Randomized LU Decomposition

Gil Shabat\textsuperscript{1}\footnote{Corresponding author: gil@eng.tau.ac.il} Yaniv Shmueli\textsuperscript{2} Yariv Aizenbud\textsuperscript{3} Amir Averbuch\textsuperscript{2}

\textsuperscript{1}School of Electrical Engineering, Tel Aviv University, Israel  
\textsuperscript{2}School of Computer Science, Tel Aviv University, Israel  
\textsuperscript{3}School of Applied Mathematics, Tel Aviv University, Israel

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Abstract

A fast randomized algorithm, which computes a low rank LU decomposition, is presented. The algorithm uses random projections type techniques to efficiently compute a low rank approximation of large matrices. The randomized LU algorithm can be parallelized and further accelerated by structured random matrices in its projection step. Several error bounds for the algorithm’s approximations are proved. To prove these bounds, recent results from random matrix theory related to subgaussian matrices are used. The algorithm is fully parallelized and thus can utilize efficiently GPUs. Numerical examples, which illustrate the performance of the algorithm and compare it to other decomposition methods, are presented.

Keywords. LU decomposition, matrix factorizations, random matrices, randomized algorithms.

1 Introduction

Matrix factorizations and low rank approximations play a major role in many of today’s applications [39]. In mathematics, matrix decompositions are used for low rank approximations that often reveal interesting properties of a matrix. Matrix decompositions are used for example in solving linear equations and in finding least squares solutions. In engineering, matrix decompositions are used in computer vision [18], machine learning [31], collaborative filtering and Big Data analytics [25]. As the size of the data grows exponentially, analysis of large datasets has gained an increasing interest. Such an analysis can involve a factorization step of the input data given as a large sample-by-feature matrix or by a sample affinity matrix. Two main reasons for the difficulties in analyzing huge
data structures are high memory consumption and the computational complexity of the factorization step. Recently, there is an on-going interest in applying mathematical tools that are based on randomized algorithms to overcome these difficulties.

Some of the randomized algorithms use random projections that project the matrix to a set of random vectors. Formally, given a matrix $A$ of size $m \times n$ (suppose $m \geq n$) and a random matrix $G$ of size $n \times k$, then the product $AG$ is computed to obtain a smaller matrix that potentially captures most of the data activities in $A$. In most of these applications, $k$ is set to be much smaller than $n$ to obtain a compact approximation for $A$.

Fast randomized matrix decomposition algorithms are used for tracking objects in videos [37], multiscale extensions for data [4] and detecting anomalies in network traffic for finding cyber attacks [12], to name some. There are randomized versions for many different matrix factorization algorithms [23], compressed sensing methods [15] and least squares problems [3].

In this paper, we develop a randomized version of the LU decomposition. Given an $m \times n$ matrix $A$, we seek a lower triangular $m \times k$ matrix $L$ and an upper triangular $k \times n$ matrix $U$ such that

$$
\|LU - PAQ\|_2 = C(m, n, k)\sigma_{k+1}(A),
$$

where $P$ and $Q$ are orthogonal permutation matrices, $\sigma_{k+1}(A)$ is the $k+1$ largest singular value of $A$ and $C(m, n, k)$ is a constant depending on $m$, $n$ and $k$.

The interest in a randomized LU decomposition can be motivated (computationally wise) by two important properties of the classical LU decomposition: First, it can be applied efficiently to sparse matrices with computation time depending on the number of non-zero elements. LU decomposition with full pivoting on sparse matrices can generate large regions of zeros in the factorized matrices [13, 14, 36]. Processing of sparse matrices will be treated in a separate paper. Second, LU decomposition can be fully parallelized that makes it applicable for running on Graphics Processing Units (GPU). GPUs are mostly used for computer games, graphics and visualization such as movies and 3D display. Their powerful computation capabilities can be used for fast matrix computations [24].

The contributions of the paper are the following: We develop a randomized version for LU decomposition. Such an algorithm does not appear in the literature. We provide several error bounds for the error $\|LU - PAQ\|_2$. In addition, we implemented the randomized LU to fully run on a standard GPU card. We present numerical results that compare our algorithm with other decomposition methods and show it superiority.

The paper is organized as follows: In Section 2, we overview related work on matrix decomposition and approximation using randomized methods. Section 3 reviews some mathematical results that are needed for the development of the randomized LU. Section 4 presents several randomized LU algorithms and proves several error bounds on their approximation. Section 5 presents numerical results on the approximation error, the computational complexity of the
algorithm and compares it with other methods. The performance comparison was done on random matrices, images and large sparse matrices.

2 Related Work

Efficient matrix decomposition serves as a basis for many studies and algorithms on data analysis applications. There is a variety of methods and algorithms that factorize a matrix into several matrices. Typically, the factorized terms have properties such as being triangular, orthogonal, diagonal, sparse or low rank. It is possible to have a certain control on the desired approximation error for a factorized matrix.

Rank revealing factorization uses permutation matrices on the columns and rows of $A$ so that the factorized matrices structure have a strong rank portion and a rank deficient portion. The most known example for approximating an $m \times n$ matrix $A$ by a low rank $k$ matrix is the truncated SVD. Other rank revealing factorizations can be used to achieve low rank approximations. For example, both QR and LU factorizations have rank revealing versions such as RRQR decomposition [7], strong RRQR [22] decomposition, RRLU decomposition [33] and strong RRLU decomposition [32].

Other matrix factorization methods such as Interpolative Decomposition (ID) [9] and CUR decomposition [17], use columns and rows of the original matrix $A$ in the factorization process. Such a property exposes the most important terms that construct $A$. An ID factorization of order $k$ of an $m \times n$ matrix $A$ consists of an $m \times k$ matrix $B$ whose columns consist of a subset of the columns of $A$, as well as a $K \times n$ matrix $P$, such that a subset of the columns of $P$ becomes a $k \times k$ identity matrix and $A \approx BP$ such that $\|A - BP\| \lesssim O(n, \sigma_{k+1}(A))$. Usually, $k = \#\{j : \sigma_j(A) \geq \delta \sigma_1(A)\}$ is the numerical rank of $A$ up to a certain accuracy $\delta > 0$. This selection of $k$ guarantees that the columns of $B$ constitute a well-conditioned basis to the range of $A$ [9].

Randomized version for many important algorithms have been developed in order to reduce the computational complexity by approximating the solution to a desired rank. These include SVD, QR and ID factorizations [30], CUR decomposition as a randomized version [17] of the pseudo-skeleton decomposition, methods for solving least squares problems [3, 11, 34] and low rank approximations [1, 11].

In general, randomization methods for matrix factorization have two steps. First, a low-dimensional space, which captures most of the “energy” of $A$, is found using randomization. Then, $A$ is projected onto the retrieved subspace and the projected matrix is factorized [23].

Several different selections exist when random projection matrix is used in Step 1. For example, it can be a matrix of random signs $(\pm 1)$ [10, 29], a matrix of i.i.d Gaussian random variables with zero mean and unit variance [30], a matrix whose columns are selected randomly from the identity matrix with either uniform or non-uniform probability [16, 19], a random sparse matrix designed to enable fast multiplication with a sparse input matrix $A$ [1, 11], random
structured matrices, which use orthogonal transforms, such as discrete Fourier transform, Walsh-Hadamard transform and more ([3, 6, 34]). In our algorithm, we use Gaussian matrices in Step 1 as well as structured Fourier matrices to achieve accelerate the computation.

3 Preliminaries

In this section, we review the rank revealing LU (RRLU) decomposition and several singular values bounds for random matrices that will be used to prove the error bounds for the randomized LU algorithm. Throughout the paper, we use the following notation: for any matrix $A$, $\sigma_j(A)$ is the $j$th largest singular value and $\|A\|$ is the spectral norm (the largest singular value or $l_2$ operator norm). If $x$ is a vector then $\|x\|$ is the standard $l_2$ (Euclidean) norm. $A^\dagger$ denotes the pseudo-inverse of $A$.

3.1 Rank Revealing LU (RRLU)

The following theorem is adapted from [33] (Theorem 1.2):

**Theorem 3.1 ([33]).** Let $A$ be an $m \times n$ matrix ($m \geq n$). Given an integer $1 \leq k < n$, then the following factorization

$$PAQ = \begin{pmatrix} L_{11} & 0 \\ L_{21} & I_{n-k} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix},$$

holds where $L_{11}$ is a unit lower triangular, $U_{11}$ is an upper triangular, $P$ and $Q$ are orthogonal permutation matrices. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$ be the singular values of $A$, then:

$$\sigma_k \geq \sigma_{\min}(L_{11}U_{11}) \geq \frac{\sigma_k}{k(n-k)+1},$$

and

$$\sigma_{k+1} \leq \|U_{22}\| \leq (k(n-k)+1)\sigma_{k+1}.$$ 

This is called RRLU decomposition. Based on Theorem 3.1, we have the following definition:

**Definition 3.1 (RRLU Rank $k$ Approximation denoted RRLU$_k$).** Given a RRLU decomposition (Theorem 3.1) of a matrix $A$ with an integer $k$ (as in Eq. 3.1) such that $PAQ = LU$, then the RRLU rank $k$ approximation is defined by taking $k$ columns from $L$ and $k$ rows from $U$ such that

$$RRLU_k(PAQ) = \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \end{pmatrix}$$

where $L_{11}, L_{21}, U_{11}, U_{12}, P$ and $Q$ are defined in Theorem 3.1.
Lemma 3.2 (RRLU Approximation Error). The error of the RRLU\(_k\) approximation of \(A\) is

\[
\|PAQ - RRLU_k(PAQ)\| \leq (k(n - k) + 1)\sigma_{k+1}.
\] (3.5)

Proof. From Eqs. 3.1 and 3.4 we have

\[
\|PAQ - RRLU_k(PAQ)\| = \left\| \left( \begin{array}{cc} L_{11} & 0 \\ L_{21} & I_{n-k} \end{array} \right) \left( \begin{array}{cc} U_{11} & U_{12} \\ 0 & U_{22} \end{array} \right) - \left( \begin{array}{cc} L_{11} & 0 \\ L_{21} & I_{n-k} \end{array} \right) \left( \begin{array}{c} U_{11} \\ U_{12} \end{array} \right) \right\| = \|U_{22}\| \leq (k(n - k) + 1)\sigma_{k+1}.
\] (3.6)

The last inequality is derived from Eq. 3.3. \(\square\)

Lemma 3.3 appears in [5], page 75:

Lemma 3.3 ([5]). Let \(A\) and \(B\) be two matrices and let \(\sigma_j(\cdot)\) denotes the \(j\)th singular value of a matrix. Then, \(\sigma_j(AB) \leq \|A\|\sigma_j(B)\) and \(\sigma_j(AB) \leq \|B\|\sigma_j(A)\).

Lemma 3.4 was taken from [30] and it is an equivalent formulation for Eq. 8.8 in [20].

Lemma 3.4 ([30]). Suppose that \(G\) is a real \(n \times l\) matrix whose entries are i.i.d Gaussian random variables with zero mean and unit variance and let \(m\) be an integer such that \(m \geq l, m \geq n, \gamma > 1\) and

\[
1 - \frac{1}{4(\gamma^2 - 1)\sqrt{\pi m} \gamma^2} \left( \frac{2\gamma^2}{e\gamma^2 - 1} \right)^m
\] (3.7)

is non-negative. Then, \(\|G\| \leq \sqrt{2m\gamma}\) with probability not less than the value in Eq. 3.7.

3.2 Subgaussian Random Matrices

Definition 3.2. A real valued random variable \(X\) is called subgaussian if there exists \(b > 0\) such that for all \(t > 0\) we have \(\mathbb{E}e^{tX} \leq e^{bt^2/2}\) where \(\mathbb{E}\) is expectation.

We review several results adapted from [28] and [35] about random matrices whose entries are subgaussian. We focus on the case where \(A\) is a tall \(m \times n\) matrix \((m > (1 + \frac{1}{\ln n}) n)\). Similar results can be found in [27] for square and almost square matrices.

Definition 3.3. Assume that \(\mu \geq 1, a_1 > 0\) and \(a_2 > 0\). \(A(\mu, a_1, a_2, m, n)\) is the set of all \(m \times n\) \((m > n)\) random matrices \(A = (\xi_{ij})\) whose entries are centered i.i.d real valued random variables satisfying the following conditions:

1. Moments: \(\mathbb{E}|\xi_{ij}|^3 \leq \mu^3\);

2. Norm: \(\mathbb{P}(\|A\| > a_1\sqrt{m}) \leq e^{-a_2m}\) where \(\mathbb{P}\) is the probability function;
3. Variance: \( \mathbb{E} \xi_{ij}^2 \geq 1. \)

It is shown in [28] that if \( A \) is subgaussian then \( A \in \mathcal{A} \).

The following theorems are taken from Section 2 in [28]:

**Theorem 3.5** ([28]). Every matrix \( A \) of size \( m \times n \) \( (m \geq n) \) whose entries are subgaussian with \( \mu \geq 1 \) and \( a_2 \geq 0 \) satisfies:

\[
\mathbb{P} (\| A \| \geq a_1 \sqrt{m} ) \leq e^{-a_2 m}
\]

(3.8)

with \( a_1 = 6\mu \sqrt{a_2 + 4} \).

Theorem 3.5 provides an upper bound for the largest singular value that depends on the desired probability. Theorem 3.6 is used to bound from below the smallest singular value of random Gaussian matrices.

**Theorem 3.6** ([28]). Let \( \mu \geq 1 \), \( a_1, a_2 > 0 \). Let \( A \) be an \( m \times n \) matrix with \( m > (1 + \frac{1}{\ln n})n \). \( m \) can be written as \( m = (1 + \delta)n \). Suppose that the entries of \( A \) are centered independent random variables such that conditions 1, 2, 3 in Definition 3.3 hold. Then, there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
\mathbb{P}(\sigma_n(A) \leq c_1 \sqrt{m}) \leq e^{-m} + e^{-c'' m/(2\mu^6)} + e^{-a_2 m} \leq e^{-c_2 m}.
\]

(3.9)

From Theorem 3.6, the exact values of constants \( c_1 \), \( c_2 \) and \( c'' \) are

\[
c_1 = \frac{b}{e^2 c_3} \left( \frac{b}{3e^2 c_3 a_1} \right)^{1 \over 3}, \quad c'' = \frac{27}{2^{11}}
\]

(3.10)

where \( c_3 = 4\sqrt{2\pi} \left( \frac{2a_1^9}{a_1^3} + \sqrt{\pi} \right) \), \( b = \min \left( \frac{1}{4}, \frac{c'}{5a_1 \mu^4} \right) \) and \( c' = \left( \frac{27}{2^{11}} \right)^{1 \over 3} \). For the constant \( c_2 \), we need a small enough constant to satisfy the inequality in Eq. 3.9 and set it, for simplification, as

\[
c_2 = \min \left( 1, \frac{c'}{(2\mu^6)}, a_2 \right) - \frac{\ln 3}{m}.
\]

(3.11)

3.3 The SRFT matrix

The Subsampled Random Fourier Transform (SRFT), which was presented in [2,43], is a random matrix \( R \) with the structure \( R = DFS \) where \( D \) is an \( n \times n \) diagonal matrix whose entries are i.i.d. random variables drawn from a uniform distribution on the unit circle in \( \mathbb{C} \), \( F \) is an \( n \times n \) discrete Fourier transform such that \( F_{jk} = \frac{1}{\sqrt{n}} e^{-2\pi i (j-1)(k-1)/n} \) and \( S \) is an \( n \times l \) matrix whose entries are all zeros except for a single randomly placed \( 1 \) in each column.

**Lemma 3.7** ([43]). For any \( m \times n \) matrix \( A \), let \( R \) be the \( n \times l \) SRFT matrix. Then, \( Y = AR \) can be computed in \( O(mn \log l) \) floating point operations.
3.4 Interpolative decomposition (ID)

Let $A$ be an $m \times n$ of rank $r$. $A \approx A(:,J)X$ is the Interpolative Decomposition (ID) of rank $r$ of $A$ if:

1. $J$ is a subset of $r$ indices from $1, \ldots, n$.
2. The $r \times n$ matrix $A(:,J)$ is a subset of $J$ columns from $A$.
3. $X$ is an $r \times n$ matrix that all its entries are less than 2 in magnitude and contains $r$ columns of the identity matrix.

Similarly, it is possible to compute the ID with row selection such that $A \approx XA(J,:)$. The ID is based on [22] and it is introduced in [9,23,30] for deterministic and non-deterministic (random) algorithms. It is possible to compute ID with LU instead of using QR. This might increase the reconstruction error, while reducing the computational cost.

4 Randomized LU

In this section, we present the randomized LU algorithm (Algorithm 4.1) that computes the LU rank $k$ approximation of a full matrix. In addition, we present a version (Algorithm 4.4) that utilizes the SRFT matrix for achieving a faster processing. Error bounds are derived for each algorithm.

The algorithm starts by projecting the input matrix on a random matrix. The resulting matrix captures most of the information of the input matrix. Then, we compute a triangular basis for this matrix and project the input matrix on it. Last, we find a second triangular basis for the projected columns and multiply it with the original basis. The product leads to a lower triangular matrix $L$ and the upper triangular matrix $U$ obtained from the second LU factorization.
Algorithm 4.1: Randomized LU Decomposition

**Input:** A matrix of size $m \times n$ to decompose, $k$ desired rank, $l \geq k$ number of columns to use.

**Output:** Matrices $P, Q, L, U$ such that $\|PAQ - LU\| \leq O(\sigma_{k+1}(A))$ where $P$ and $Q$ are orthogonal permutation matrices, $L$ and $U$ are the lower and upper triangular matrices, respectively.

1: Create a matrix $G$ of size $n \times l$ whose entries are i.i.d. Gaussian random variables with zero mean and unit standard deviation.
2: $Y \leftarrow AG.$
3: Apply RRLU decomposition (Theorem 3.1) to $Y$ such that $PYQ_y = L_yU_y.$
4: Truncate $L_y$ and $U_y$ by choosing the first $k$ columns and the first $k$ rows, respectively, such that $L_y \leftarrow L_y(:, 1 : k)$ and $U_y \leftarrow U_y(1 : k, :)$
5: $B \leftarrow L_y^T PA.$
6: Apply LU decomposition to $B$ with column pivoting $BQ = L_bU_b.$
7: $L \leftarrow L_yL_b.$
8: $U \leftarrow U_b.$

**Remark 4.1.** The pseudo-inverse of $L_y$ in step 5 can be computed by $L_y^T = (L_y^T L_y)^{-1} L_y^T.$ This can be done efficiently when it is computed on platforms such as GPUs that can multiply matrices efficiently via parallelization. Usually, the inversion is done on a small matrix since in many cases $k \ll n$ and it can be done by the application of Gaussian elimination.

**Remark 4.2.** In practice, it is sufficient to perform step 3 in Algorithm 4.1 using standard LU decomposition with partial pivoting instead of applying RRLU. The cases where $U$ grows exponentially are extremely rare - see section 3.4.5 in [21] and [40].

Theorem 4.3 presents an error bound for Algorithm 4.1:

**Theorem 4.3.** Given a matrix $A$ of size $m \times n$. Then, its randomized LU decomposition produced by Algorithm 4.1 with integers $k$ and $l$ ($l \geq k$) satisfies:

$$\|LU - PAQ\| \leq \left(2\sqrt{2nl\beta^2\gamma^2 + 1} + 2\sqrt{2nl\beta\gamma(k(n-k) + 1)}\right)\sigma_{k+1}(A), \quad (4.1)$$

with probability not less than

$$\xi \triangleq 1 - \frac{1}{\sqrt{2\pi(l-k+1)}} \left(\frac{e}{(l-k+1)\beta}\right)^{l-k+1} - \frac{1}{4(\gamma^2 - 1)\sqrt{\pi n\gamma^2}} \left(\frac{2\gamma^2}{e\gamma^2 - 1}\right)^n,$$

where $\beta > 0$ and $\gamma > 1$.

The proof of Theorem 4.3 is given in Section 4.2. To show that the success probability $\xi$ in Eq. 4.2 is sufficiently high, we present in Table 4.1 several calculated values of $\xi$. We omitted the value of $n$ from Table 4.1 since it does
not affect the value of $\xi$ due to the fact that the second term in Eq. 4.2 decays fast.

Table 4.1: Calculated values for the success probability $\xi$ (Eq. 4.2). The terms $l-k$, $\beta$ and $\gamma$ appear in Eq. 4.2.

<table>
<thead>
<tr>
<th>$l-k$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>5</td>
<td>$1 - 6.8 \times 10^{-5}$</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>$1 - 9.0 \times 10^{-8}$</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>5</td>
<td>$1 - 5.2 \times 10^{-16}$</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>5</td>
<td>$1 - 5.2 \times 10^{-8}$</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>5</td>
<td>$1 - 1.9 \times 10^{-12}$</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>10</td>
<td>$1 - 1.9 \times 10^{-12}$</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>10</td>
<td>$1 - 1.4 \times 10^{-24}$</td>
</tr>
</tbody>
</table>

In Section 5, we show that in practice, Algorithm 4.1 produces comparable results to other well-known randomized factorization methods of low rank matrices such as randomized SVD and randomized ID.

4.1 Computational Complexity Analysis

To compute the number of floating points operations in Algorithm 4.1, we evaluate the complexity of each step:

1. Generating an $n \times l$ random matrix requires $O(nl)$ operations.

2. Multiplying $A$ by $G$ to form $Y$ requires $lC_A$ operations, where $C_A$ is the complexity of applying $A$ to an $n \times 1$ column vector.

3. Partial pivoting computation of LU for $Y$ requires $O(ml^2)$ operations.

4. Selecting the first $k$ columns (we do not modify them) requires $O(1)$ operations.

5. Computing the pseudo inverse of $L_y$ requires $O(k^2m + k^3 + k^2m)$ operations and multiplying it by $A$ requires $kC_{AT}$ operations. Note that $P$ is a permutation matrix that does not modify the rows of $A$.

6. Computing the partial pivoting LU for $B$ requires $O(k^2n)$ operations.

7. Computing $L$ requires $O(k^2m)$ operations.

8. Computing $U$ requires $O(1)$ operations.

By summing up the complexities of all the steps above, then Algorithm 4.1 necessitated

$$C_{RandLU} = lC_A + kC_{AT} + O(l^2m + k^3 + k^2n)$$  \hspace{1cm} (4.3)
operations. Here, we used $C_A$ (and $C_{AT}$) to denote the complexity of applying $A$ (and $A^T$) to a vector, respectively. For a general $A$, $C_A = C_{AT} = O(mn)$.

4.2 Bounds for the Randomized LU (Proof of Theorem 4.3)

In this section, we prove Theorem 4.3 and provide an additional complementary bound. This is done by finding a basis to a smaller matrix $AG$, which is achieved in practice by using RRLU. The assumptions are that $L$ is numerically stable so its pseudo-inverse can be computed accurately, there exists a matrix $U$ such that $LU$ is a good approximation to $AG$ and there exists a matrix $F$ such that $\|AGF - A\|$ is small. $L$ is always numerically stable since it has a small condition number [38].

Several lemmas are needed for the proof of Theorem 4.3. Lemma 4.4 states that a given basis $L$ can form a basis for the columns $A$ by bounding the error $\|LL^\dagger A - A\|$.

Lemma 4.4. Assume that $A$ is an $m \times n$ matrix, $L$ is an $m \times k$ matrix with rank $k$, $G$ is an $n \times l$ matrix, $l$ is an integer ($l \geq k$), $U$ is a $k \times l$ matrix and $F$ is $l \times n$ ($k \leq m$) matrix. Then,

$$\|LL^\dagger A - A\| \leq 2\|AGF - A\| + 2\|F\||LU - AG||. \quad (4.4)$$

Proof. By using the triangular inequality we get

$$\|LL^\dagger A - A\| \leq \|LL^\dagger A - LL^\dagger AGF\| + \|LL^\dagger AGF - AGF\| + \|AGF - A\|. \quad (4.5)$$

Clearly, the first term can also be bounded by

$$\|LL^\dagger A - LL^\dagger AGF\| \leq \|LL^\dagger\|\|A - AGF\| \leq \|A - AGF\|. \quad (4.6)$$

The second term can be bounded by

$$\|LL^\dagger AGF - AGF\| \leq \|F\||LL^\dagger AG - AG||. \quad (4.7)$$

Also,

$$\|LL^\dagger AG - AG\| \leq \|LL^\dagger AG - LL^\dagger LU\| + \|LL^\dagger LU - LU\| + \|LU - AG\|. \quad (4.8)$$

Since $L^\dagger L = I$, it follows that $\|LL^\dagger LU - LU\| = 0$ and that $\|LL^\dagger AG - LL^\dagger LU\| \leq \|AG - LU\|$. When combined with Eq. 4.8 we obtain:

$$\|LL^\dagger AG - AG\| \leq 2\|LU - AG\|. \quad (4.9)$$

By substituting Eq. 4.9 in Eq. 4.7 we get

$$\|LL^\dagger AGF - AGF\| \leq 2\|F\||LU - AG||. \quad (4.10)$$

By substituting Eqs. 4.6 and 4.10 in Eq. 4.5 we get

$$\|LL^\dagger A - A\| \leq 2\|AGF - A\| + 2\|F\||LU - AG||. \quad (4.11)$$

\square
Lemma 4.5 appears in [30]. It uses a lower bound for the least singular value of a Gaussian matrix with zero mean and unit variance. This bound can be found in [8].

**Lemma 4.5 ([30]).** Assume that $k, l, m$ and $n$ are positive integers such that $k \leq l$, $l \leq m$ and $l \leq n$. Assume that $A$ is a real $m \times n$ matrix, $G$ is $n \times l$ whose entries are i.i.d Gaussian random variables of zero mean and unit variance, $\beta$ and $\gamma$ are real numbers, such that $\beta > 0$, $\gamma > 1$ and the quantity

\[
1 - \frac{1}{\sqrt{2\pi(l-k+1)}} \left( \frac{e}{(l-k+1)\beta} \right)^{l-k+1} - \frac{1}{4(\gamma^2 - 1)\sqrt{\pi n\gamma^2}} \left( \frac{2\gamma^2}{e^{\gamma^2-1}} \right)^{n} \tag{4.12}
\]

is non-negative. Then, there exists a real $l \times n$ matrix $F$ such that

\[
\|AGF - A\| \leq \sqrt{2nl\beta^2\gamma^2 + 1}\sigma_{k+1}(A) \tag{4.13}
\]

and

\[
\|F\| \leq \sqrt{l}\beta \tag{4.14}
\]

with probability not less than the value in Eq. 4.12.

Lemma 4.6 rephrases Lemma 4.5 by utilizing the bounds that appear in Section 3.2. The proof is close to the argumentation that appear in the proof of Lemma 4.5.

**Lemma 4.6.** Let $A$ be a real $m \times n$ ($m \geq n$) matrix. Let $G$ be a real $n \times l$ matrix whose entries are Gaussian i.i.d with zero mean and unit variance. Let $k$ and $l$ be integers such that $l < m$, $l < n$ and $l > (1 + \frac{1}{\ln k})k$. We define $a_1, a_2, c_1$ and $c_2$ as in Theorem 3.6. Then, there exists a real matrix $F$ of size $l \times n$ such that

\[
\|AGF - A\| \leq \sqrt{\frac{a_2^2n}{c_1l} + 1}\sigma_{k+1}(A), \tag{4.15}
\]

and

\[
\|F\| \leq \frac{1}{c_1\sqrt{l}} \tag{4.16}
\]

with probability not less than $1 - e^{-c_2l} - e^{-a_2n}$.

**Proof.** We begin by forming the SVD of $A$

\[
A = USV^T, \tag{4.17}
\]

where $U$ is orthogonal $m \times m$ matrix, $\Sigma$ is $m \times n$ diagonal matrix with non-negative entries and $V$ is orthogonal matrix $n \times n$. Given $V^T$ and $G$, suppose that

\[
V^TG = \begin{pmatrix} H \\ R \end{pmatrix}, \tag{4.18}
\]

where $H$ is $k \times l$ and $R$ is $(n-k) \times l$. Since $G$ is a Gaussian i.i.d. matrix and $V$ is an orthogonal matrix, $V^TG$ is also a Gaussian i.i.d. matrix. Therefore, $H$ is
a Gaussian i.i.d. matrix. Let us define \( F = PV^T \), where \( P \) is of size \( l \times n \) such that
\[
P = (H^\dagger \quad 0),
\]
Therefore,
\[
F = (H^\dagger \quad 0)V^T. \tag{4.19}
\]
Computing \( \|F\| \) using Theorem 3.6 gives:
\[
\|F\| = \|PV^T\| = \|H^\dagger\| = \|H^T(HH^T)^{-1}\| = \frac{1}{\sigma_k(H)} \leq \frac{1}{c_1\sqrt{l}} \tag{4.20}
\]
with probability not less than \( 1 - e^{-c_2l} \). Now we can bound \( \|AGF - A\| \). By using Eqs. 4.17, 4.18 and 4.19 we get
\[
AGF - A = U\Sigma \left( \begin{pmatrix} H \\ R \end{pmatrix} (H^\dagger \quad 0) - I \right) V^T. \tag{4.21}
\]
We define \( S \) to be the upper-left \( k \times k \) block of \( \Sigma \) and let \( T \) to be the lower-right \((n - k) \times (n - k)\) block. Then,
\[
\Sigma \left( \begin{pmatrix} H \\ R \end{pmatrix} (H^\dagger \quad 0) - I \right) = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} 0 & 0 \\ RH^\dagger & -I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ TRH^\dagger & -T \end{pmatrix}.
\]
The norm of the last term can be rewritten as:
\[
\left\| \begin{pmatrix} 0 & 0 \\ TRH^\dagger & -T \end{pmatrix} \right\|^2 \leq \|TRH^\dagger\|^2 + \|T\|^2. \tag{4.22}
\]
Therefore, by using Eqs. 4.21, 4.22 and the fact that \( \|T\| = \sigma_{k+1}(A) \), we get
\[
\|AGF - A\| \leq \sqrt{\|TRH^\dagger\|^2 + \|T\|^2} \leq \sqrt{\|H^\dagger\|^2 \|R\|^2 + 1\sigma_{k+1}(A)}. \tag{4.23}
\]
Also we know that
\[
\|R\| \leq \|V^TG\| = \|G\| \leq a_1\sqrt{n}
\]
with probability not less than \( 1 - e^{-a_2n} \). Combining Eq. 4.23 with the fact that \( \|H^\dagger\| \leq \frac{1}{c_1\sqrt{l}} \) and \( \|R\| \leq a_1\sqrt{n} \) gives:
\[
\|AGF - A\| \leq \sigma_{k+1}(A)\sqrt{\frac{a_1^2n}{c_1^2l}} + 1. \tag{4.24}
\]
\[\square\]

**Remark 4.7.** In contrast to Lemma 4.5 where \( \|AGF - A\| = \mathcal{O}(\sqrt{n}) \), Lemma 4.6 provides the bound \( \|AGF - A\| = \mathcal{O}(\sqrt{\frac{n}{l}}) \) that is sharper for large values of \( l \).
Remark 4.8. The condition \( l > (1 + \frac{1}{\ln k}) k \) in Lemma 4.6 is satisfied without a dramatic increase in the computational complexity of Algorithm 4.1. However, there are bounds for the case where \( H \) is almost square \( (l \approx k) \) and square \( (l = k) \) and they are given in [27].

Proof of Theorem 4.3. The error is given by the expression \( \|LU − PAQ\| \) where \( L, U, P \) and \( Q \) are the outputs of Algorithm 4.1 whose inputs are the matrix \( A \), integers \( k \) and \( l \). From Steps 7 and 8 in Algorithm 4.1 we have

\[
\|LU − PAQ\| = \|L_yL_y U_y − PAQ\|. \tag{4.25}
\]

Here, \( L_y \) is the \( m \times k \) matrix in step 4. By using the fact that \( BQ = L_y U_y = L_y^1 PAQ \), we get

\[
\|LU − PAQ\| = \|L_yL_y U_y − PAQ\| = \|L_yL_y^1 PAQ − PAQ\|. \tag{4.26}
\]

Applying Lemma 4.4 to Eq. 4.26 gives

\[
\|LU − PAQ\| = \|L_yL_y^1 PAQ − PAQ\| \leq 2\|PAQ\tilde{G}F − PAQ\| + 2\|F\|\|L_yU_y − PAQ\tilde{G}\|. \tag{4.27}
\]

Here, \( U_y \) is the \( k \times n \) matrix in step 4 in Algorithm 4.1. This holds for any matrix \( \tilde{G} \). In particular, for a matrix \( \tilde{G} \) satisfying \( Q \tilde{G} = GQ_y \), where \( G \) is a random Gaussian i.i.d. matrix. After row and columns permutations \( G \) is in fact \( \tilde{G} \). Therefore, the last term in Eq. 4.27 can be reformulated as \( \|L_yU_y − PAQ\tilde{G}\| = \|L_yU_y − PAGQ_y\| \) where \( G \) is the random matrix in Algorithm 4.1. By applying Lemmas 3.2 and 3.3 to \( \|L_yU_y − PAQ\tilde{G}\| \) we get

\[
\|L_yU_y − PAQ\tilde{G}\| = \|L_yU_y − PAGQ_y\| \leq (k(n − k) + 1)\sigma_{k+1}(AG) \tag{4.28}
\]

\[
\|L_yU_y − PAQ\tilde{G}\| \leq (k(n − k) + 1)\|G\|\sigma_{k+1}(A).
\]

Lemma 4.5 gives that \( \|PAQ\tilde{G}F − PAQ\| \leq \sqrt{2nl\beta^2\gamma^2 + 1\sigma_{k+1}(A)} \) and \( \|F\| \leq \sqrt{l\beta} \). By combining Lemmas 4.5 and 3.4 we get

\[
\|LU − PAQ\| \leq \left(2\sqrt{2nl\beta^2\gamma^2 + 1} + 2\sqrt{2nl\beta\gamma(k(n − k) + 1)}\right)\sigma_{k+1}(A), \tag{4.29}
\]

which completes the proof. \( \square \)

Remark 4.9. The error in Theorem 4.3 may appear large, especially for the case where \( k \approx \frac{n}{2} \) and \( n \) is large. Yet, we performed extensive numerical experiments showing that the actual error was much smaller when using Gaussian elimination with partial pivoting. Note that the error can decrease by increasing \( k \). Numerical illustrations appear in section 5.
We now present an additional error bound that relies on [28]. Asymptotically, this is a sharper bound for large values of \( n \) and \( l \), since it contains the term \( \sqrt{\frac{n}{l}} \), which is smaller than the term \( \sqrt{n l} \) appears in Theorem 4.3. See also Remark 4.7.

**Theorem 4.10.** Given a matrix \( A \) of size \( m \times n \), integers \( k \) and \( l \) such that \( l > (1 + \frac{1}{\ln k}) k \) and \( a_2 > 0 \). By applying Algorithm 4.1 with \( A, k \) and \( l \) as its input parameters, we get a randomized LU decomposition that satisfies

\[
\|L U - PAQ\| \leq \left( 2 \sqrt{\frac{a_1^2 n}{c_1^2 l}} + 1 + \frac{2a_1 \sqrt{n}}{c_1 \sqrt{l}} (k(n - k) + 1) \right) \sigma_{k+1}(A),
\]

(4.30)

with probability not less than \( 1 - e^{-a_2 n} - e^{-c_2 l} \). The value of \( c_1 \) is given in Eq. 3.10 and the value of \( c_2 \) is given in Eq. 3.11. Both values depend on \( a_2 \).

**Proof.** By using steps 5,6,7 and 8 in Algorithm 4.1, we get that

\[
\|L U - PAQ\| = \|L_y L_y^\dagger PAQ - PAQ\|.
\]

(4.31)

Then, from Lemma 4.4,

\[
\|L_y L_y^\dagger PAQ - PAQ\| \leq 2 \|PAQ \tilde{G}F - PAQ\| + 2 \|F\|\|L_y U_y - PAQ \tilde{G}\|.
\]

(4.32)

From Lemma 4.6 we get that

\[
\|PAQ \tilde{G}F - PAQ\| \leq \sqrt{\frac{a_1^2 n}{c_1^2 l}} + 1 \sigma_{k+1}(A).
\]

(4.33)

Using the same argumentation given in Theorem 4.3, we get

\[
\|L_y U_y - PAQ \tilde{G}\| = \|L_y U_y - PAQ G_y\| \leq (k(n - k) + 1) \|G\| \sigma_{k+1}(A)
\]

(4.34)

where \( G \) is the matrix used in Algorithm 4.1 Step 1. Combining Eqs. 4.32, 4.33, 4.34 and since \( \|F\| \leq \frac{1}{c_1 \sqrt{l}} \), \( \|G\| \leq a_1 \sqrt{n} \) (see Lemma 4.6 and Theorem 3.5, respectively), we get that

\[
\|L U - PAQ\| \leq 2 \sqrt{\frac{a_1^2 n}{c_1^2 l}} + 1 \sigma_{k+1}(A) + \frac{2a_1 \sqrt{n}}{c_1 \sqrt{l}} (k(n - k) + 1) \sigma_{k+1}(A).
\]

(4.35)

Here, \( \mu = \left( \frac{4}{\sqrt{2\pi}} \right)^{\frac{3}{2}} \), \( a_1 \) is given by Theorem 3.5 and \( c_1 \) is given by Eq. 3.10. \( \square \)

**4.3 Analysis of the bound in Theorem 4.10**

In this section, we analyze the bound in Eq. 4.35. Although Eq. 4.35 focuses on the randomized LU decomposition, this approach can be used for analyzing other randomized algorithms such as eigenvalue or singular value decompositions that find an orthogonal approximated basis for the range of the input matrix.
SVD is used in [30], to find an orthogonal basis for the range of the matrix \( Y \) instead of computing an LU decomposition in step 3 of Algorithm 4.1. The first steps of Algorithm 4.1 (steps 1-3) can be rewritten as:

**Algorithm 4.2:** Randomized algorithm with orthogonal basis

**Input:** A matrix of size \( m \times n \) to decompose, \( k \) matrix rank, \( l \geq k \) number of columns to use.

**Output:** Matrix \( Q \) of size \( m \times k \) such that \( \| A - QQ^*A \| \) is bounded, and \( QQ^* = I \) are orthogonal permutation matrices, \( L \) and \( U \) are the lower and upper triangular matrices, respectively.

1. Create a matrix \( G \) of size \( n \times l \) whose entries are i.i.d. Gaussian random variables with zero mean and unit standard deviation.
2. \( Y \leftarrow AG \).
3. Construct a matrix \( U \) whose columns form an orthonormal basis for the range of \( Y \) using SVD.
4. Construct a matrix \( Q \) by grouping the first \( k \) vectors from \( U \).

By estimating \( \| QQ^*A - A \| \) in the same way as was done in Eq. 4.35 and Lemma 4.4 instead of estimating \( \| LL^\dagger A - A \| \) we get

\[
\| QQ^*A - A \| \leq 2 \sqrt{\frac{a_2^2 n}{c_1^2 l}} + 1\sigma_{k+1}(A) + \frac{2a_1 \sqrt{n}}{c_1 \sqrt{l}} \sigma_{k+1}(A) \tag{4.36}
\]

with probability not less than \( 1 - e^{-a_2 n} - e^{-c_2 l} \). The value of \( c_1 \) is given in Eq. 3.10, the value of \( c_2 \) is given in Eq. 3.11 and the value of \( a_2 \) is given in Theorem 3.5. All of them depend on \( a_2 \).

Equation 4.36 provides an alternative bound to the randomized SVD algorithm. By neglecting constants and analyzing the asymptotic behavior of Eq. 4.36 we get that for \( n \gg l \)

\[
\| QQ^*A - A \| \leq 2 \sqrt{\frac{a_2^2 n}{c_1^2 l}} + 1\sigma_{k+1}(A) + \frac{2a_1 \sqrt{n}}{c_1 \sqrt{l}} \sigma_{k+1}(A) \propto \sqrt{\frac{n}{l}} \sigma_{k+1}(A) \tag{4.37}
\]

with asymptotic failure probability of \( e^{-c_2 l} \). Two bounds are given in [23]. One uses expectation and one probability. The expectation bound was proven to be sharp (see [42]) and the probability bound is better than previously developed bounds in [30]. This bound is given in [23], Corollary 10.9:

**Corollary 4.11** ([23]). For \( Q \) from Algorithm 4.2 and \( p \geq 4 \) (\( p = l - k \)):

\[
\| QQ^*A - A \| \leq (1 + 17 \sqrt{1 + k/p})\sigma_{k+1} + \frac{8 \sqrt{k + p}}{p + 1} \left( \sum_{j > k} \sigma_j^2 \right)^{1/2} \tag{4.38}
\]

with failure probability at most \( 6e^{-p} \).
We now compare the case of fixed $\sigma_j$, $j > k$, $\sigma_{k+1} = \sigma_{k+2} = \cdots = \sigma_{\min(m,n)}$ between the asymptotic behavior of Eqs. 4.38 and 4.36. The asymptotic behavior for $n \gg k + p$ of Eq. 4.38 is given by

$$\|QQ^*A - A\| \leq (1 + 17 \sqrt{1 + k/p}) \sigma_{k+1} + \frac{8 \sqrt{k + p}}{p + 1} \left( \sum_{j > k} \sigma_j^2 \right)^{1/2} \propto \frac{\sqrt{(k + p)(n - k)}}{p + 1} \sigma_{k+1}. \quad (4.39)$$

Comparing between Eqs. 4.39 and 4.37 shows that Eq. 4.37 provides a sharper bound since Eq. 4.39 has an additional factor of $\sqrt{k + p}$ in the numerator compared to Eq. 4.37 and a smaller denominator than the one in Eq. 4.37. Also, the failure probability is better in Eq. 4.37 since the exponents depend on $l$ instead of $p$.

The bound in Eq. 4.36 is useful especially for large values of $l$. We also assume that $n \gg l$ and $\sigma_j = \sigma$ for $j > k$. Here is a numerical example that illustrates the bounds. $m = 2 \cdot 10^8, n = 10^8, k = 990, l = 1000$ and $a_2 = 1$. Computing $a_1, c_1$ and $c_2$ by using Theorems 3.5 and 3.6 provides $a_1 = 15.68$, $c_1 = 0.022$, $c_2 = 0.011$. Substituting these values in Eq. 4.36 gives

$$\|QQ^*A - A\| \leq 2.9 \cdot 10^5 \sigma_{k+1} \quad (4.40)$$

with failure probability $1.1 \cdot 10^{-49}$. The same setup for Eq. 4.38 gives

$$\|QQ^*A - A\| \leq 7.28 \cdot 10^5 \sigma_{k+1} \quad (4.41)$$

with failure probability $2.72 \cdot 10^{-4}$. Clearly, in this example Eq. 4.36 provides a sharper bound both in accuracy and failure probability.

Figure 4.1 compares between the asymptotic behaviors of the bounds in Eqs. 4.39 and 4.37. This figure shows that when there is a small oversampling (small $p$), then the bound in Eq. 4.36, which is indicated by the red line, is asymptotically sharper in comparison to the bound in Eq. 4.39 which is indicated by the dashed blue line. As the oversampling increases the bounds coincide.
Figure 4.1: Bound factor vs. oversampling: \( k = 3, \ p = 4, 5, \ldots, 100, \ l = k + p \).

Figure 4.2 shows the asymptotic behavior of Eq. 4.36 for different values of \( k \) and a fixed \( p \). The red line illustrates Eqs. 4.37 and 4.36 and the blue dashed line illustrates Eqs. 4.39 and 4.38.

Figure 4.2: Bound factor for fixed \( p = 10 \) and \( k = 3, 4, \ldots, 100 \).
4.4 Rank Deficient Least Squares

In this section, we present an algorithm that uses the randomized LU and show how it can be used to solve efficiently the Rank Deficient Least Squares (RDLS) problem. Assume that $A$ is an $m \times n$ matrix ($m \geq n$) with rank($A$) = $k$, $k < n$ and $b$ is a column vector of size $m \times 1$. We want to minimize $\|Ax - b\|$. Because $A$ is a rank deficient matrix, then the problem has an infinite number of solutions since if $x$ is a minimizer and $z \in \text{null}(A)$, then $x + z$ is also a minimizer (i.e. a valid solution). We now show that the complexity of the solution depends on the rank of $A$ and that the problem is equivalent to solving the following two problems: a full rank Least Square (LS) problem of size $m \times k$ and a simplified underdetermined linear system of equations that requires a matrix inversion of size $k \times k$.

The solution is derived by the application of Algorithm 4.1 to $A$ to get

$$\|Ax - b\| = \|P^T L U Q^T x - b\| = \|L U Q^T x - P b\|,$$  \hspace{1cm} (4.42)

where $L$ is an $m \times k$ matrix, $U$ is a $k \times n$ matrix and both $L$ and $U$ are of rank $k$. Let $y = U Q^T x$ and $c = Pb$. Then, the problem is reformulated as

$$\min \|L y - c\|.$$  \hspace{1cm} (4.43)

Note that $L$ is a full rank matrix and the problem to be solved becomes a standard full rank LS problem. The solution is given by $y = L^T c$. Next, we solve

$$U z = y,$$  \hspace{1cm} (4.44)

where $z = Q^T x$. Since $U$ is a $k \times n$ matrix, Eq. 4.44 is an underdetermined system. Assume that $U = [U_1 \ U_2]$ and $z = [z_1 \ z_2]^T$, where $U_1$ is a $k \times k$ matrix, $z_1$ is a $k \times 1$ vector and $z_2$ is a $(n - k) \times 1$ vector. Then, the solution is given by setting any value to $z_2$ and solving

$$U_1 z_1 = y - U_2 z_2.$$  \hspace{1cm} (4.45)

For simplicity, we choose $z_2 = 0$. Therefore, we get $z_1 = U_1^{-1} y$. The final solution is given by $x = Q z$. This procedure is summarized in Algorithm 4.3 that finds the solution to the deficient least squares problem that uses Algorithm 4.1.

---

**Algorithm 4.3: Solving Rank Deficient Least Squares with Randomized LU**

**Input:** matrix $A$ of size $m \times n$ with rank $k$, $l$ integer such that $l \geq k$, $b$ vector of size $m \times 1$.

**Output:** Solution $x$ that minimizes $\|Ax - b\|$.

1. Apply Algorithm 4.1 to $A$ with parameters $k$ and $l$.
2. $y \leftarrow L^T Pb$. 
3. $z_1 \leftarrow U_1^{-1} y$.
4. $z \leftarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_2$ is an $n - k$ zero vector.
5. $x \leftarrow Q z$. 

18
The complexity of Algorithm 4.3 is equal to the randomized LU complexity (Algorithm 4.1) with an additional inversion cost of the matrix $U_1$ in Step 3, which is of size $k \times k$. Note that the solution given by Algorithm 4.3 is sparse in the sense that $x$ contains at most $k$ non-zero entries.

### 4.5 Fast Randomized LU

Algorithm 4.1 describes the randomized LU algorithm. This algorithm computes an LU approximation the matrix $A$ of rank $k$ whose computational complexity is $C_{\text{RandLU}} = O(lmn + l^2m + k^3 + k^2n)$ operations. We present now an asymptotic improvement to Algorithm 4.1 whose complexity is

$$C_{\text{FastRandLU}} = O(mn \log l + mk + nkl + mk + k^3).$$  \hspace{1cm} (4.46)

In order to achieve it, we use the SRFT and the ID [9], which were presented in sections 3.3 and 3.4, respectively.

The most computationally expensive procedures are steps 2 and 5 in Algorithm 4.1. Step 2 involves matrix multiplication with the matrix $A$ where $A$ is projected by a random matrix. Instead of projecting it with a Gaussian random matrix, we use the SRFT matrix. Due to the special structure of the SRFT matrix $R$, as was shown in Lemma 3.7, the multiplication of an $m \times n$ matrix $A$ by an $n \times l$ matrix $R$ necessitates $O(mn \log l)$ floating point operations.

Instead of direct computation of $L_y^T P A$ in step 5 in Algorithm 4.1, $A$ is approximated by the ID of $Y$, namely, if $Y = XY_{(J,:)}$ is the full rank ID of $Y$, then $A \approx XA_{(J,:)}$.

---

**Algorithm 4.4:** Fast Randomized LU Decomposition

**Input:** Matrix $A$ of size $m \times n$ to decompose, $k$ desired rank, $l$ number of columns to use.

**Output:** Matrices $P, Q, L, U$ such that $\|PAQ - LU\| \leq O(\sigma_{k+1}(A))$ where $P$ and $Q$ are orthogonal permutation matrices, $L$ and $U$ are the lower and upper triangular matrices, respectively.

1. Create a random SRFT matrix $R$ of size $n \times l$ (Lemma 3.7).
2. $Y \leftarrow AR$.
3. Apply RRLU decomposition to $Y$ such that $PYQ_y = L_yU_y$.
4. Truncate $L_y$ and $U_y$ by choosing the first $k$ columns and the first $k$ rows, respectively, such that $L_y \leftarrow L_y(1:k)$ and $U_y \leftarrow U_y(1:k,:)$
5. Compute the full rank ID decomposition of $Y$: $Y = XY_{(J,:)}$ (Section 3.4).
6. $B \leftarrow L_y^T PXA_{(J,:)}$.
7. Apply the LU decomposition to $B$ with column pivoting $BQ = L_bU_b$.
8. $L \leftarrow L_yL_b$.
9. $U \leftarrow U_b$. 

19
4.5.1 Computational complexity

To compute the number of floating points operations in Algorithm 4.4, we evaluate the complexity of each step:

1. Multiplying an $m \times n$ matrix $A$ by an $n \times l$ matrix $R$ requires $O(nm \log l)$ operations;
2. Computing the RRLU decomposition of an $m \times l$ $Y$ requires $O(ml^2)$ operations;
3. Truncating $L_y$ and $U_y$ requires $O(1)$ operations;
4. Computing the ID decomposition of $Y$ requires $O(ml^2)$ operations;
5. Computing the pseudo inverse $L_y^\dagger$ requires $O(k^2m + k^3)$ operations;
6. Multiplying $L_y^\dagger PXA_{(ij,:)}$ requires $O(mlk + nkl)$ operations;
7. Computing partial pivoting LU of $B$ requires $O(nk^2)$ operations;
8. Computing $L$ requires $O(mk^2)$ operations.

The total computational complexity of the algorithm is $O(mn \log l + mlk + nkl + mk^2 + k^3)$. By simplifying this expression while assuming that $k$ and $l$ are of the same magnitude, we get that the total computational complexity of Algorithm 4.4 is $O(mn \log k + (m + n)k^2 + k^3)$.

4.5.2 Correctness of the Algorithm

We now prove that Algorithm 4.4 approximates the LU-factorization and provide an error bound.

**Theorem 4.12.** Given a matrix $A$ of size $m \times n$. Then, its fast randomized LU decomposition in Algorithm 4.4 with integers $k$ and $l$ ($n, m \geq l \geq k$ sufficiently large) satisfies:

$$\|LU - PAQ\| \leq \left( \left[ 1 + \sqrt{1 + 4k(n - k)} \right] \sqrt{1 + \frac{7n}{l}} \right) \sigma_{k+1}(A) + 2 \left( \sqrt{\alpha n + 1} + \sqrt{\frac{\pi}{l}} (k(n - k) + 1) \right) \sigma_{k+1}(A)$$

with probability not less than $1 - 3\frac{1}{\beta k}$ where $\beta > 1$ is a constant.

The proof of Theorem 4.12 uses Lemmas 4.13-4.15.

**Lemma 4.13.** Let $A$ be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(m,n)}$. Let $k$ and $l$ be integers such that

$$4 \left( \sqrt{k} + \sqrt{8 \ln(kn)} \right)^2 \ln k \leq l \leq n.$$
Let $R$ be an $n \times l$ SRFT matrix and $Y = AR$. Denote by $Q$ the $m \times l$ matrix whose columns form an orthonormal basis for the range of $Y$. Then, with a failure probability of at most $3k^{-1}$ we have

$$\|A - QQ^* A\| \leq \sqrt{1 + 7n/l}\sigma_{k+1}.$$ 

Lemma 4.13 appears in slightly different formulation in [23] as Theorem 11.2 and in [41] as Theorem 3.1.

**Lemma 4.14.** Let $A$ be an $m \times n$ matrix, let $R$ be an $n \times l$ SRFT random matrix and let $Y = XY_{(J,:)}$ be the full rank ID of $Y = AR$. Then,

$$\|A - XA(J,:)\| \leq \left[1 + \sqrt{1 + 4k(n-k)}\right] \sqrt{1 + 7n/l}\sigma_{k+1}(A)$$

with failure probability at most $3k^{-1}$ when $l \geq 4 \left[\sqrt{k} + \sqrt{8\ln(kn)}\right]^2 \ln k$.

The proof is as the proof of Lemma 5.1 in [23].

**Proof.** Denote by $Q$ a matrix whose columns form an orthonormal basis for the range of $Y$. By using Lemma 4.13 we have

$$\|A - QQ^* A\| \leq \sqrt{1 + 7n/l}\sigma_{k+1}(A) \quad \text{(4.47)}$$

except with probability $3k^{-1}$.

Denote $\hat{A} = QQ^* A$. Since $\hat{A} = XQ_{(J,:)}Q^* A$ and $X_{(J,:)} = I$, we have $\hat{A}_{(J,:)} = Q_{(J,:)}Q^* A$, and thus $\hat{A} = X\hat{A}_{(J,:)}$.

$$\begin{align*}
\|A - XA(J,:)\| & = \|A - X\hat{A}_{(J,:)} + X\hat{A}_{(J,:)} - XA(J,:)\| \\
& \leq \|A - \hat{A}\| + \|X\hat{A}_{(J,:)} - XA(J,:))\| \\
& = \|A - \hat{A}\| + \|X\|\|\hat{A}_{(J,:)} - A(J,:))\| \\
& \leq (1 + \|X\|)\|A - \hat{A}\|.
\end{align*}$$

By using Eq. 4.47 we have

$$\|A - XA(J,:)\| \leq (1 + \|X\|) \sqrt{1 + 7n/l}\sigma_{k+1}(A).$$

The proof is completed since $X$ contains a $k \times k$ identity matrix and that the remaining $(n-k) \times k$ submatrix are bounded in magnitude by 2. \qed

**Lemma 4.15** (appears in [43] as Lemma 4.6). Suppose that $k, l, n$ and $m$ are positive integers with $k \leq l$ such that $l < m$ and $l < n$. Suppose further that $\alpha$ and $\beta$ are real numbers greater than 1 such that

$$m > l \geq \frac{\alpha^2 \beta}{(\alpha - 1)^2} k^2.$$

Suppose that $A$ is an $m \times n$ complex matrix and $Q$ is the $n \times l$ SRFT matrix. Then, there exists an $l \times n$ complex matrix $F$ such that $\|AQF - A\| \leq \sqrt{\alpha n + 1}\sigma_{k+1}$ and $\|F\| \leq \sqrt{\frac{2}{\beta}}$ with probability at least $1 - \frac{1}{\beta}$ where $\sigma_{k+1}$ is the $(k+1)$th greatest singular value of $A$.  

21
Lemma 4.16. Let $A, P, Q, L_y$ and $L_y^\dagger$ be as in Algorithm 4.4, then

$$\|L_yL_y^\dagger PAQ - PAQ\| \leq 2 \left( \sqrt{\alpha n + 1} + \sqrt{\frac{\alpha}{l}} (k(n - k) + 1) \right) \sigma_{k+1}(A)$$

with probability of at least $1 - \frac{1}{\beta}$ where $m > l \geq \frac{\alpha^2 \beta}{(\alpha - 1)^2} k^2$.

Proof. By applying Lemma 4.4 we get

$$\|L_yL_y^\dagger PAQ - PAQ\| \leq 2 \|PAQ\tilde{G}F - PAQ\| + 2 \|F\| \|L_yU_y - PAQ\tilde{G}\|. \quad (4.48)$$

$U_y$ is the $k \times n$ matrix in step 4 in Algorithm 4.4. This holds for any matrix $\tilde{G}$. In particular, for a matrix $\tilde{G}$ that satisfies $Q\tilde{G} = RQ_y$, where $R$ is the SRFT matrix in Algorithm 4.4. After row and column permutations $R$ is in fact $\tilde{G}$. Therefore, the last term in Eq. 4.48 can be reformulated by $\|L_yU_y - PAQ\tilde{G}\| = \|L_yU_y - PARQ_y\|$. By applying Lemmas 3.2 and 3.3 to $\|L_yU_y - PAQ\tilde{G}\|$ we get

$$\|L_yU_y - PAQ\tilde{G}\| = \|L_yU_y - PARQ_y\|$$

$$\leq (k(n - k) + 1) \sigma_{k+1}(AR) \quad (4.49)$$

$$\leq (k(n - k) + 1) \|R\| \sigma_{k+1}(A).$$

Since $R$ is an SRFT matrix, it is orthogonal, and thus $\|R\| = 1$. Lemma 4.15 proves that $\|PAQ\tilde{G}F - PAQ\| \leq \sqrt{\alpha n + T} \sigma_{k+1}(A)$ and $\|F\| \leq \sqrt{T}$. By summing up, we have:

$$\|L_yL_y^\dagger PAQ - PAQ\| \leq 2 \|PAQ\tilde{G}F - PAQ\| + 2 \|F\| \|L_yU_y - PAQ\tilde{G}\|$$

$$\leq 2 \sqrt{\alpha n + T} \sigma_{k+1}(A) + 2 \|F\| (k(n - k) + 1) \sigma_{k+1}(A)$$

$$\leq 2 \left( \sqrt{\alpha n + T} \sqrt{T} (k(n - k) + 1) \right) \sigma_{k+1}(A).$$

Proof of Theorem 4.12.

Proof. By substituting $L$ and $U$ from Algorithm 4.4 we have

$$\|LU - PAQ\| = \|L_y L_y U_y - PAQ\| = \|L_y BQ - PAQ\| = \|L_y L_y^\dagger PXA_{(J,\cdot)} Q - PAQ\| \leq \|L_y L_y^\dagger PXA_{(J,\cdot)} Q - L_y L_y^\dagger PAQ\| + \|L_y L_y^\dagger PAQ - PAQ\|. \quad (4.50)$$

The first term in the last inequality in Eq. 4.50 is bounded in the following way:

$$\|L_y L_y^\dagger PXA_{(J,\cdot)} Q - L_y L_y^\dagger PAQ\| \leq \|L_y L_y^\dagger PXA_{(J,\cdot)} Q - XA_{(J,\cdot)} - A\| \|Q\| = \|XA_{(J,\cdot)} - A\|.$$

By using Lemma 4.14 we get

$$\|LU - PAQ\| = \|L_y L_y U_y - PAQ\| \leq \left( 1 + \sqrt{1 + 4k(n - k)} \right) \sqrt{1 + 7n/l} \sigma_{k+1}(A)$$
with probability of not less than $1 - 3k^{-1}$.

The second term $\|L_y L^*_y PAQ - PAQ\|$ in the last inequality of Eq. 4.50 is bounded by Lemma 4.16.

By combining these results we get

$$\|LU - PAQ\| \leq \left(1 + \sqrt{1 + 4k(n - k)} \right) \sqrt{1 + 7n/l} \sigma_{k+1}(A) + 2 \left(\sqrt{\alpha n + 1} + \sqrt{\frac{n}{T}}(k(n - k) + 1)\right) \sigma_{k+1}(A)$$

which completes the proof.

\[ \square \]

5 Numerical Results

In order to evaluate Algorithm 4.1, we present the numerical results by comparing between the performances of several randomized low rank approximation algorithms. We tested the algorithms and compared between them by applying them to random and sparse matrices and to images. All the results were computed using the standard MATLAB libraries on a machine with two Intel Xeon CPUs X5560 2.8GHz that contains an nVidia GPU GTX TITAN card.

5.1 Error Rate and Computational Time Comparisons

The performance of the randomized LU (Algorithm 4.1) was tested and compared to the randomized SVD and to the randomized ID (see [23, 30]). The tests compare between the normalized (relative) error of the low rank approximation obtained by the examined methods. In addition, the computational time of each method was measured. If $A$ is the original matrix and $\hat{A}$ is a low rank approximation of $A$, then the relative approximation error is given by:

$$\text{err} = \frac{\|A - \hat{A}\|}{\|A\|}.$$  \hspace{1cm} (5.1)

We compared between the low rank approximation achieved by the application of the randomized SVD, randomized ID and randomized LU with different ranks $k$. Throughout the experiments, we chose $l = k + 3$ and the test matrix was a random matrix of size $3000 \times 3000$ with exponentially decaying singular values. The computations of the algorithms were done in a single precision. The results are presented in Fig. 5.1. The experiment shows that the error of the randomized ID is larger than the error obtained from both the randomized SVD and the randomized LU (Algorithm 4.1), which are almost identical. In addition, we compared between the execution time of these algorithms. The results are presented in Fig. 5.2. The results show that the execution time of the randomized LU (Algorithm 4.1) is lower than the execution time of the randomized SVD and the randomized ID algorithms. The LU factorization has a parallel implementation (see [21] section 3.6). To see the impact of the parallel LU factorization implementation, the execution time for computing the randomized LU of a matrix of size $3000 \times 3000$ was measured on an nVidia GTX TITAN. 

23
GPU device and it is shown in Fig. 5.3. The execution time on the GPU was up to \( \times 10 \) faster than running it on an eight cores CPU. Thus, the algorithm scales well. For larger matrices \((n \text{ and } k \text{ are large})\), the differences between the performances running on CPU and on GPU are more significant.

![Comparison between the low rank approximation error of different algorithms](image)

**Figure 5.1:** Comparison between the low rank approximation error of different algorithms: Randomized SVD, Randomized ID and Randomized LU. Randomized LU achieves the lowest error.
Figure 5.2: Comparison between the execution times of the same algorithms as in Fig. 5.1 running on a CPU. Randomized LU achieved the lowest execution time.

Figure 5.3: Comparison between the execution times from running Algorithm 4.1 on different computational platforms: CPU with 8 cores and GPU. Randomized LU achieved the lowest execution time on GPU.

5.2 Image Matrix Factorization

Algorithm 4.1 was applied to images given in a matrix format. The factorization error and the execution time were compared with the performances of
the randomized SVD and the randomized ID. We also added the SVD error computation and execution time as a benchmark computed using Lanczos bidiagonalization [21] implemented in the PROPACK package [26]. The image size was 2124 × 7225 pixels and it has 256 gray levels. The parameters were $k = 200$ and $l = 203$. The approximation quality (error) was measured in PSNR defined by

$$
\text{PSNR} = 20 \log_{10} \frac{\max_A \sqrt{N}}{\|A - \hat{A}\|_F}
$$

(5.2)

where $A$ is the original image, $\hat{A}$ is the approximated image (the output from Algorithm 4.1), $\max_A$ is the maximal pixel value of $A$ and $N$ is the total number of pixels and $\| \cdot \|_F$ is the Frobenius norm.

![Figure 5.4: The original input image of size 2124 × 7225 that was factorized by the application of the randomized LU, randomized ID and randomized SVD algorithms.](image)

![Figure 5.5: The reconstructed image from the application of the randomized LU factorization with $k = 200$ and $l = 203$.](image)

Figures 5.4 and 5.5 show the original and the reconstructed images, respectively. The image approximation quality (measured in PSNR) related to the rank $k$ is shown in Fig. 5.6 where for the same $k$ the PSNR from the application of Algorithm 4.1 is higher than the PSNR generated by the application of randomized ID and almost identical to the randomized SVD. In addition, the
PSNR values are close to the result achieved by the application of the Lanczos SVD which is the best possible rank $k$ approximation. The execution time of each algorithm is shown in Fig. 5.7. All the computations were done in double precision. Here, the randomized LU is faster than all the other compared methods making it applicable for real time applications.

Figure 5.6: Comparison between the PSNR values from image reconstruction application using randomized LU, randomized ID, randomized SVD and Lanczos SVD algorithms.
Figure 5.7: Comparison between the execution time of the randomized LU, randomized ID, randomized SVD and Lanczos SVD algorithms.

5.3 Fast Randomized LU

In order to compare the decomposition running time between Algorithms 4.4 and 4.1, we apply these algorithms to different matrix sizes.

The y-axis in Fig. 5.8 shows the time (in seconds) for decomposing an $n \times n$ matrix with $l = 3 \log_2 n$ where $n$ is the x-axis.
Figure 5.8: Comparison between the running time of the fast randomized LU and the randomized LU algorithms

In addition, we see in Fig. 5.9 that although the error of Algorithm 4.4 is larger than the error that Algorithm 4.1 generates. Figure 5.9, like Figure 5.1, shows the relative error (Eq. 5.1) for a randomly chosen matrix of size \(3000 \times 3000\) with exponentially decaying singular values where \(l = k + 3\) for different values of \(k\).
Figure 5.9: Comparison between the normalized error (Eq. 5.1) of the fast randomized LU and the randomized LU algorithms.

Conclusion

In this work, we presented a randomized algorithm for computing an LU rank $k$ decomposition. Given an integer $k$, the algorithm finds an LU decomposition such that both $L$ and $U$ are of rank $k$ with negligible failure probability. We constructed error bounds for the approximation of the input matrix and proved that they are proportional to the $(k + 1)$th singular value. We also compared the performance of the algorithm with regard to the error rate and to the computational time. We compared the results with randomized SVD, randomized ID and to the application of Lanczos SVD running on sparse matrices. We also showed that our algorithm can be parallelized since it consists mostly of matrix multiplication and pivoted LU. The results on GPU show that it was possible to accelerate the computational time significantly even by using only the standard MATLAB libraries.

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