discrete splines of second and fourth order. The two left columns: synthesis framelets $\psi^2(t)$ and filters $G^2(z)$ for the various modes of factorization of $V(z)$. Two right columns: corresponding analysis framelets $\tilde{\psi}^2(t)$ and filters $\tilde{G}^2(z)$.

$\psi^2(t)$ and $\tilde{\psi}^2(t)$ are symmetric and have two vanishing moments. The middle row illustrates the symmetric factorization of Eq. (4.21). The analysis framelet $\tilde{\psi}^2(t)$ has four vanishing moments at the expense of $\psi^2(t)$, which has none. In the antisymmetric factorization of Eq. (4.22), which is illustrated in the upper row, $\tilde{\psi}^2$ has three vanishing moments. The synthesis framelet $\psi^2(t) = 2\varphi(2t)$ has one.

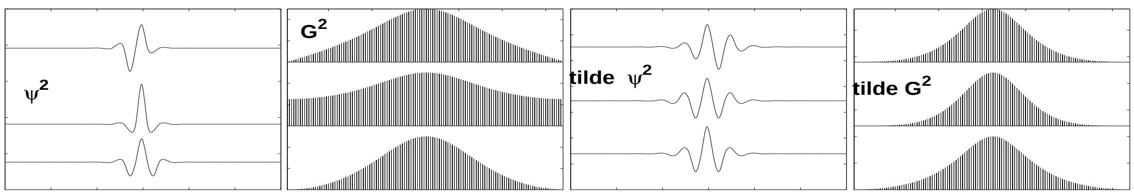

Figure 5: Filters and framelets for the bi-frames resulting from a pair of discrete splines of sixth and fourth order. The left two column: synthesis framelets $\psi^2(t)$ and filters $G^2(z)$ for the various modes of factorization of $V(z)$. The right two columns: corresponding analysis framelets $\tilde{\psi}^2(t)$ and filters $\tilde{G}^2(z)$.

Figure 6: We display the filters and the waveforms for a pair of dual frames, which is defined in Eqs. (4.25) and (4.26). All waveforms are symmetric and all framelets have two vanishing moments. The analysis filters are FIR and the waveforms are piece-wise linear and compactly supported. Unlike all the above examples, the frequency response of the synthesis filters $H(z)$ and $G^1(z)$ are not localized in the half-bands.

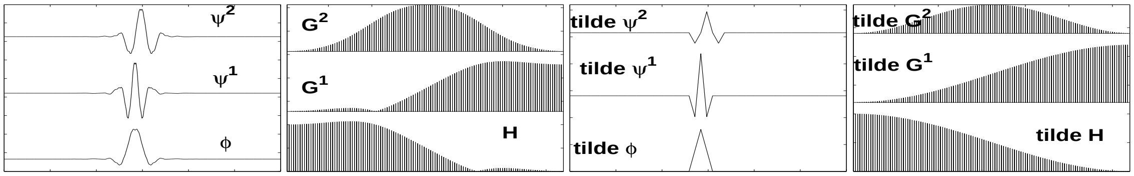

Figure 6: Filters and wavelets associated with the dual pair of frames. Two left columns: synthesis waveforms and filters. Two right columns: analysis waveforms and filters.

5 Discussion

5.1 Comparison with wavelet transform

We used similar filters to construct biorthogonal wavelet transforms. We compare wavelet and frame transforms that are based on the same Butterworth filter. As an example we use the tight frame originated from the fourth order discrete spline, which is defined in Eq. (4.3) and displayed in Figure 2. For the biorthogonal wavelet transform we use the synthesis low-pass filter and the analysis high-pass filter

$$H_w(z) = \frac{(z+2+z^{-1})^2}{2\,(z^{-2}+6+z^2)} \quad \text{and} \quad \tilde{G}_w(z) = z^{-1}\frac{(z-2+z^{-1})^2}{2\,(z^{-2}+6+z^2)}, \tag{5.27}$$

respectively. They coincide with the filters $H(z)$ and $G^1(z)$ of the frame transform (up to factor z^{-1}). The analysis low-pass filter and the synthesis high-pass filter have more complicated structure:

$$\tilde{H}_w(z) = H(z) + V(z^2) \quad \text{and} \quad G_w(z) = z^{-1}\left(G^1(z) + V(z^2)\right), \tag{5.28}$$

$$\text{where } V(z^2) = \frac{(z-z^{-1})^4}{2\,(z^{-2}+6+z^2)^2} = G^2(z)G^2(1/z).$$

The frequency response of the wavelet filters and their corresponding waveforms are displayed in Figure 7. The analysis waveforms belong to C^1 and the synthesis waveforms and the framelets belong

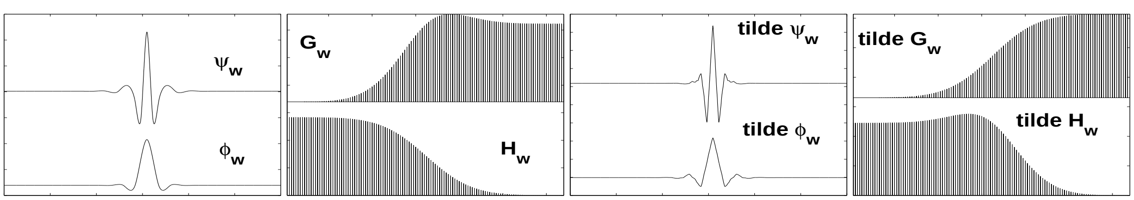

Figure 7: Filters and wavelets for the biorthogonal wavelet transform. Left pair of columns: synthesis waveforms and filters. Right pair of columns: analysis waveforms and filters.

to C^2. We conclude from Eqs. (5.27) and (5.28) that the move from biorthogonal wavelet transform to frame transform simplifies the structure of filters and enhances the smoothness of waveforms. The additional filter $G^2(z)$ removes the difference between the analysis and synthesis wavelet filters. By comparing between Fig. 7 and Fig. 2, where the tight frame is displayed, we see that the "bumps", which are present at the displays of the frequency response of the wavelet filters $\tilde{H}_w(z)$ and $G_w(z)$ are removed when the filter $G^2(z)$ is introduced. In the tight frame case, the frequency response of the filters $H(z)$ and $G^1(z)$ are mirrored versions of each other. The scaling function $\varphi(t)$ and the framelet $\psi^1(t)$ are smoother than their wavelet counterparts $\tilde{\varphi}_w(t)$ and $\psi_w(t)$. Loosely speaking, by moving from wavelet to frame transforms we split the complex filters $\tilde{H}_w(z)$ and $G_w(z)$ and wavelets $\tilde{\varphi}_w(t)$ and $\psi_w(t)$ into simpler components. By varying the order of the generating discrete splines and modes of factorization of the function $V(z)$, we can efficiently control the properties of the transforms such as smoothness and time-domain localization of waveforms, shape of spectra, to name a few, which are important for signal and image processing applications.

5.2 Practical application of the designed frames: example

We anticipate that this new family of transforms will affect and enhance the performance of classical signal processing applications. Our first results in this direction confirm it. We propose to perform error corrections of images that are transmitted through noisy channels. Initially, the image is transformed into four levels of the framelet transform using the filter banks in the paper. As a result the transformed image is inflated by factor of 2.66 of the original image. Then, a number of randomly chosen coefficients are set to zero. The errors are corrected by a method that was introduced by Petukhov [19]. The example in Fig. 9 demonstrates the performance of the method. We decompose the image (see Fig. 8) using the symmetric bi-frames given in Eqs. (4.15) and (4.17). 60% of randomly chosen coefficients are set to zero. The result is shown in the left image of Fig. 9. The reconstructed image is shown in Fig. 9 (right). It utilizes the introduced redundancy of the symmetric bi-frames.

Figure 8: Original Image.

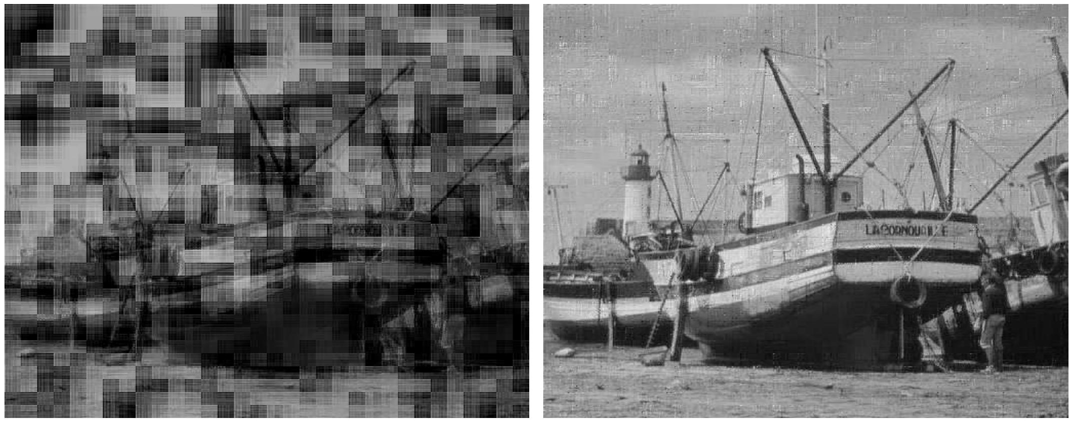

Figure 9: Left: corrupted image with 60% of random erasures of coefficients. Right: recovered image.

A detailed report how the redundancy in the designed transforms helps in error correction will be given in [3].

5.3 Conclusion

We presented a new family of frames, which are generated by perfect reconstruction filter banks consisting of linear phase filters. The filter banks are based on the discrete interpolatory splines and are related to the Butterworth filters. A similar scheme for filter design is possible on the base of the continuous interpolatory and quasi-interpolatory splines. Each designed filter bank contains one interpolatory symmetric low-pass filter and two high-pass filters, one of which is also interpolatory and symmetric. The second high-pass filter may be symmetric or antisymmetric. These filter banks generate analysis and synthesis scaling functions and pairs of framelets. The scaling function and one of the framelets in either of the analysis and synthesis sets are symmetric, whereas the second framelet is symmetric or antisymmetric. One step in the framelet transform of a signal of length N produces $1.5N$ coefficients. Thus, the full transform of this signal consisting of $J = \log 2(N)$ steps produces $2N$ coefficients.

We introduced also the concept of *semi-tight* frame. While in the case of a tight frame the canonical synthesis filter bank coincides (up to a constant factor) with its analysis counterpart, in the semi-tight frame we can modify the second filters making them different for the synthesis and the analysis cases. Therefore, we can, for example, move the vanishing moments from the synthesis to the analysis framelets or to add smoothness to the synthesis framelet. This concept provides additional flexibility to the design of frames generated by filter banks. We constructed dual pairs of frames, where all the waveforms are symmetric and all the framelets have the same number of vanishing moments.

Although most of the designed filters are IIR, they allow fast implementation via recursive procedures. The waveforms are well localized in time domain despite their infinite support. The frequency response of the designed filters are flat due to their relation to Butterworth filters.

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