A Framework for Discrete Integral Transformations II – the 2D discrete Radon transform

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Abstract

The Radon transform is a fundamental tool in many areas. For example, in reconstruction of an image from its projections (CT scanning). Although it is situated in the core of many modern physical computations, the Radon transform lacks a coherent discrete definition for 2D discrete images which is algebraically exact, invertible, and rapidly computable.

We define a notion of 2D discrete Radon transforms for discrete 2D images, which is based on summation along lines of absolute slope less than 1. Values at non-grid locations are defined using trigonometric interpolation on a zero-padded grid. The discrete 2D definition of the Radon transform

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is shown to be geometrically faithful as the lines used for summation exhibit no wraparound effects. There exists a special set of lines in $\mathbb{R}^2$ for which the transform is rapidly computable and invertible. We describe an algorithm that computes the 2D discrete Radon transform and uses $O(N \log N)$ operations, where $N = n^2$ is the number of pixels in the image. The algorithm relies on a discrete Fourier slice theorem, which associates the discrete Radon transform with the pseudo-polar Fourier transform [14]. Finally, we prove that our definition provides a faithful description of the continuum, as it converges to the continuous Radon transform as the discretization step goes to zero.

1 Introduction

An important problem in image processing is how to reconstruct a cross section of an object from several images of its projections. A projection is defined as the line integral of some function on the object. For example, in CT scanning this function is the attenuation coefficient at each point, and the line integral (projection) corresponds to the attenuation of a X-ray beam that passes through the object. The Radon transform is the underlying mathematical tool used for CT scanning, as well as for a wide range of other disciplines, including radar imaging, geophysical imaging, nondestructive testing and medical imaging [6].

For the 2D case, the Radon transform of a function $f(x, y)$, denoted by $\Re f(\theta, s)$, is defined as the line integral of f along a line L inclined at an angle θ and at distance s from the origin. Formally,

$$\begin{aligned} \Re f(\theta, s) &= \int_L f(x, y) du \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos\theta + y \sin\theta - s) dx\, dy, \end{aligned} \tag{1}$$

where $\delta(x)$ is Dirac's delta function.

There is a fundamental relationship between the 2D Fourier transform of a function and the 1D Fourier transform of its Radon transform. This relation is called the "Fourier slice theorem", and for the continuous case it stats that the 1D Fourier transform with respect to s of the projection $\Re f(\theta, s)$ is equal to a central slice, at angle θ, of the 2D Fourier transform of the function $f(x, y)$. That is,

$$\widehat{\Re f}(\theta, \xi) = \hat{f}(\xi \cos\theta, \xi \sin\theta), \tag{2}$$

where $\hat{f}$ is the Fourier transform of f. For the proof, see for example [6, 10].

For modern applications it is important to have a discrete analog of $\Re f$ for 2D digital images $I = (I(u, v) : u, v = -n/2, \ldots, n/2 - 1)$. This has been the object of attention of many authors over the last twenty years; a very large literature has ensued [1, 2, 3, 11, 12, 9]. Despite many attempts at defining a "digital Radon transform", we believe that there is at present no definition which is both intellectually and practically satisfying.

1.1 Desiderate

To support our assertion, we propose the following desiderata for a notion of digital Radon transform.

P1. Algebraic exactness The transform should be based on a clear and rigorous definition, not for example on analogy to Eq. 1, e.g., formulations such as 'we approximate the integral in Eq. 1 by a sum'.

P2. Geometric fidelity The transform should be based on true geometric lines rather than lines which wrap around or are otherwise non-geometric.

P3. Speed The transform should be rapidly computable, for example, admit an $O(N \log N)$ algorithm where N is the size of the data in I, i.e., $N = n^2$ for in the 2D case.

P4. Invertibility The transform should be invertible on its range. Moreover, there should be a fast reconstruction algorithm.

P5. Parallels with continuum theory The transform should obey relations which parallel with those of the continuum theory (for example, relations with the Fourier transform).

The many existing contributions to the literature do not offer these properties simultaneously. A complete discussion would take space we do not have, so we content ourselves with three examples, which also help to illustrate the meaning of our desiderata above.

- Fourier Approaches. Some authors [7, 11] have attempted to exploit the projection-slice theorem (Eq. 2). In the continuum theory, this says that $\Re f(\theta, \cdot)$ can be obtained by (a) performing a 2D Fourier transform, (b) obtaining a radial slice of the Fourier transform, and (c) applying a 1D inverse Fourier transform to the obtained slice. This suggests an algorithm for discrete data, by replacing steps (a) and (c) by fast Fourier transforms for data on 2D and 1D Cartesian grids, respectively. However, strictly speaking, this continuum approach is problematic since step (b) is not naturally defined on digital data: the 2D FFT outputs data in a Cartesian format, while the radial slices of the Fourier domain typically do not intersect the Cartesian grid. Therefore, some sort of interpolation is required, and so the transform is not algebraically exact. Also, even if the transform should turn out to be invertible (which may be very difficult to determine) the transform is typically not invertible by any straightforward algorithm.

- Multiscale Approaches. Other authors [2, 3, 4, 8] have attempted to exploit two-scale relations, which say that if one knows the Radon transform over four dyadic subsquares of a dyadic square these can be combined to obtain the Radon transform over the larger square. This suggests a recursive algorithm, in which the problem is broken up to the problem of computing Radon transforms over squares of smaller sizes which are then recombined. Strictly speaking, however, the driving identity is a fact about the continuum and does not directly apply to digital arrays, so that when this principle is operationalized, the results involve interpolation and other approximations, and end up being quite crude compared to what we have in mind here. Finally, the use of two-scale relations means that summation along lines is approximated by summation along line segments which are not exactly parallel and so the results can lack a certain degree of geometric fidelity.

- Algebraic Approaches. When n is a prime p, the data grid $G = \{(u, v) : 0 \leq u, v < n\}$ may be considered as the group Z_p^2, which has very special properties [12]. The "lines" $\{(ka + b \mod p,\ kc \mod p + d) : 0 \leq k < p\}$ for appropriate a, b, c, d have a very special structure: pairs of "lines" either do not intersect at all, or intersect in just one point. This property makes it possible to define an algebraically exact Radon transform for integration along "lines" which operates in $O(N \log(N))$ flops and is invertible. However, the "lines" have, for most parameters (a, b, c, d), very little connection with lines of $\mathbb{R}^2$; simple plots of such "lines" reveal that they are scattered point sets roughly equidistributed through the grid G. In effect, the "lines" wrap around (owing to the $\mod p$ in their definition), which destroys the geometric fidelity of the transform.

In this paper, we describe a notion of Radon transform for digital data which has all of our desired properties [P1]–[P5]. The notion we discuss belongs to a fourth stream of Radon research complementing the three streams of research just mentioned (Fourier-based methods, multiscale methods, and Algebraic Approaches) and represents in a certain sense the culmination of that stream. In effect, this fourth stream says that, to really make sense for digital data, the appropriate notions of continuum Radon transform and of discrete 2D Fourier domain are subtly different than the usual ones. Nevertheless, we prove that the discrete Radon transform converges to the continuous Radon transform. This property is of major theoretical and computational importance since it shows that the discrete transform is indeed an approximation of the the continuous transform, and thus can be used to replace the continuous transform in digital implementations.

The organization of this paper is as follows. In Section 2 we define the 2D discrete Radon transform. The definition in Section 2 is derived for discrete images and a continuous set of lines. Section 3 provides a detailed proof of the Fourier slice theorem, which associates the discrete Radon transform with the 2D Fourier transform. Section 4 then discritizes the parameters used by the definition of Section 2, presents the relation between the discrete definition and the pseudo-polar Fourier transform [14], and describes fast and accurate forward and inverse algorithms that are based on the pseudo-polar Fourier transform. Finally, Section 5 proves that the discrete Radon transform converges to the continuous Radon transform as the discretization step goes to zero.

2 Construction of the transform

We are looking for a definition for the Radon transform for discrete data that satisfies properties [P1]–[P5]. Loosely speaking, the discrete Radon transform is defined by summing the samples of $I(u,v)$ along lines. The two key issues of the construction are how to process lines of the discrete transform when they do not pass through grid points, and how to choose the set of lines which we sum in order to achieve geometric fidelity, rapid computation, and invertibility?

Generally speaking, the discrete Radon transform is defined by summing the discrete samples of the image $I(u,v)$ along lines with absolute slope less than 1, for a reason that will be explained later. For lines of the form $y = sx + t$ ($|s| \leq 1$), we traverse each line by unit horizontal steps $x = -n/2, \ldots, n/2-1$, and for each x, we interpolate the image sample at position (x, y) by using trigonometric interpolation along the corresponding image column. For lines of the form $y = sx + t$ ($|s| \geq 1$), we rephrase the line equation as $x = s'y + t'$ ($|s'| \leq 1$). In this case, we traverse the line by unit vertical steps, and for each integer y, we interpolate the value at the x coordinate $x = s'y + t'$ by using trigonometric interpolation along the corresponding row.

Note that we did not restrict the values of s to be within a discrete set of values. As for now, we allow s to take any real value in $[-1, 1]$. The requirement for slopes less than 1 induces two families of lines:

basically horizontal line is a line of the form $y = sx + t$ where $|s| \leq 1$.

basically vertical line is a line of the form $x = sy + t$ where $|s| \leq 1$.

Using these line families we define the 2D Radon transform for discrete images as

Definition 2.1 (2D Radon transform for discrete images)**.** Let $I(u, v)$, $u, v = -n/2, \ldots, n/2-1$, be a $n \times n$ array. Let s be a slope such that $|s| \leq 1$, and let t be an intercept such that $t = -n, \ldots, n$. Then,

$$Radon(\{y = sx + t\}, I) = \sum_{u=-n/2}^{n/2-1} \widetilde{I}^1(u, su + t), \tag{3}$$

$$Radon(\{x = sy + t\}, I) = \sum_{v=-n/2}^{n/2-1} \widetilde{I}^2(sv + t, v), \tag{4}$$

where

$$\widetilde{I}^1(u, y) = \sum_{v=-n/2}^{n/2-1} I(u, v) D_m(y - v), \quad u = -n/2, \ldots, n/2-1, \; y \in \mathbb{R}, \tag{5}$$

$$\widetilde{I}^2(x, v) = \sum_{u=-n/2}^{n/2-1} I(u, v) D_m(x - u), \quad v = -n/2, \ldots, n/2-1, \; x \in \mathbb{R}, \tag{6}$$

and

$$D_m(t) = \frac{\sin(\pi t)}{m \sin(\pi t/m)}, \quad m = 2n+1. \tag{7}$$

Moreover, for an arbitrary line l with slope s and intercept t, such that $|s| \leq 1$ and $t = -n, \dots, n$, the Radon transform is given by

$$(RI)(s,t) = \begin{cases} Radon(\{y = sx + t\}, I) & l \text{ is a basically horizontal line} \\ Radon(\{x = sy + t\}, I) & l \text{ is a basically vertical line.} \end{cases} \tag{8}$$

$\widetilde{I}^1$ and $\widetilde{I}^2$ in Eqs. 5 and 6 are column-wise and row-wise interpolated versions of I, respectively, using the *Dirichlet kernel* D_m with $m = 2n+1$.

Next, we explain the selection of the parameters t and m in Definition 2.1.

2.1 Selection of the parameters t and m

Let $y = sx + t$ be a basically horizontal line, that is $|s| \leq 1$. This selection of s partitions the plane between basically horizontal and vertical lines, and moreover, gives a complete symmetry in handling basically horizontal and vertical lines. Each result applies to basically horizontal lines will apply analogously to basically vertical lines, and vice versa. Hence, once s is fixed, it remains to find the range t of relevant intercepts. As we see in Section 4, using $t = -n, \dots, n$ assures that the transform is invertible, and moreover, when considering the analogy to the continuous case, this range of t provides us with all the lines that intersect the bounding box of $I(u,v)$ (non-trivial projections).

Next, we explain the requirement $m = 2n+1$. Suppose we were using the kernel D_n of length n instead of D_m. Taking some line $y = sx + t$ (Fig. 1) and evaluating y_1 at x_1, we obtain from Eq. 5

$$\widetilde{I}^1(x_1, y_1) = \sum_{v=-n/2}^{n/2-1} I(x_1, v) D_n(y_1 - v).$$

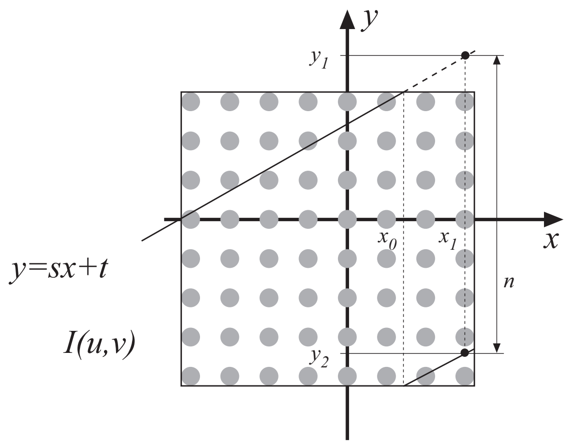

Figure 1: Summation of wraparound line without padding

It is easy to verify that if $y_2 = y_1 - n$, then (see Fig. 1)

$$\widetilde{I}^1(x_1, y_1) = \pm \sum_{v=-n/2}^{n/2-1} I(x_1, v) D_n(y_2 - v). \tag{9}$$

Equation 9 states that evaluating $\widetilde{I}^1$ at a point y_1 is the same (in absolute value) as evaluating $\widetilde{I}^1$ at point $y_1 - n$. Taking a concrete example, evaluating $\widetilde{I}^1$ at $y_1 = n/2$, which is outside the domain of $I(u, v)$ and therefore should be interpolated as a small value, is the same as evaluating $\widetilde{I}^1$ at $y = -n/2$, which is a true sample of $I(u, v)$. This means that our line exhibits a wraparound, and the summation is not over true straight geometric lines. In other words, when we evaluate a sample (x_0, y_0), $y_0 \geq n/2$, the wraparound effect causes the interpolation to take into account non-zero samples of $I(u, v)$ that are geometrically far from y_0. Specifically, we are actually evaluating $y_0 - n$, and the geometric meaning is that our summation is not over true geometric lines, but over some other “broken” lines (see Fig. 1 for the solid broken line).

To eliminate the wraparound effect we pad I with zeros (see Fig. 2), so that although wraparound occurs due to the periodic nature of the Dirichlet kernel, it will be over zeros and not over true samples of $I(u, v)$. See Fig. 2 for an

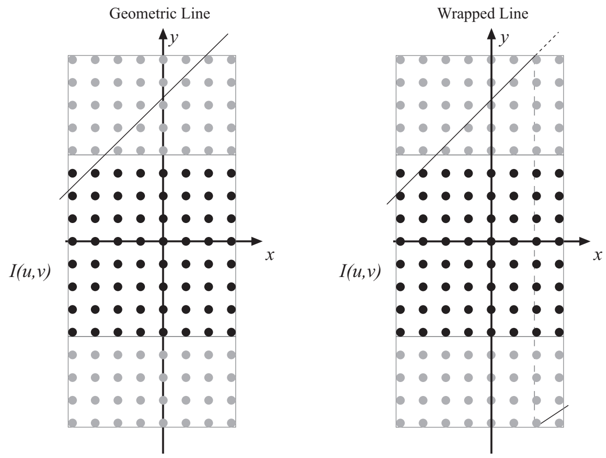

Figure 2: Summation of wraparound line with padding

illustration of geometric and wraparound lines in a padded image. Although the line on the right image exhibits wraparound, the wraparound is over zeros and not over true samples of $I(u, v)$. According to our choice of s and t, it follows that a line $y = sx+t$ with $x = -n/2, \ldots, n/2-1$ satisfies $y_0 < 3n/2$. When wraparound is taken into consideration (due to periodicity of length $m = 2n + 1$), this y_0 is equal to $y_1 = y_0 - (2n + 1) < -n/2$. Hence, we have to extend I with at least $n + 1$ zeros to separate y_1 (the wrapped around version of y_0) from true samples of I. Thus, setting $m = 2n + 1$ assures that samples with $n/2 \leq y_0 < 3n/2$ not to overlap with samples of I with $-n/2 \leq y < n/2$.

When basically vertical lines are considered, we obtain (due to symmetry) the same result, i.e., we have to use an interpolation kernel D_m with $m = 2n + 1$.

2.2 Representation of lines using angles

In order to derive the 2D Radon transform (Definition 2.1) in a more natural way, we rephrase it using angles instead of slopes. For a basically horizontal line $y = sx + t$ with $|s| \leq 1$, we express s as $s = \tan\theta$ with $\theta \in [-\pi/4, \pi/4]$, where θ is the angle between the line and the positive direction of the x-axis.

Using this notation, a basically horizontal line has the form $y = (\tan\theta)x + t$ with $\theta \in [-\pi/4, \pi/4]$. Given a basically vertical line $x = sy + t$ with $|s| \leq 1$, we express s as $s = \cot\theta$ with $\theta \in [\pi/4, 3\pi/4]$, where θ is again the angle between the line and the positive direction of the x-axis. Hence, a basically vertical line has the form $x = (\cot\theta)y + t$ with $\theta \in [\pi/4, 3\pi/4]$.

Using this parametric representation we rephrase the definition of the Radon transform (Definition 2.1) as

Definition 2.2. Let $I(u, v)$, $u, v = -n/2, \dots, n/2 - 1$, be a $n \times n$ array. Let $\theta \in [-\pi/4, 3\pi/4]$, and let t be an intercept such that $t = -n, \dots, n$. Then,

$$R_\theta I(t) = \begin{cases} Radon(\{y = (\tan\theta)x + t\}, I) & \theta \in [-\pi/4, \pi/4] \\ Radon(\{x = (\cot\theta)y + t\}, I) & \theta \in [\pi/4, 3\pi/4], \end{cases} \tag{10}$$

where $Radon(\{y = sx + t\}, I)$ and $Radon(\{x = sy + t\}, I)$ are given in Eqs. 3 and 4, respectively.

The Radon transform in Definition 2.2 operates on discrete images $I(u, v)$ while θ is continuous. In Section 4 we discretize the parameter θ in order to get a "discrete Radon transform" that satisfies properties [P1]–[P5] given in Section 1.

3 Fourier slice theorem

The continuous Fourier slice theorem (Eq. 2) allows to evaluate the continuous Radon transform using the 2D Fourier transform. We are looking for a similar relation between the discrete Radon transform and the discrete Fourier transform of an image. We can then use such a relation to compute the discrete Radon transform.

To construct such a relation we define some auxiliary operators which enable us to rephrase the definition of the discrete Radon transform (Definition 2.2) in a more convenient way. Throughout this section we denote by $\mathbb{C}^m$ the set of complex-valued vectors of length m, indexed from $-\lfloor m/2 \rfloor$ to $\lfloor (m-1)/2 \rfloor$. Also, we denote by $\mathbb{C}^{k\times l}$ the set of 2D complex-valued images of dimensions $k \times l$. Each image $I \in \mathbb{C}^{k\times l}$ is indexed as $I(u,v)$, where $u = -\lfloor k/2 \rfloor, \ldots, \lfloor (k-1)/2 \rfloor$ and $v = -\lfloor l/2 \rfloor, \ldots, \lfloor (l-1)/2 \rfloor$.

Definition 3.1 (Translation operator)**.** Let $\alpha \in \mathbb{C}^m$ and $\tau \in \mathbb{R}$. The *translation* operator $T_\tau : \mathbb{C}^m \to \mathbb{C}^m$ is given by

$$(T_\tau \alpha)_u = \sum_{i=-n}^{n} \alpha_i D_m(u - i - \tau), \quad m = 2n+1, \tag{11}$$

where $u = -n, \ldots, n$ and D_m is given by Eq. 7.

The translation operator T_τ takes a vector of length m and translates it by τ by using trigonometric interpolation.

Lemma 3.2. *Let T_τ be the translation operator given by Definition 3.1. Then,*

$$\operatorname{adj}\, T_\tau = T_{-\tau}.$$

The proof easily follows from Definition 3.1 and the fact that $D_m(t)$ is an even function.

An important property of the translation operator T_τ is that the translation of exponentials is algebraically exact. In other words, translating a vector of samples of the exponential $e^{2\pi\imath kx/m}$ is the same as resampling the exponential at the translated points. This observation is of great importance for proving the Fourier slice theorem.

Lemma 3.3. *Let $m = 2n+1$, $\varphi(x) = e^{2\pi\imath kx/m}$. Define the vector $\phi \in \mathbb{C}^m$ as $\phi = (\varphi(t) : t = -n, \ldots, n)$. Then, for arbitrary $\tau \in \mathbb{R}$*

$$(T_\tau \phi)_t = \varphi(t - \tau), \quad t = -n, \ldots, n.$$

The proof follows since $(T_\tau \phi)_x$ and $\varphi(x-\tau)$, $x \in \mathbb{R}$, are two trigonometric polynomials of degree m that coincide on $m = 2n+1$ points $t = -n, \ldots, n$ [15].

Definition 3.4. The *extension* operators $E^1 : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times m}$ and $E^2 : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times n}$ are given by

$$E^i I(u,v) = \begin{cases} I(u,v) & u,v = -n/2, \ldots, n/2-1 \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where $i = 1,2$, and $m = 2n+1$.

E^1 is an operator that takes an array of size $n \times n$ and produces an array of size $(2n+1) \times n$ ($2n+1$ rows and n columns) by adding $n/2+1$ zero rows at the top of the array and $n/2$ zero rows at the bottom of the array. Similarly, E^2 corresponds to padding the array I with $n+1$ zero columns, $n/2+1$ at the right and $n/2$ at the left.

Definition 3.5. The *truncation* operators $U^1 : \mathbb{C}^{n\times m} \to \mathbb{C}^{n\times n}$ and $U^2 : \mathbb{C}^{m\times n} \to \mathbb{C}^{n\times n}$ are given by

$$U^i I(u,v) = I(u,v), \quad u,v = -n/2, \ldots, n/2-1, \tag{13}$$

where $i = 1,2$ and $m = 2n+1$.

The operator U^1 removes $n/2+1$ rows from the top of the array and $n/2$ rows from the bottom of the array, recovering a $n \times n$ image. Similarly, U^2 removes $n/2+1$ columns from the right and $n/2$ columns from the left of the array.

Lemma 3.6. *Let E^1 and U^1 be the extension and truncation operators, given by Definitions 3.4 and 3.5, respectively. Then,*

$$\text{adj } E^1 = U^1, \qquad U^2 = \text{adj } E^2.$$

Definition 3.7. Let $I(u,v)$, $u,v = -n/2,\ldots,n/2-1$, be a $n \times n$ image. The *shearing operators* $S^1 : \mathbb{C}^{n\times m} \to \mathbb{C}^{n\times m}$ and $S^2 : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$, $m = 2n+1$, are given by

$$(S^1_\theta I)(u,v) = (T_{-u\tan\theta} I(u,\cdot))_v, \quad \theta \in [-\pi/4, \pi/4], \tag{14}$$

$$(S^2_\theta I)(u,v) = (T_{-v\cot\theta} I(\cdot,v))_u, \quad \theta \in [\pi/4, 3\pi/4]. \tag{15}$$

Lemma 3.8. *For* $t = -n,\ldots,n$*,*

$$RI(\theta,t) = \begin{cases} \sum_{u=-n/2}^{n/2-1} (S^1_\theta(E^1 I))(u,t), & \theta \in [-\pi/4,\pi/4], \\ \sum_{v=-n/2}^{n/2-1} (S^2_\theta(E^2 I))(t,v), & \theta \in [\pi/4, 3\pi/4]. \end{cases}$$

Proof. For simplicity we denote $\dot{I}^1 \triangleq E^1 I$. For $\theta \in [-\pi/4,\pi/4]$ in Definition 2.2 we have

$$RI(\theta,t) = Radon\{y = s_1 x + t, I\} = \sum_{u=-n/2}^{n/2-1} \widetilde{I}^1(u, s_1 u + t), \tag{16}$$

where $s = \tan\theta$ and $\widetilde{I}^1$ is given by Eq. 5. By substituting Eq. 5 in Eq. 16 we get

$$RI(\theta,t) = \sum_{u=-n/2}^{n/2-1} \sum_{v=-n/2}^{n/2-1} I(u,v) D_m(s_1 u + t - v).$$

From Definition 3.1

$$(T_{-\tau}\dot{I}^1(u,\cdot))_t = \sum_{v=-n}^{n} \dot{I}^1(u,v) D_m(t - v + \tau), \quad u = -n/2,\ldots,n/2-1,$$

and by substituting $\tau = u\tan\theta$ we get

$$(T_{-u\tan\theta}\dot{I}^1(u,\cdot))_t = \sum_{v=-n}^{n} \dot{I}^1(u,v) D_m(t - v + u\tan\theta). \tag{17}$$

From Eq. 17 and the definition of S^1_θ we get

$$\begin{aligned}
\sum_{u=-n/2}^{n/2-1} (S^1_\theta \dot{I}^1)(u,t) &= \sum_{u=-n/2}^{n/2-1} (T_{-u\tan\theta}\dot{I}(u,\cdot))_t \\
&= \sum_{u=-n/2}^{n/2-1} \sum_{v=-n}^{n} \dot{I}^1(u,v) D_m(t-v+u\tan\theta) \qquad (18) \\
&= \sum_{u=-n/2}^{n/2-1} \sum_{v=-n/2}^{n/2-1} I(u,v) D_m(t-v+u\tan\theta), \qquad (19)
\end{aligned}$$

where Eq. 19 follows from Eq. 18 since $\dot{I}^1(u,v) = 0$ for $v < -n/2$ or $v \geq n/2$ and $\dot{I}^1(u,v) = I(u,v)$ otherwise. Hence, for $\theta \in [-\pi/4, \pi/4]$

$$(RI)(\theta,t) = \sum_{u=-n/2}^{n/2-1} (S^1_\theta(E^1 I))(u,t).$$

The proof for $\theta \in [\pi/4, 3\pi/4]$ is similar. □

Definition 3.9. Let $\psi \in \mathbb{C}^m$, $m = 2n+1$. The backprojection operators $B^1_\theta : \mathbb{C}^m \to \mathbb{C}^{n\times m}$ and $B^2_\theta : \mathbb{C}^m \to \mathbb{C}^{m\times n}$ are given by

$$\begin{aligned}
(B^1_\theta \psi)(u,\cdot) &= T_{u\tan\theta}\psi, \quad \theta \in [-\pi/4, \pi/4], \\
(B^2_\theta \psi)(\cdot,v) &= T_{v\cot\theta}\psi, \quad \theta \in [\pi/4, 3\pi/4].
\end{aligned} \qquad (20)$$

Lemma 3.10.

$$\text{adj } B^1_\theta = \sum_u S^1_\theta, \qquad (21)$$

$$\text{adj } B^2_\theta = \sum_v S^2_\theta. \qquad (22)$$

The proof follows easily from Definitions 3.7 and 3.9, Lemma 3.1, and the definition of the inner product $\langle \cdot, \cdot \rangle$.

We can now given an explicit formula for the adjoint Radon transform.

Theorem 3.11. *The adjoint Radon transform* adj $R_\theta : \mathbb{C}^m \to \mathbb{C}^{n\times n}$ *is given by*

$$\text{adj } R_\theta = \begin{cases} U^1 \circ B^1_\theta & \theta \in [-\pi/4, \pi/4] \\ U^2 \circ B^2_\theta & \theta \in [\pi/4, 3\pi/4], \end{cases} \tag{23}$$

where U^i, $i = 1, 2$, is given by Definition 3.5 and B^i_θ, $i = 1, 2$, is given by Definition 3.9.

Proof. From Lemma 3.8 it follows that the Radon transform $(RI)(\theta, t)$ for $\theta \in [-\pi/4, \pi/4]$ and $t = -n, \dots, n$, can be written as

$$R_\theta I(t) \triangleq (RI)(\theta, t) = \sum_{u=-n/2}^{n/2-1} (S^1_\theta(E^1 I))(u, t),$$

or,

$$R_\theta = (\sum_u S^1_\theta) \circ E^1. \tag{24}$$

Equation 24 implies that R_θ is a superposition of two operators: the extension operator E^1, given by Definition 3.4, and the summation of the shearing operator $\sum_u S^1_\theta$, where S^1_θ is given by Definition 3.7. Therefore,

$$\text{adj } R_\theta = \text{adj } E^1 \circ \text{adj } (\sum_u S^1_\theta).$$

Thus, by using Lemmas 3.6 and 3.10 we get

$$\text{adj } R_\theta = U^1 \circ B^1_\theta.$$

The proof for $\theta \in [\pi/4, 3\pi/4]$ is similar. ☐

We next examine how the adjoint Radon transform operates on the vector $\phi^{(k)} = (\varphi^{(k)}(t) : t = -n, \dots, n)$, where $\varphi^{(k)}(t) = e^{2\pi \imath kt/m}$. For $\theta \in [-\pi/4, \pi/4]$ and $u, v = -n/2, \dots, n/2 - 1$ we have

$$\begin{aligned} (\text{adj } R_\theta \phi^{(k)})(u, v) &= (U^1 B^1_\theta \phi^{(k)})(u, v) && \text{(by Theorem 3.11)} \\ &= (B^1_\theta \phi^{(k)})(u, v) = (T_{u \tan\theta} \phi^{(k)})_v && \text{(by Eq. 20)} \\ &= \varphi^{(k)}(v - u\tan\theta) && \text{(by Lemma 3.3)} \\ &= e^{\frac{2\pi\imath}{m} k(v - u\tan\theta)}, \end{aligned}$$

and hence,

$$(\text{adj } R_\theta \phi^{(k)})(u,v) = e^{\frac{2\pi\imath}{m}k(v-s_1u)}. \tag{25}$$

Similarly, for $\theta \in [\pi/4, 3\pi/4]$ and $u, v = -n/2, \dots, n/2-1$ we get

$$(\text{adj } R_\theta \phi^{(k)})(u,v) = e^{\frac{2\pi\imath}{m}k(u-v\cot\theta)}. \tag{26}$$

Theorem 3.12 (Fourier slice theorem)**.** *Let $I(u,v)$ be a $n \times n$ image, $u, v = -n/2, \dots, n/2-1$, and let $m = 2n+1$. Then,*

$$(\widehat{R_\theta I})(k) = \begin{cases} \hat{I}(-s_1k, k), & s_1 = \tan\theta, \quad \theta \in [-\pi/4, \pi/4], \\ \hat{I}(k, -s_2k), & s_2 = \cot\theta, \quad \theta \in [\pi/4, 3\pi/4], \end{cases}$$

where $\widehat{R_\theta I}(k)$ is the 1D DFT of the discrete Radon transform, given by Definition 2.2, with respect to the parameter t, $k = -n, \dots, n$, and $\hat{I}$ is the trigonometric polynomial

$$\hat{I}(\xi_1, \xi_2) = \sum_{u=-n/2}^{n/2-1} \sum_{v=-n/2}^{n/2-1} I(u,v) e^{-\frac{2\pi\imath}{m}(\xi_1 u + \xi_2 v)}. \tag{27}$$

Proof. Denote $\phi_j^{(k)} = \varphi^{(k)}(j) = e^{\frac{2\pi\imath}{m}kj}$. For $\theta \in [-\pi/4, \pi/4]$ we have

$$\begin{aligned}
\widehat{R_\theta I}(k) &= \sum_{j=-n}^{n} R_\theta I(j) e^{-2\pi\imath kj/m} \\
&= \sum_{j=-n}^{n} R_\theta I(j) \left(\phi_j^{(k)}\right)^* \\
&= \langle R_\theta I, \phi^{(k)} \rangle \\
&= \langle I, \text{adj } R_\theta \phi^{(k)} \rangle \\
&= \sum_{u=-n/2}^{n/2-1} \sum_{v=-n/2}^{n/2-1} I(u,v) \left(e^{\frac{2\pi\imath}{m}k(v-s_1u)}\right)^* \qquad \text{(by Eq. 25)} \\
&= \sum_{u=-n/2}^{n/2-1} \sum_{v=-n/2}^{n/2-1} I(u,v) e^{\frac{-2\pi\imath}{m}k(v-s_1u)} \\
&= \hat{I}(-s_1k, k),
\end{aligned}$$

where $\hat{I}$ is the trigonometric polynomial given by Eq. 27.

The proof for $\theta \in [\pi/4, 3\pi/4]$ is similar. □

4 Discretization and fast algorithms

The Radon transform, given by Definition 2.2, and the Fourier slice theorem, given by Theorem 3.12, were defined for discrete images and a continuous set of lines. Specifically, $R_\theta I(t)$ is defined for any angle in the range $[-\pi/4, 3\pi/4]$. For our transform to be fully discrete, we must discretize the set of angles. We denote such a discrete set of angles by Θ. By using the discrete set of intercepts

$$T \triangleq \{-n, \ldots, n\}, \tag{28}$$

we define the discrete Radon transform as

$$RI = \{R_\theta I(t) \mid \theta \in \Theta,\ t \in T\}, \tag{29}$$

where R_θ is given by Definition 2.2 and Θ is the set of angles induced by lines with equally spaced slopes. Specifically,

$$\Theta \triangleq \Theta_1 \cup \Theta_2, \tag{30}$$

where

$$\Theta_2 = \{\arctan(2l/n) \quad \mid l = -n/2, \ldots, n/2\}, \tag{31}$$

$$\Theta_1 = \{\pi/2 - \arctan(2l/n) \mid l = -n/2, \ldots, n/2\}. \tag{32}$$

If we take an arbitrary element from Θ_2, i.e. $\theta_2^l = \arctan(2l/n)$, then, the slope of the line corresponding to θ_2^l is $s = \tan\theta_2^l = 2l/n$. By inspecting two elements θ_2^l and θ_2^{l+1} we can see that the difference between the slopes that correspond to the two angles is

$$s_2 - s_1 = \tan\theta_2^{l+1} - \tan\theta_2^l = \frac{2(l+1)}{n} - \frac{2l}{n} = \frac{2}{n},$$

which means that our angles define a set of equally spaced slopes.

Theorem 3.12 holds also for the discrete set Θ. For $\theta \in \Theta^2$ (Eq. 31) we have from Theorem 3.12 that $\widehat{R_\theta I}(k) = \hat{I}(-s_1 k, k)$, where $s_1 = \tan\theta$. Since $\theta \in \Theta^2$ has the form $\theta = \arctan(2l/n)$, it follows that $s_1 = \tan(\arctan(2l/n)) = 2l/n$ and

$$\widehat{R_\theta I}(k) = \hat{I}(-\frac{2l}{n}k, k), \quad k = -n, \ldots, n. \tag{33}$$

For $\theta \in \Theta^1$ (Eq. 32), we have from Theorem 3.12 that $\widehat{R_\theta I}(k) = \hat{I}(k, -s_2 k)$, where $s_2 = \cot\theta$. Since $\theta \in \Theta^1$ has the form $\theta = \pi/2 - \arctan(2l/n)$, it follows that $s_2 = \cot(\pi/2 - \arctan(2l/n)) = 2l/n$ and

$$\widehat{R_\theta I}(k) = \hat{I}(k, -\frac{2l}{n}k), \quad k = -n, \ldots, n. \tag{34}$$

Equations 33 and 34 show that $R_\theta I$ is obtained by resampling the trigonometric polynomial $\hat{I}$, given by Eq. 27, on the pseudo-polar grid Ω_{pp}, defined in [14] as

$$\Omega_{pp} \triangleq \Omega_{pp}^1 \cup \Omega_{pp}^2, \tag{35}$$

where

$$\Omega_{pp}^1 \triangleq \{(-\frac{2l}{n}k, k) \mid -n/2 \le l \le n/2, \ -n \le k \le n\} \tag{36}$$

$$\Omega_{pp}^2 \triangleq \{(k, -\frac{2l}{n}k) \mid -n/2 \le l \le n/2, \ -n \le k \le n\}. \tag{37}$$

Specifically, if we use the notation $\hat{I}_{\Omega_{pp}^1}$ and $\hat{I}_{\Omega_{pp}^2}$ to denote the samples of $\hat{I}$ (Eq. 27) on the sets Ω_{pp}^1 and Ω_{pp}^2, respectively, then,

$$\widehat{R_\theta I}(k) = \hat{I}_{\Omega_{pp}^1}(k, l), \quad \theta \in \Theta^2, \tag{38}$$

$$\widehat{R_\theta I}(k) = \hat{I}_{\Omega_{pp}^2}(k, l), \quad \theta \in \Theta^1. \tag{39}$$

From Eqs. 38 and 39 we derive the relation

$$\begin{aligned} R_\theta I &= F_k^{-1} \circ \hat{I}_{\Omega_{pp}^1}, \qquad \theta \in \Theta^2, \\ R_\theta I &= F_k^{-1} \circ \hat{I}_{\Omega_{pp}^2}, \qquad \theta \in \Theta^1, \end{aligned} \tag{40}$$

where F_k^{-1} is the 1D inverse Fourier transform along each row of the array $\hat{I}_{\Omega_{pp}^s}$, $s = 1, 2$ (along the parameter k).

The fast algorithm that computes the pseudo-polar Fourier transform [14] immediately gives an algorithm for computing the discrete Radon transform. To see this, consider a row in RI (Eq. 29), which corresponds to a constant θ. The values of the discrete Radon transform that correspond to θ are computed by applying F_k^{-1} to a row of $\hat{I}_{\Omega_{pp}^s}$, $s = 1, 2$. Thus, the computation of RI requires to apply F_k^{-1} to all rows of $\hat{I}_{\Omega_{pp}^s}$, $s = 1, 2$ ($2n+2$ rows). Since each application of F_k^{-1} operates on a vector of length $2n+1$ (a row of $\hat{I}_{\Omega_{pp}^1}$ or $\hat{I}_{\Omega_{pp}^2}$), it requires $O(n \log n)$ operations for a single application, and a total of $O(n^2 \log n)$ operations for the required $2n + 2$ calls to F_k^{-1}. Thus, once we compute the arrays $\hat{I}_{\Omega_{pp}^s}$, $s = 1, 2$, it requires $O(n^2 \log n)$ operations to compute RI. Since computing $\hat{I}_{\Omega_{pp}^s}$, $s = 1, 2$, requires $O(n^2 \log n)$ operations [14], the total complexity of computing the values of RI is $O(n^2 \log n)$ operations.

Invertibility of the 2D discrete Radon transform RI also follows from Eq. 40. F_k^{-1} is invertible and can be rapidly computed by using the inverse fast Fourier transform. $\hat{I}_{\Omega_{pp}}$ is invertible and can be rapidly inverted as described in [14]. Thus, the discrete Radon transform is invertible, and can be inverted by applying the inverse FFT on each row of RI ($O(n^2 \log n)$ operations), followed by an inversion of $\hat{I}_{\Omega_{pp}}$ ($O(n^2 \log n)$ operations). Hence, inverting the discrete Radon transform requires $O(n^2 \log n)$ operations.

5 Convergence

In this section we prove that the discrete Radon transform converges to the continuous Radon transform as the discretization step goes to zero.

5.1 Alternative representation of the continuous Radon transform

Let $f(x,y)$ be the X-ray attenuation coefficient of some physical object. The continuous two-dimensional Radon transform along a straight line L is defined as the line integral along L. Usually L is specified by the equation $x\cos\theta + y\sin\theta = t$ and the Radon transform is defined by equation

$$\Re f(\theta,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - t)\,dx\,dy. \tag{41}$$

In this section we use another (equivalent) definition. We divide the set of all lines in $\mathbb{R}^2$ into basically horizontal and basically vertical lines (Section 2). We denote the Radon transform of f along basically horizontal lines by $\Re_x f$ and along basically vertical lines by $\Re_y f$. Any basically horizontal line can be specified by $y = sx + t$, $s \in [-1,1]$, and the integral of f along this line is given by

$$\Re_x f(s,t) = \sqrt{1+s^2}\int_{-\infty}^{\infty} f(x, t+sx)\,dx. \tag{42}$$

Any basically vertical line can be specified by $x = sy + t$, $s \in [-1,1]$, and the integral of f along this line is given by

$$\Re_y f(s,t) = \sqrt{1+s^2}\int_{-\infty}^{\infty} f(t+sy, y)\,dy. \tag{43}$$

When the function $f(x,y)$ represents an object that is bounded in space, the limits of integration in Eqs. 42 and 43 can be made finite. Indeed, assume that there exists a positive constant D such that $f(x,y) = 0$ whenever $|x| \geq \frac{D}{2}$ or $|y| \geq \frac{D}{2}$. This ensures that all vertical and horizontal shears of $f(x,y)$ fit into a square of size $D \times D$. Then, for any $s \in [-1,1]$ we have

$$\Re_x f(s,t) = \sqrt{1+s^2}\int_{-D}^{D} f(x, t+sx)dx \tag{44}$$

and $\Re_x f(s,t) = 0$ whenever $t \notin [-D, D]$. Similarly,

$$\Re_y f(s,t) = \sqrt{1+s^2}\int_{-D}^{D} f(t+sy, y)dy \tag{45}$$

and $\Re_y f(s,t) = 0$ whenever $t \notin [-D, D]$.

5.2 Mathematical preliminaries

In this section we present several definitions and theorems used in subsequent derivations.

Definition 5.1. A trigonometric polynomial of order N is an expression of the form

$$T(x) = \sum_{n=-N}^{N} c_n e^{inx}, \tag{46}$$

where c_n are complex numbers.

Theorem 5.2. *([15] Uniqueness of a trigonometric interpolating polynomial)*

Given $2N+1$ points $x_{-N}, \ldots, x_0, \ldots, x_N$, which are distinct modulo 2π, and arbitrary numbers $y_{-N}, \ldots, y_0, \ldots, y_N$, there always exists a unique polynomial (Eq. 46) such that $T(x_k) = y_k$, $k = -N, \ldots, N$.

The polynomial $T(x)$ is called the *(trigonometric) interpolating polynomial* that corresponds to points x_k and values y_k. The points $x_{-N}, \ldots, x_0, \ldots, x_N$ are often called *fundamental* or *nodal* points of the interpolation, or "interpolation nodes".

The trigonometric interpolating polynomial that corresponds to values $\{x_n\}_{n=-N}^{N}$ and points $\{\frac{2\pi}{M}n\}_{n=-N}^{N}$ is explicitly given by

$$x(t) = \frac{1}{M} \sum_{k=-N}^{N} \widehat{x}_k e^{ikt} = \sum_{n=-N}^{N} x_n D_M \left(n - \frac{M}{2\pi} t \right),$$

where $M = 2N+1$, $\{\widehat{x}_k\}_{k=-N}^{N}$ is the 1D DFT of $\{x_n\}_{n=-N}^{N}$, and

$$D_M(t) = \frac{1}{M} \frac{\sin(\pi t)}{\sin(\frac{\pi}{M} t)} = \frac{1}{M} \sum_{k=-N}^{N} e^{i \frac{2\pi}{M} kt}$$

is the Dirichlet kernel of order N.

Definition 5.3. A two-dimensional trigonometric polynomial of order N is an expression of the form

$$T(x, y) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} c_{k,l} e^{i[kx+ly]},$$

where $c_{k,l}$ are complex numbers.

A two-dimensional trigonometric polynomial of degree N that interpolates an arbitrary set of values $\{x_{u,v}\}_{u,v=-N}^{N}$ at nodes $\left\{\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v\right)\right\}_{u,v=-N}^{N}$ is explicitly given by

$$x(t,s) = \frac{1}{M^2}\sum_{k,l=-N}^{N} \widehat{x}_{k,l} e^{\imath[kt+ls]} = \sum_{u,v=-N}^{N} x_{u,v} D_M\left(u - \frac{M}{2\pi}t\right) D_M\left(v - \frac{M}{2\pi}s\right),$$

where $M = 2N+1$ and $\{\widehat{x}_{k,l}\}_{u,v=-N}^{N}$ is the 2D DFT of $\{x_{u,v}\}_{u,v=-N}^{N}$. Such an interpolating polynomial is unique.

Definition 5.4. Let $f : \mathbb{R} \to \mathbb{R}$. We denote the one-dimensional trigonometric interpolating polynomial of degree N corresponding to points $\left\{\frac{2\pi}{M}u\right\}_{u=-N}^{N}$ and values $\left\{f\left(\frac{2\pi}{M}u\right)\right\}_{u=-N}^{N}$ by $f_N(x)$. This function is explicitly given by

$$f_N(x) = \sum_{n=-N}^{N} f\left(\frac{2\pi}{M}n\right) D_M\left(n - \frac{M}{2\pi}t\right).$$

Definition 5.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$. We denote the two-dimensional trigonometric interpolating polynomial of degree N corresponding to points $\left\{\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v\right)\right\}_{u,v=-N}^{N}$ and values $\left\{f\left(\frac{2\pi}{M}u, \frac{2\pi}{M}u\right)\right\}_{u,v=-N}^{N}$ by $f_N(x,y)$. This function is explicitly given by

$$f_N(x,y) = \sum_{u=-N}^{N}\sum_{u=-N}^{N} f\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v\right) D_M\left(u - \frac{M}{2\pi}x\right) D_M\left(v - \frac{M}{2\pi}y\right).$$

Definition 5.6. Let $D \in \mathbb{R}^+$, $f : \mathbb{R}^2 \to \mathbb{R}$. We define

$$f_N^D(x,y) \triangleq \sum_{u=-N}^{N}\sum_{v=-N}^{N} f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) D_M\left(u - \frac{M}{2D}x, v - \frac{M}{2D}y\right).$$

Note that when $D = \pi$, we have $f_N^D(x,y) = f_N(x,y)$.

Lemma 5.7. *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $N \in \mathbb{N}$, $v \in [-N : N]$. Then, for $g(x) \triangleq f\left(x, \frac{2\pi}{M}v\right)$ we have $g_N(x) = f_N\left(x, \frac{2\pi}{M}v\right)$.*

Proof. The function $f_N\left(x, \frac{2\pi}{M}v\right)$ is a trigonometric polynomial of degree N in x. For any $u \in [-N : N]$ we have $f_N\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v\right) = f\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v\right) = g\left(\frac{2\pi}{M}u\right)$. The lemma now follows from Theorem 5.2 (uniqueness of the one-dimensional trigonometric interpolating polynomial). □

Definition 5.8. (Lipschitz class $Lip_C(\alpha, \Omega)$) Let $\Omega \subseteq \mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the condition

$$|f(x) - f(y)| \le C\|x - y\|^{\alpha}, \quad 0 < \alpha \le 1,$$

for all $x, y \in \Omega$, then we say that f belongs to the class $Lip_C(\alpha, \Omega)$. When the value of the constant C is not important we say that f is Lipschitz α on Ω.

Lemma 5.9. *[13](Uniform convergence of a shifted interpolation)*

Let $A \in [0, \pi]$, $C \in \mathbb{R}^+$, $\alpha \in (0, 1]$, $N \in \mathbb{N}$. Let $f(x) \in Lip_C(\alpha, \mathbb{R})$ such that $f(x) = 0$ whenever $|x| \ge A$. Then, for any $|\delta| \le \pi - A$ we have

$$|f_N(x - \delta) - f(x - \delta)| \le \Phi(C, \alpha, N), \quad x \in [-\pi, \pi],$$

where $f_N(x)$ is given by Definition 5.4 and $\Phi(C, \alpha, N)$ is a function independent of both f and A, such that $\lim_{N\to\infty} \Phi(C, \alpha, N) = 0$.

Definition 5.10. Let $s \in [-1, 1]$, $f : \mathbb{R}^2 \to \mathbb{R}$. The horizontal shear of f, denoted $f_s(x, y)$, is defined as

$$f_s(x, y) \triangleq f(x + sy, y).$$

Lemma 5.11. *Let $f \in Lip_C(\alpha, \mathbb{R}^2)$. Then, for any $s \in [-1, 1]$ we have $f_s \in Lip_{(\sqrt{3})^{\alpha}C}(\alpha, \mathbb{R}^2)$.*

Proof. Since $f(x, y)$ is Lipschitz (see Definition 5.8), then, for any two points in $\mathbb{R}^2$ we have

$$\begin{aligned} |f_s(x_1, y_1) - f_s(x_2, y_2)| &= |f(x_1 + sy_1, y_1) - f(x_2 + sy_2, y_2)| \\ &\le C\left(\sqrt{((x_1 - x_2) + s(y_1 - y_2))^2 + (y_1 - y_2)^2}\right)^{\alpha}. \end{aligned}$$

For $s \in [-1,1]$ and for any $x, y \in \mathbb{R}^2$ we have

$$(x+sy)^2 + y^2 \le x^2 + 2|x||y| + y^2 + y^2 \le 3(x^2+y^2).$$

Therefore,

$$|f_s(x_1,y_1) - f_s(x_2,y_2)| \le (\sqrt{3})^\alpha C \left(\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}\right)^\alpha .$$

Thus, for $C_1 = (\sqrt{3})^\alpha C$ we have $f_s \in Lip_{C_1}(\alpha, \mathbb{R}^2)$. □

Lemma 5.12. *Let $C \in \mathbb{R}^+$, $\alpha \in (0,1]$, $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$. Consider $f \in Lip_C(\alpha, [t_1,t_2])$ and $\xi \in [t_1,t_2]$. Then,*

$$\left|\int_{t_1}^{t_2} f(x)dx - f(\xi)(t_2-t_1)\right| \le C|t_2-t_1|^{1+\alpha}.$$

Proof. By the mean value theorem for integrals [5], there exists $\xi_0 \in [t_1,t_2]$ such that $\int_{t_1}^{t_2} f(x)dx = f(\xi_0)(t_2-t_1)$. Therefore,

$$\left|\int_{t_1}^{t_2} f(x)dx - f(\xi)(t_2-t_1)\right| \le |f(\xi) - f(\xi_0)| \cdot |t_2-t_1|.$$

Since $f \in Lip_C(\alpha, [t_1,t_2])$, we have

$$|f(\xi) - f(\xi_0)| \le C|\xi - \xi_0|^\alpha \le C|t_2-t_1|^\alpha.$$

□

5.3 Discretization of the continuous Radon transform

In this section we define a discretization of the integral from Eq. 45. First, we consider this integral for $s = 0$.

Definition 5.13.

Let $D \in \mathbb{R}^+$, $f \in \mathbb{C}^0(\mathbb{R}^2)$. For arbitrary $x \in \mathbb{R}$ we define

$$T[D,f](x) \triangleq \int_{-D}^{D} f(x,y)\,dy.$$

Definition 5.14. Let $N \in \mathbb{N}$, $M = 2N + 1$. Let $D \in \mathbb{R}^+$, $f : \mathbb{R}^2 \to \mathbb{R}$. For arbitrary $x \in \mathbb{R}$ we define

$$T_N[D, f](x) \triangleq \frac{2D}{M} \sum_{v=-N}^{N} f\left(x, \frac{2D}{M} v\right).$$

Theorem 5.15. *(Approximation of the vertical Radon transform)*

Let $D \in \mathbb{R}^+$, $C \in \mathbb{R}^+, \alpha \in (0, 1]$. Let $N \in \mathbb{N}, M = 2N + 1$. Denote $\Omega = \{(x, y) \mid x, y \in [-D, D]\}$. Then, $T_N[D, f](x)$ converges to $T[D, f](x)$ uniformly in both $f \in Lip_C(\alpha; \Omega)$ and $x \in [-D, D]$.

Proof. Let us fix $f \in Lip_C(\alpha; \Omega)$ and $x \in [-D, D]$. We define $f_x(y) \triangleq f(x, y)$. Then,

$$|T[D, f](x) - T_N[D, f](x)| = \left| \int_{-D}^{D} f_x(y)\, dy - \frac{2D}{M} \sum_{v=-N}^{N} f_x\left(\frac{2D}{M} v\right) \right|.$$

For $v \in [-N : N]$ we define $\xi_v \triangleq \frac{2D}{M} v$. The interval $[-D, D]$ is a union of the subintervals $\left[\xi_v - \frac{D}{M}, \xi_v + \frac{D}{M}\right]$, $v \in [-N : N]$. Then,

$$|T[D, f](x) - T_N[D, f](x)| = \left| \sum_{v=-N}^{N} \int_{\xi_v - \frac{D}{M}}^{\xi_v + \frac{D}{M}} f_x(y)\, dy - \sum_{v=-N}^{N} \frac{2D}{M} f_x(\xi_v) \right|. \tag{47}$$

Since f belongs to $Lip_C(\alpha; \Omega)$, we have $f_x \in Lip_C(\alpha; [-D, D])$. By the application of Lemma 5.12, we have for arbitrary $v \in [-N : N]$

$$\left| \int_{\xi_v - \frac{D}{M}}^{\xi_v + \frac{D}{M}} f_x(y)\, dy - \frac{2D}{M} f_x(\xi_v) \right| \le C \left(\frac{2D}{M}\right)^{1+\alpha}. \tag{48}$$

Applying the triangle inequality to Eq. 47 and using Eq. 48 we get

$$\begin{aligned} |T[D, f](x) - T_N[D, f](x)| &\le \sum_{v=-N}^{N} \left| \int_{\xi_v - \frac{D}{M}}^{\xi_v + \frac{D}{M}} f_x(y)\, dy - \frac{2D}{M} f_x(\xi_v) \right| \\ &\le MC \left(\frac{2D}{M}\right)^{1+\alpha} = 2DC \left(\frac{2D}{M}\right)^{\alpha}. \end{aligned}$$

The last expression tends to zero as N grows, and it depends neither on x nor on f. □

Definition 5.16.

Let $s \in [-1, 1]$. Let $D \in \mathbb{R}^+$, $f \in \mathbb{C}^0(\mathbb{R}^2)$. For arbitrary x we define

$$T^s[D, f](x) \triangleq \int_{-D}^{D} f(x + sy, y)\, dy.$$

Note that $T^s[D, f](x) = T[D, f_s](x)$.

Definition 5.17.

Let $s \in [-1, 1]$, $N \in \mathbb{N}$, $M = 2N + 1$. Let $f : \mathbb{R}^2 \to \mathbb{R}$. For arbitrary $x \in \mathbb{R}$ we define

$$T_N^s[D, f](x) \triangleq \frac{2D}{M} \sum_{v=-N}^{N} f\left(x + s\frac{2D}{M}v, \frac{2D}{M}v\right). \tag{49}$$

Note that $T_N^s[D, f](x) = T_N[D, f_s](x)$.

Theorem 5.18. *(Approximation of the vertical Radon transform of a sheared object)*

Let $D \in \mathbb{R}^+$, let $f \in Lip_C(\alpha; \mathbb{R}^2)$ such that $f(x, y) = 0$ whenever $|x|+|y| \geq D$. Then, $T_N^s[D, f_N^D](x)$, where f_N^D is given by Definition 5.6, converges to $T^s[D, f](x)$ uniformly in $x \in [-D, D]$ and $s \in [-1, 1]$.

Proof.

It is sufficient to prove the theorem for $D = \pi$ since we can always scale the variables x and y. This affects the constant C in $Lip_C(\alpha, \mathbb{R}^2)$, but does not affect α. In the context of X-ray tomography, $f(x, y)$ describes a physical object, and therefore, scaling x and y corresponds to a change in the metric units.

By definition, $T^s[D, f](x', \pi) = T[D, f_s](x', \pi)$. By the note from Definition 5.6, for any function $g : \mathbb{R}^2 \to \mathbb{R}$ we have $g_N^D = g_N$ when $D = \pi$. Therefore, for

$D = \pi$ we have

$$\left|T^s[D,f](x) - T_N^s[D,f_N^D](x)\right| = \left|T[D,f_s](x) - \frac{2\pi}{M}\sum_{v=-N}^{N} f_N\left(x + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right)\right| \tag{50}$$

$$\leq \left|T[D,f_s](x) - \frac{2\pi}{M}\sum_{v=-N}^{N} f\left(x + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right)\right| + \tag{51}$$

$$+\left|\frac{2\pi}{M}\sum_{v=-N}^{N}\left(f\left(x + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right) - f_N\left(x + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right)\right)\right|. \tag{52}$$

The expression in Eq. 51 can be written as $|T[D,f_s](x) - T_N[D,f_s](x)|$. By Lemma 5.11, for any $s \in [-1,1]$, the function f_s belongs to $Lip_{C(\sqrt{3})^\alpha}(\alpha;\mathbb{R}^2)$. By Theorem 5.15, the expression in Eq. 51 tends to zero uniformly in both $x \in [-\pi,\pi]$ and $s \in [-1,1]$ as N grows.

Next we consider Eq. 52. For a fixed $v \in [-N : N]$, we denote $g_v(x) = f\left(x, \frac{2\pi}{M}v\right)$. This function belongs to $Lip_C(\alpha;\mathbb{R})$. By Lemma 5.7 we have $g_{v_N}(x) = f_N\left(x, \frac{2\pi}{M}v\right)$. The function $g_v(x)$ satisfies $g_v(x) = 0$ for $|x| \geq \pi - \frac{2\pi}{M}v$. From Lemma 5.9 there exists a function $\Phi(C,\alpha,N)$ that tends to zero as N grows such that for any $s \in [-1,1]$ and $x \in [-\pi,\pi]$ we have $\left|g_v\left(x + s\frac{2\pi}{M}v\right) - g_{v_N}\left(x + s\frac{2\pi}{M}v\right)\right| \leq \Phi(C,\alpha,N)$. It is important to note that the function $\Phi(C,\alpha,N)$ does not depend on v. By expanding the definition of $g_v(x)$, we see that for any $v \in [-N,N]$, $s \in [-1,1]$ and $x \in [-\pi,\pi]$ the following inequality holds

$$\left|f\left(x + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right) - f_N\left(x + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right)\right| \leq \Phi(C,\alpha,N).$$

For any $\varepsilon > 0$ there exists N_0 such that for any $N > N_0$ we have $|\Phi(C,\alpha,N)| < \frac{\varepsilon}{2\pi}$. Therefore, for $N > N_0$ the expression in Eq. 52 is less than ε for any $x \in [-\pi,\pi]$ and $s \in [-1,1]$.

□

5.4 2D discrete Radon transform as an approximation of the continuous Radon transform

Let $D \in \mathbb{R}^+$ and let $A(D) = \left\{(x,y) \mid -\frac{D}{2} < x < \frac{D}{3},\ -\frac{D}{2} < y < \frac{D}{3}\right\}$. Throughout this section we assume that the function $f(x,y)$ satisfies $f(x,y) = 0$ for $(x,y) \notin A(D)$. This assumption is made only for convenience to simplify subsequent proofs. Technically, this assumption ensures that $f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) = 0$ whenever $u \notin \left[-\frac{N}{2} : \frac{N}{2} - 1\right]$ or $v \notin \left[-\frac{N}{2} : \frac{N}{2} - 1\right]$.

Theorem 5.19. *Let $D \in \mathbb{R}^+$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y) = 0$ whenever $(x,y) \notin A(D)$. Consider the discrete object*

$$I(u,v) = f\left(\frac{2D}{M}u, \frac{2D}{M}v\right), \quad u,v \in \left[-\frac{N}{2} : \frac{N}{2}\right]. \tag{53}$$

Then, for any $u \in [-N : N]$ and $s \in [-1,1]$ we have

$$T_N^s[D, f_N^D]\left(\frac{2D}{M}u\right) = \frac{2D}{M}\, Radon(\{x = u + sy\}, I).$$

Proof. By the definition,

$$T_N^s[D,f]\left(\frac{2D}{M}u\right) \triangleq \frac{2D}{M} \sum_{v=-N}^{N} f_N^D\left(\frac{2D}{M}u + s\frac{2D}{M}v, \frac{2D}{M}v\right). \tag{54}$$

From the definition of $f_N^D(x,y)$ (Definition 5.6) we get

$$\begin{aligned}
f_N^D\left(\frac{2D}{M}u + s\frac{2D}{M}v, \frac{2D}{M}v\right) &= \sum_{n,k=-N}^{N} f\left(\frac{2D}{M}n, \frac{2D}{M}k\right) D_M(n-u-sv) D_M(k-v) \\
&= \sum_{n=-N}^{N} D_M\left(n-u-sv\right) \sum_{k=-N}^{N} f\left(\frac{2D}{M}n, \frac{2D}{M}k\right) D_M\left(k-v\right) \\
&= \sum_{n=-N}^{N} f\left(\frac{2D}{M}n, \frac{2D}{M}v\right) D_M\left(n-u-sv\right),
\end{aligned} \tag{55}$$

where Eq. 55 follows since $D_M(n)$ equals one for $n = Mk$, $k \in \mathbb{Z}$, and zero otherwise.

Note that $f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) = 0$ for $u \notin \left[-\frac{N}{2} : \frac{N}{2} - 1\right]$ or $v \notin \left[-\frac{N}{2} : \frac{N}{2} - 1\right]$. Note also that $f(x, y) = 0$ whenever $|x| \geq \frac{D}{2}$ or $|y| \geq \frac{D}{2}$. Based on this observation and Eq. 55 we get

$$f_N^D\left(\frac{2D}{M}u + s\frac{2D}{M}v, \frac{2D}{M}v\right) = \sum_{n=-N/2}^{N/2-1} f\left(\frac{2D}{M}n, \frac{2D}{M}v\right) D_M\left(n - u - sv\right).$$

We substitute this expression into Eq. 54

$$T_N^s[D, f]\left(\frac{2D}{M}u\right) = \frac{2D}{M}\sum_{v=-N}^{N}\sum_{n=-N/2}^{N/2-1} f\left(\frac{2D}{M}n, \frac{2D}{M}v\right) D_M\left(n - u - sv\right).$$

As we already observed, $f\left(\frac{2D}{M}n, \frac{2D}{M}v\right) = 0$ whenever $v \notin [-N/2 : N/2 - 1]$. Therefore,

$$T_N^s[D, f]\left(\frac{2D}{M}u\right) = \frac{2D}{M}\sum_{v=-N/2}^{N/2-1}\sum_{n=-N/2}^{N/2-1} f\left(\frac{2D}{M}n, \frac{2D}{M}v\right) D_M\left(n - u - sv\right). \quad (56)$$

Now, by definition of the 2D discrete Radon transform (Definition 2.1) we have

$$\begin{aligned} Radon(\{x = u + sy\}, I) &= \sum_{v=-N/2}^{N/2-1} \widetilde{I}^2(sv + u, v) \\ &= \sum_{v=-N/2}^{N/2-1}\sum_{n=-N/2}^{N/2-1} I(n, v) D_M(sv + u - n). \end{aligned} \quad (57)$$

We substitute Eq. 53 into Eq. 57 to get

$$Radon(\{x = u + sy\}, I) = \sum_{v=-N/2}^{N/2-1}\sum_{n=-N/2}^{N/2-1} f\left(\frac{2D}{M}n, \frac{2D}{M}v\right) D_M(sv + u - n).$$

It remains to compare this expression to Eq. 56, bearing in mind that $D_M(x)$ is an even function. □

When the function $f(x, y)$ represents an object that is bounded in space, it is always possible to find D such that conditions of Theorem 5.19 are satisfied. If, in addition, we assume that $f \in Lip_C(\alpha; \mathbb{R}^2)$ for some C and α, then we can use the 2D discrete Radon transform to approximate the continuous Radon transform of this object. We prove it formally in Theorem 5.20.

Theorem 5.20. *Let $D \in \mathbb{R}^+$, $C \in \mathbb{R}^+$, and $\alpha \in (0,1]$. Let $f \in Lip_C(\alpha, \mathbb{R}^2)$ such that $f(x,y) = 0$ whenever $(x,y) \notin A(D)$. For arbitrary $N \in \mathbb{N}$ we define a discrete object I_N by*

$$I_N(u,v) \triangleq f\left(\frac{2D}{M}u, \frac{2D}{M}v\right), \quad u,v \in \left[-\frac{N}{2}, \frac{N}{2} - 1\right]. \tag{58}$$

Then, for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for any $N > N_0$

$$\left|\Re_y f\left(s, \frac{2D}{M}u\right) - \sqrt{1+s^2}\,\frac{2D}{M} Radon(\{x = u + sy\}, I_N)\right| < \varepsilon$$

for any $u \in [-N : N]$ and any $s \in [-1,1]$.

Proof. Fix $\varepsilon > 0$ and let $s \in [-1,1]$. Under the conditions imposed on f in the statement of the theorem, $\Re_y f(s,t)$ is given by Eq. 45 when $t \in [-D, D]$ and equals zero outside this interval. Using Definition 5.16 we rewrite Eq. 45 as

$$\Re_y f(s,t) = \sqrt{1+s^2}\, T^s[D,f](t), \quad t \in [-D, D]. \tag{59}$$

By Theorem 5.18, there exists N_0 such that for any $N > N_0$

$$\left|T^s[D,f](x) - T^s_N[f^D_N](x)\right| < \frac{\varepsilon}{\sqrt{2}}$$

for any $x \in [-D, D]$ and $s \in [-1,1]$. Then,

$$\left|\sqrt{1+s^2}\, T^s[D,f]\left(\frac{2D}{M}u\right) - \sqrt{1+s^2}\, T^s_N[f^D_N]\left(\frac{2D}{M}u\right)\right| < \varepsilon$$

for any $u \in [-N : N]$ and $s \in [-1,1]$. Replacing the first term in this inequality by $\Re_y f\left(s, \frac{2D}{M}u\right)$ (see Eq. 59) and applying Theorem 5.19 we get the required result. $\square$

Theorem 5.20 states that the continuous Radon transform of an object for lines with intercepts $\left\{\frac{2D}{M}u \mid u \in [-N : N]\right\}$ can be approximated by means of the discrete Radon transform applied to a Cartesian set of samples this object.

Theorem 5.20 was formulated and proved for $\Re_y f(s,t)$, which corresponds to projections along basically vertical lines. An analogous theorem can be proved for $\Re_x f(s,t)$ and basically horizontal lines by minor modifications of the proofs in Sections 5.3 – 5.4.

6 Conclusions

We developed a 2D discrete Radon transform, derived its main properties, and presented fast algorithms for its computation and inversion. The discrete transform is based on a rigorous definition, where each step in the construction is chosen to achieve some desirable property. As a result, the transform does not use arbitrary interpolation schemes, preserves the concept of summation along straight lines, and is rapidly computable and invertible.

We proved a Fourier slice theorem that relates the discrete Radon transform to the pseudo-polar Fourier transform of the underlying image. The fast algorithms for the pseudo-polar Fourier transform provide fast forward and inverse algorithms for the discrete Radon transform.

Finally, we proved that the discrete Radon transform converges to the continuous Radon transform. This enables the discrete Radon transform to be used as an approximation to the continuous Radon transform in digital implementations.

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