On Efficient Computation of Multidimensional Oscillatory Integrals with Local Fourier Bases

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Abstract

The integral

$$\int_0^L \int_0^L e^{i\nu\phi(x,y,s,t)} f(s,t)\; ds\; dt$$

with a highly oscillatory kernel (large ν) is represented using local Fourier basis. The representation of the oscillatory kernel in this basis is sparse.

Key words: Multidimensional oscillatory integrals, Local Fourier Basis, sparse representations
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1 Introduction

Computation of oscillatory integrals occurring in scientific and engineering simulations faces difficult problems as compared to the non-oscillatory case, since accurate evaluation of such integrals requires at least a few grid points per oscillation.

For a one-dimensional integral

$$\int_a^b e^{i\lambda\phi(s)} f(s) ds, \tag{1}$$

with large λ and smooth (non-oscillating) f there exist methods for fast evaluation, for example, the method of stationary phase first developed in [7] and justified in [8]. Upon assuming large λ, we find that the rapid oscillation of $\exp\{i\lambda\phi(s)\}$ produces cancellation of the integral everywhere except near the stationary points, where the derivative $\phi'(s)$ vanishes. The method proved to be an effective tool in asymptotic expansions and evaluations. However, it requires analytic work in each case (cannot be applied automatically), does not reach high accuracy and it is not easily extended to higher dimensions. There exist fast algorithms for special kinds of kernels without regularity assumptions such as [6].

In [2,3] a numerical method for fast computation of the integral

$$\int_a^b dt \int_c^d K(t,s) f(s)\ ds, \quad K(t,s) = e^{i\lambda\phi(t,s)}$$

was described and it was shown analytically and numerically that $K(t,s)$ becomes sparse when represented in appropriate local Fourier basis (LFB). The following integral operator was chosen in [3] as a model problem

$$\int_0^L e^{i\nu|z(s)-z(t)|} f(s)\ ds. \tag{2}$$

Here $\nu \geq 100$, the number of grid points $N \geq 2\nu L$ and $z = z(s)$, $0 \leq s \leq L$, is the parametric representation of a closed smooth curve. In [2] it was found that the number of elements above a threshold in a LFB decomposition of the kernel in (2) is $O(N)$ provided the curve is smooth with several bounded derivatives. Here N is the number of discretization points in each direction. The present paper is an extension of [2] to higher dimensions.

We consider the problem of fast evaluation of the double integral

$$(Tf)(x,y) = \int_a^b dt \int_c^d K(x,y,t,s) f(t,s)\ ds \tag{3}$$

where $K(x,y,t,s)$ is an oscillatory kernel and f is a smooth function. The kernel is of the type

$$K(x,y,t,s) = e^{i\lambda\phi(x,y,t,s)}. \tag{4}$$

Whereas 2D models of acoustic scattering lead to integrals of type (2), integrals of type (3) with kernel (4) arise in 3D models of acoustic scattering [3] and similar applications. Methods, usually effective for evaluation of (1), become very expensive in the multidimensional case (3). If the oscillatory integral (3) is computed for each x, y, even the choice of a minimal number of grid points results in a large scale matrix problem. Computation time can be reduced if a sparse representation of $K(x,y,t,s)$ can be achieved in some basis. A sparse representation of the kernels (4) with wavelets (which appeared to be an efficient tool for sparsification of matrix-vector multiplications in many cases) could be expected. However, wavelet representation of an oscillating matrix appears to be as dense as the original, i.e. oscillatory kernels cannot be handled efficiently by representing them in wavelet bases.

The paper is organized as follows. In Section 2 the Local Fourier Bases are described and various bell functions for LFB are introduced. Section 3 presents a theoretical estimation for the number of significant coefficients in the LFB decomposition of oscillatory kernels of type (4). Section 4 contains numerical results. The dependence of the sparsity of the matrix which is obtained by the decomposition of oscillatory kernel in LFB on the form of the bell is analyzed.

2 Local Fourier Bases

The local Fourier basis is known to capture well oscillatory patterns (see, for example, [1–3]) in contrast to wavelets or multiwavelets. We attempt to achieve an effective computation of the oscillatory integral by representing its kernel in LFB, i.e. we computed the following coefficients:

$$A^{i_1,i_2,i_3,i_4}_{k_1,k_2,k_3,k_4} = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} K(x,y,t,s) C^{i_1}_{k_1}(x) C^{i_2}_{k_2}(y) C^{i_3}_{k_3}(t) C^{i_4}_{k_4}(s)\ dx\ dy\ dt\ ds, \tag{5}$$

$$B^{i_1,i_2,i_3,i_4}_{k_1,k_2,k_3,k_4} = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} K(x,y,t,s) S^{i_1}_{k_1}(x) S^{i_2}_{k_2}(y) S^{i_3}_{k_3}(t) S^{i_4}_{k_4}(s)\ dx\ dy\ dt\ ds, \tag{6}$$

where

$$C^i_k(s) = b_i(s)\left(\frac{2}{a_{i+1}-a_i}\right)^{1/2} \cos\left(\left(k+\frac{1}{2}\right)\pi\frac{s-a_i}{a_{i+1}-a_i}\right). \tag{7}$$

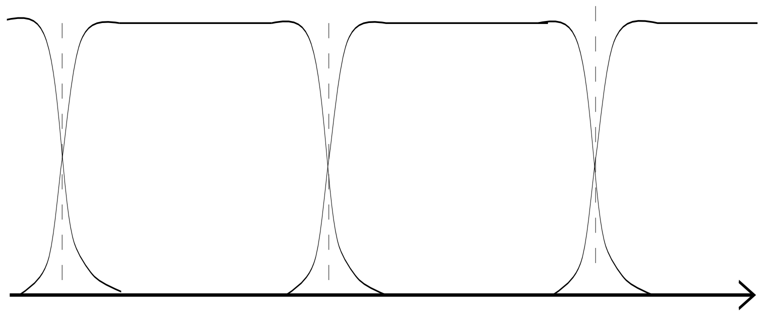

Fig. 1. The collection of bell functions. Each bell is defined on three adjacent intervals.

$$S_k^i(s) = b_i(s)\left(\frac{2}{a_{i+1}-a_i}\right)^{1/2}\sin\left(\left(k+\frac{1}{2}\right)\pi\frac{s-a_i}{a_{i+1}-a_i}\right). \qquad (8)$$

Let the interval $[0, 2\pi]$ be divided into M subintervals $I_i = (a_i, a_i + 1)$, also let $b_i(s)$ be a collection of smooth window (bell) functions supported on $I_{i-1} \cup I_i \cup I_{i+1}$ such that

$$\sum_{i=1}^{M} b_i^2(s) = 1, \qquad b_i(s) = b_{i-1}(2a_i - s).$$

A discrete system of basis functions is constructed. Let N be the number of discretization points. Following [2,3] we choose $N \geq 2\nu L$, i.e. $N \geq 4\pi\nu$. Usually in the numerical implementations we assumed $N = 8\nu$.

To achieve sparser representation we use the optimized bells developed in [5] which minimize the number of local Fourier coefficients necessary for their representation (9). These bells generalize the Coifman-Meyer (CM) construction [4] by extending the choice of the bell functions. Each bell is defined on three adjacent intervals as shown in figure 1.

Each bell is given by:

$$b_k(x) = \begin{cases} \frac{1}{2}\left(1 + \sum_{n=0}^{k-1} g_n \sin(n+\frac{1}{2})\pi x\right), & -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \frac{1}{2}\left(1 + \sum_{n=0}^{k-1} (-1)^n g_n \cos(n+\frac{1}{2})\pi x\right), & \frac{1}{2} \leq x \leq \frac{3}{2} \\ 0 & \text{otherwise} \end{cases} \qquad (9)$$

where the coefficients g_n are the solutions of a linear system and its values for $k = 0, \ldots, 11$ are tabulated in [5]. Here k is a parameter that characterizes the smoothness of the bell and its steepness on the boundaries of the window. Figure 2 illustrates the bells for $k = 1, 6, 11$.

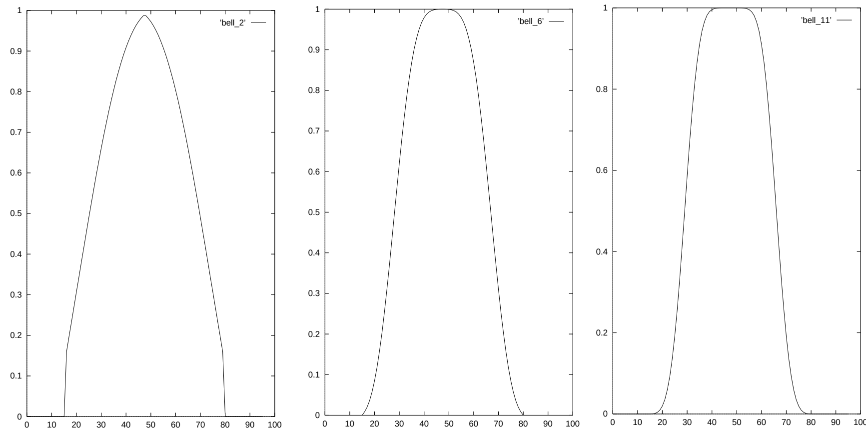

Fig. 2. The form of the bell for the steepness $k = 2, 6, 11$, respectively.

The basis which was built corresponds to the discretization of the continuous basis functions where N discretization points x_j are taken such that a_i are midpoints between adjacent x_j.

3 Sparsification of the kernel by local Fourier bases: theoretical estimates

We compute the following integral

$$(Tf)(x,y) = \int_0^1 \int_0^1 e^{i\nu\phi(x,y,s,t)} f(s,t)\ ds\ dt \tag{10}$$

where ϕ is a smooth function satisfying

$$\left| \frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \phi(x,y,s,t)}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial s^{\alpha_3} \partial t^{\alpha_4}} \right| \leq C. \tag{11}$$

If a one-dimensional function is decomposed with local cosine basis, each coefficient of such a decomposition corresponds to a certain location (window) q and a certain frequency p. Let us assume that we have $N = 4\nu$ discretization points (less than two points per oscillation would not work), $\sqrt{N} = 2\sqrt{\nu}$ bells (windows for local cosine transform). Then, the number of the window q is such that $0 \leq q < 2\sqrt{\nu}$, and frequency $p: \quad 0 \leq p < 2\sqrt{\nu}$.

Let us denote by $c_q = q/(2\sqrt{\nu})$ the origin of the q-th interval and by $d_q = (q{+}1/2)/(2\sqrt{\nu})$ the center of the q-th window and bell. Then the basis function corresponding to location q and frequency p is

$$C_{q,p}(x) = 2\sqrt{\nu}b(2\sqrt{\nu}(x-d_q))\cos\left(\left(p+\frac{1}{2}\right)\pi(x-c_q)2\sqrt{\nu}\right) \tag{12}$$

where $b(x)$ is the bell function.

The collection of functions $C_{q,p}$ forms an orthogonal basis. Thus, the interaction between two windows with 2D locations q_1, q_2 and q_1', q_2' and frequencies p_1, p_2 and p_1', p_2' is given by the expression

$$G_{(q_1',q_2';p_1',p_2')(q_1,q_2;p_1,p_2)} =$$

$$2\sqrt{\nu}\int_0^1\int_0^1\int_0^1\int_0^1 \exp\{i\nu\phi(x,y,s,t)\}b\left((x-d_{q_1'})2\sqrt{\nu}\right)b\left((y-d_{q_2'})2\sqrt{\nu}\right)$$

$$\times b\left((s-d_{q_1})2\sqrt{\nu}\right)b\left((t-d_{q_2})2\sqrt{\nu}\right) \quad dx\ dy\ ds\ dt \tag{13}$$

$$\times\cos\left(\left(p_1'+\frac{1}{2}\right)\pi\left(x-\frac{q_1'}{2\sqrt{\nu}}\right)2\sqrt{\nu}\right)\cos\left(\left(p_2'+\frac{1}{2}\right)\pi\left(y-\frac{q_2'}{2\sqrt{\nu}}\right)2\sqrt{\nu}\right)$$
$$\times\cos\left(\left(p_1+\frac{1}{2}\right)\pi\left(s-\frac{q_1}{2\sqrt{\nu}}\right)2\sqrt{\nu}\right)\cos\left(\left(p_2+\frac{1}{2}\right)\pi\left(t-\frac{q_2}{2\sqrt{\nu}}\right)2\sqrt{\nu}\right).$$

After change of variables

$$z_1 = (x-d_{q_1'})2\sqrt{\nu}, z_2 = (y-d_{q_2'})2\sqrt{\nu},$$
$$w_1 = (s-d_{q_1})2\sqrt{\nu}, w_2 = (t-d_{q_2})2\sqrt{\nu} \tag{14}$$

we obtain

$$x = \frac{z_1}{2\sqrt{\nu}}+\frac{q_1'+1/2}{2\sqrt{\nu}},\quad y = \frac{z_2}{2\sqrt{\nu}}+\frac{q_2'+1/2}{2\sqrt{\nu}},$$
$$s = \frac{w_1}{2\sqrt{\nu}}+\frac{q_1+1/2}{2\sqrt{\nu}},\quad t = \frac{w_2}{2\sqrt{\nu}}+\frac{q_2+1/2}{2\sqrt{\nu}}.$$

The boundaries for z_1, z_2 and w_1, w_2 are

$$m_i^z = -q_i'-1/2 \le z_i \le 2\sqrt{N}-q_i'-1/2 = M_i^z,$$

$$m_i^w = -q_i-1/2 \le d_i \le 2\sqrt{N}-q-1/2 = M_i^w,\quad i=1,2, \tag{15}$$

and

$$x-\frac{q_1'}{2\sqrt{\nu}} = x-\frac{q_1'+1/2}{2\sqrt{\nu}}+\frac{1}{4\sqrt{\nu}} = \frac{z_1}{2\sqrt{\nu}}+\frac{1}{4\sqrt{\nu}}. \tag{16}$$

Formulas similar to (16) are valid for the variables y,s,t.

The change of variables (14) turns (13) into:

$$\frac{1}{2\sqrt{\nu}}\int\limits_{m_1^w}^{M_1^w}\int\limits_{m_2^w}^{M_2^w}\int\limits_{m_1^z}^{M_1^z}\int\limits_{m_2^z}^{M_2^z}\exp\left\{i\nu\phi\left(\frac{q_1'}{2\sqrt{\nu}}+\frac{z_1}{2\sqrt{\nu}},\frac{q_2'}{2\sqrt{\nu}}+\frac{z_2}{2\sqrt{\nu}},\right.\right.$$

$$\left.\left.\frac{q_1}{2\sqrt{\nu}}+\frac{w_1}{2\sqrt{\nu}},\frac{q_2}{2\sqrt{\nu}}+\frac{w_2}{2\sqrt{\nu}}\right)\right\}b(z_1)b(z_2)b(w_1)b(w_2)$$

$$\times\cos\left(\left(p_1'+\frac{1}{2}\right)\pi\left(z_1+\frac{1}{2}\right)\right)\cos\left(\left(p_2'+\frac{1}{2}\right)\pi\left(z_2+\frac{1}{2}\right)\right)$$

$$\times\cos\left(\left(p_1+\frac{1}{2}\right)\pi\left(w_1+\frac{1}{2}\right)\right)\cos\left(\left(p_2+\frac{1}{2}\right)\pi\left(w_2+\frac{1}{2}\right)\right)\ dz_1\ dz_2\ dw_1\ dw_2.$$

Let us expand the function $\nu\phi$ into a Taylor series and denote by $r^{\nu}_{q_1',q_2',q_1,q_2}$ the residual of the linear approximation. Then the residual has the following form:

$$r^{\nu}(z_1,z_2,w_1,w_2)=r^{\nu}_{q_1',q_2',q_1,q_2}(z_1,z_2,w_1,w_2)$$

$$=\nu\phi\left(\frac{q_1'}{2\sqrt{\nu}}+\frac{z_1}{2\sqrt{\nu}},\frac{q_2'}{2\sqrt{\nu}}+\frac{z_2}{2\sqrt{\nu}},\frac{q_1}{2\sqrt{\nu}}+\frac{w_1}{2\sqrt{\nu}},\frac{q_2}{2\sqrt{\nu}}+\frac{w_2}{2\sqrt{\nu}}\right)$$

$$-\nu\phi\left(\frac{q_1'}{2\sqrt{\nu}},\frac{q_2'}{2\sqrt{\nu}},\frac{q_1}{2\sqrt{\nu}},\frac{q_2}{2\sqrt{\nu}}\right)\tag{17}$$

$$-\frac{\sqrt{\nu}}{2}\left[\frac{\partial\phi}{\partial q_1'}\left(\frac{q_1'}{2\sqrt{\nu}},\frac{q_2'}{2\sqrt{\nu}},\frac{q_1}{2\sqrt{\nu}},\frac{q_2}{2\sqrt{\nu}}\right)z_1+\frac{\partial\phi}{\partial q_2'}\left(\frac{q_1'}{2\sqrt{\nu}},\frac{q_2'}{2\sqrt{\nu}},\frac{q_1}{2\sqrt{\nu}},\frac{q_2}{2\sqrt{\nu}}\right)z_2\right.$$
$$\left.+\frac{\partial\phi}{\partial q_1}\left(\frac{q_1'}{2\sqrt{\nu}},\frac{q_2'}{2\sqrt{\nu}},\frac{q_1}{2\sqrt{\nu}},\frac{q_2}{2\sqrt{\nu}}\right)w_1+\frac{\partial\phi}{\partial q_2}\left(\frac{q_1'}{2\sqrt{\nu}},\frac{q_2'}{2\sqrt{\nu}},\frac{q_1}{2\sqrt{\nu}},\frac{q_2}{2\sqrt{\nu}}\right)w_2\right].$$

We assume $|z_1|<1,\ |z_2|<1,\ |w_1|<1,\ |w_2|<1$ which corresponds to overlapping of less than half a window. These restriction enable the convergence of the Taylor series.

Denote

$$\beta(z_1,z_2,w_1,w_2)=b(z_1)b(z_2)b(w_1)b(w_2)\exp\left\{ir^{\nu}(z_1,z_2,w_1,w_2)\right\}\tag{18}$$

and by A_1,A_2,B_1,B_2 the first partial derivatives multiplied by $\sqrt{\nu}/2$ at the point

$$\left(\frac{q_1'}{2\sqrt{\nu}},\frac{q_2'}{2\sqrt{\nu}},\frac{q_1}{2\sqrt{\nu}},\frac{q_2}{2\sqrt{\nu}}\right)\quad :$$

$$A_1=\frac{\sqrt{\nu}}{2}\frac{\partial\phi}{\partial q_1'},\ B_1=\frac{\sqrt{\nu}}{2}\frac{\partial\phi}{\partial q_1},\ A_2=\frac{\sqrt{\nu}}{2}\frac{\partial\phi}{\partial q_2'},\ B_2=\frac{\sqrt{\nu}}{2}\frac{\partial\phi}{\partial q_2}.\tag{19}$$

Then the original integral, which defines the interaction between two windows, has the following form:

$$\frac{1}{2\sqrt{\nu}} \exp\left\{ i\nu\phi \left(\frac{q_1'}{2\sqrt{\nu}}, \frac{q_2'}{2\sqrt{\nu}}, \frac{q_1}{2\sqrt{\nu}}, \frac{q_2}{2\sqrt{\nu}} \right) \right\}$$

$$\times \int_{m_1^w}^{M_1^w}\int_{m_2^w}^{M_2^w}\int_{m_1^z}^{M_1^z}\int_{m_2^z}^{M_2^z} \beta(z_1, z_2, w_1, w_2) \exp\{A_1 z_1 + A_2 z_2 + B_1 w_1 + B_2 w_2\}$$

$$\times \cos\left(\left(p_1' + \frac{1}{2}\right) \pi \left(z_1 + \frac{1}{2}\right)\right) \cos\left(\left(p_2' + \frac{1}{2}\right) \pi \left(z_2 + \frac{1}{2}\right)\right) \; dz_1 \; dz_2 \; dw_1 \; dw_2.$$

$$\times \cos\left(\left(p_1 + \frac{1}{2}\right) \pi \left(w_1 + \frac{1}{2}\right)\right) \cos\left(\left(p_2 + \frac{1}{2}\right) \pi \left(w_2 + \frac{1}{2}\right)\right) \qquad (20)$$

In addition to local cosine, the function can be also expanded into local sine bases. We now represent the cosine as a combination of exponentials. To compute (20) we have to evaluate the following integral:

$$\frac{1}{2\sqrt{\nu}} \exp\left\{ i\nu\phi \left(\frac{q_1'}{2\sqrt{\nu}}, \frac{q_2'}{2\sqrt{\nu}}, \frac{q_1}{2\sqrt{\nu}}, \frac{q_2}{2\sqrt{\nu}} \right) \right\} \sum_{\pm} \int_{m_1^w}^{M_1^w}\int_{m_2^w}^{M_2^w}\int_{m_1^z}^{M_1^z}\int_{m_2^z}^{M_2^z} \beta(z_1, z_2, w_1, w_2)$$

$$\times \exp\left\{ i \left[\pm \left(p_1' + \frac{1}{2}\right) \pi \left(z_1 + \frac{1}{2}\right) + A_1 z_1 \right] \right\}$$

$$\times \exp\left\{ i \left[\pm \left(p_2' + \frac{1}{2}\right) \pi \left(z_2 + \frac{1}{2}\right) + A_2 z_2 \right] \right\}$$

$$\times \exp\left\{ i \left[\pm \left(p_1 + \frac{1}{2}\right) \pi \left(w_1 - \frac{1}{2}\right) B_1 w_1 \right] \right\}$$

$$\times \exp\left\{ i \left[\pm \left(p_2 + \frac{1}{2}\right) \pi \left(w_2 - \frac{1}{2}\right) B_2 w_2 \right] \right\} dz_1 \; dz_2 \; dw_1 \; dw_2$$

$$= \frac{1}{2\sqrt{\nu}} \exp\left\{ i\nu\phi \left(\frac{q_1'}{2\sqrt{\nu}}, \frac{q_2'}{2\sqrt{\nu}}, \frac{q_1}{2\sqrt{\nu}}, \frac{q_2}{2\sqrt{\nu}} \right) \right\} \sum_{\pm} \exp\left\{ \pm i \left(p_1' + \frac{1}{2}\right) \frac{\pi}{2} \right\}$$

$$\times \exp\left\{ \pm i \left(p_2' + \frac{1}{2}\right) \frac{\pi}{2} \right\} \exp\left\{ \pm i \left(p_1 + \frac{1}{2}\right) \frac{\pi}{2} \right\} \exp\left\{ \pm i \left(p_2 + \frac{1}{2}\right) \frac{\pi}{2} \right\}$$

$$\times \int_{m_1^w}^{M_1^w}\int_{m_2^w}^{M_2^w}\int_{m_1^z}^{M_1^z}\int_{m_2^z}^{M_2^z} \beta(z_1, z_2, w_1, w_2)$$

$$\times \exp\left\{ i \left[\pm \left(p_1' + \frac{1}{2}\right) \pi + A_1 \right] z_1 \right\} \exp\left\{ i \left[\pm \left(p_2' + \frac{1}{2}\right) \pi + A_2 \right] z_2 \right\}$$

$$\times \exp\left\{ i \left[\pm \left(p_1 + \frac{1}{2}\right) \pi + B_1 \right] w_1 \right\}$$

$$\times \exp\left\{ i \left[\pm \left(p_2 + \frac{1}{2}\right) \pi + B_2 \right] w_2 \right\} dz_1 dz_2 dw_1 dw_2$$

$$= \frac{1}{2\sqrt{\nu}} \exp\left\{ i\nu\phi\left(\frac{q_1'}{2\sqrt{\nu}}, \frac{q_2'}{2\sqrt{\nu}}, \frac{q_1}{2\sqrt{\nu}}, \frac{q_2}{2\sqrt{\nu}} \right) \right\} \sum_{\pm} \exp\left\{ \pm i \left(p_1' + \frac{1}{2} \right) \frac{\pi}{2} \right\}$$

$$\times \exp\left\{ \pm i \left(p_2' + \frac{1}{2} \right) \frac{\pi}{2} \right\} \exp\left\{ \pm i \left(p_1 + \frac{1}{2} \right) \frac{\pi}{2} \right\} \exp\left\{ \pm i \left(p_2 + \frac{1}{2} \right) \frac{\pi}{2} \right\}$$

$$\hat{\beta}\left(A_1 \pm \left(p_1' + \frac{1}{2} \right) \pi, A_2 \pm \left(p_2' + \frac{1}{2} \right) \pi, B_1 \pm \left(p_1 + \frac{1}{2} \right) \pi, B_2 \pm \left(p_2 + \frac{1}{2} \right) \pi \right),$$

Here $\hat{\beta}$ is the Fourier transform of

$$\beta(x, y, s, t) = b(x)b(y)b(s)b(t) \exp\left\{ i r^{\nu}_{q_1', q_2', q_1, q_2}(z_1, z_2, w_1, w_2) \right\},$$

where $r^{\nu}_{q_1', q_2', q_1, q_2}$ is the residual after the linear approximation (17).

Let us evaluate $\hat{\beta}$. We claim that there exists a constant K such that

$$\left| r^{\nu}_{q_1', q_2', q_1, q_2}(z_1, z_2, w_1, w_2) \right| \leq K$$

and

$$\left| \frac{\partial^{i+j+m+l}}{\partial x^i \partial y^j \partial s^m \partial t^l} r^{\nu}_{q_1', q_2', q_1, q_2}(z_1, z_2, w_1, w_2) \right| \leq \frac{K}{(\sqrt{\nu})^{i+j+m+l}}. \qquad (21)$$

We observe that the residual $r^{\nu}_{q_1', q_2', q_1, q_2}$ has a leading term which is the sum of ten quadratic terms, four containing the derivatives of ϕ in z_1, z_2, w_1, w_2 and six with mixed derivatives. Each of the terms is bounded, therefore the residual is bounded.

Similarly, the first differential of the residual is a sum of terms; each of them is the third partial derivative of ϕ (in z_1, z_2, w_1, w_2 or mixed) with an appropriate coefficient multiplied by $(z_1^i z_2^j w_1^m w_2^l)/\sqrt{\nu}$, $\quad i+j+m+l = 3$. The third derivatives of ϕ are bounded due to the assumptions, consequently

$$\frac{\partial r^{\nu}_{q_1', q_2', q_1, q_2}(z_1, z_2, w_1, w_2)}{\partial z_i} = O\left(\frac{1}{\sqrt{\nu}} \right), \frac{\partial r^{\nu}_{q_1', q_2', q_1, q_2}(z_1, z_2, w_1, w_2)}{\partial w_i} = O\left(\frac{1}{\sqrt{\nu}} \right).$$

Similarly all the partial derivatives of $r^{\nu}_{q_1', q_2', q_1, q_2}(z_1, z_2, w_1, w_2)$ of order k are $O(1/\nu^{k/2})$, which proves (21).

The function

$$\beta(z_1, z_2, w_1, w_2) = b(z_1)b(z_2)b(w_1)b(w_2) \exp\left\{ i r^{\nu}_{q_1', q_2', q_1, q_2}(z_1, z_2, w_1, w_2) \right\}$$

is bounded together with all the partial derivatives of β up to order M:

$$\left| \frac{\partial^{i+j+m+l}}{\partial x^i \partial y^j \partial s^m \partial t^l} \beta(z_1, z_2, w_1, w_2) \right| \leq C, \; i+j+m+l \leq M. \qquad (22)$$

It is known that if a function f is bounded together with its derivatives

$$\left|\frac{\partial^{i+j+m+l} f(x,y,s,t)}{\partial x^i \partial y^j \partial s^m \partial t^l}\right| \leq C,\ i+j \leq M, \tag{23}$$

then the Fourier transforms of these derivatives are also bounded:

$$\left|\left(\frac{\partial^{i+j+m+l} \widehat{f(x,y,s,t)}}{\partial x^i \partial y^j \partial s^m \partial t^l}\right)\right| = \left|\hat{f}(\xi_1,\xi_2,\eta_1,\eta_2)\xi_1^i \xi_2^j \eta_1^m \eta_2^l\right| \leq C_1. \tag{24}$$

Moreover,

$$\left|\hat{f}(\xi_1,\xi_2,\eta_1,\eta_2)\right| \leq \frac{C_1}{1+\max(|\xi_1|,|\xi_2|,|\eta_1|,|\eta_2|)^M}.$$

Thus

$$\left|\hat{\beta}(\xi_1,\xi_2,\eta_1,\eta_2)\right| \leq \frac{C_1}{1+\max(|\xi_1|,|\xi_2|,|\eta_1|,|\eta_2|)^M}.$$

This means that for large $\xi_1, \xi_2, \eta_1, \eta_2$ the Fourier coefficients are negligible and therefore there is only a constant number C_2 of significant Fourier coefficients for each two windows corresponding to q_1, q_2 and q_1', q_2'.

For example, if anyone of $\xi_1, \xi_2, \eta_1, \eta_2$ exceeds 10, then

$$\left|\hat{\beta}(\xi_1,\xi_2,\eta_1,\eta_2)\right| \leq \frac{C_1}{1+10^M},$$

which means that the coefficients corresponding to frequencies $|\xi_1| + |\xi_2| + |\eta_1| + |\eta_2| > 10$ are negligible.

We have $(2\sqrt{\nu})^2 = 4\nu$ windows, so there are $16\nu^2$ pairs of interaction for windows located at various pairs of frequencies, not more than C_2 coefficients above a certain threshold for each interaction lead to $16\nu^2 C_2$ significant coefficients among the total of $256\nu^4$ coefficients. Thus, the total number of coefficients above a threshold is $O(N^2)$, where N is again the number of grid points in each direction (we recall that the total number of points is N^4).

4 Numerical results

Consider the case where the smooth surface is a sphere, and the oscillatory integral has the following form:

$$\int_0^{\pi} d\alpha_1 \int_0^{2\pi} \exp\{i\nu\rho(u(\alpha_1,\beta_1),u(\alpha_2,\beta_2))\} f(\alpha_1,\beta_1)\, d\beta_1, \tag{25}$$

wind. type	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
diagonal	1	2	4	8	15	20
close	0.1 - 0.2	0.2 - 0.5	0.5 - 1.0	1.0 - 2.0	1.6 - 3.0	2.0 - 5.0
distant	0.05 - 1.0	0.1 - 0.2	0.2 - 0.4	0.4 - 0.8	0.8 - 1.4	1.4 - 2.0

Table 1
Percent of coefficients above a threshold in diagonal blocks and blocks with strong and weak interaction. Size of the block is 32^4, with 16 points of overlapping, $\nu = 128$, bell with $k = 10$

ν	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
1	0.0093	0.024	0.060	0.16	0.37	0.71
2	0.018	0.035	0.070	0.16	0.36	0.69
4	0.021	0.043	0.092	0.21	0.44	0.81
8	0.026	0.053	0.12	0.26	0.54	0.95
16	0.034	0.067	0.15	0.33	0.65	1.11
32	0.039	0.081	0.19	0.41	0.76	1.27
64	0.050	0.10	0.23	0.48	0.87	1.43
128	0.063	0.12	0.26	0.54	1.00	1.64
256	0.096	0.19	0.40	0.81	1.44	2.28

Table 2
Percent of coefficients above a threshold in a diagonal block 32^4, with 16 points of overlapping for various ν, bell with $k = 10$

where $\rho(u_1, u_2)$ is the distance between two three dimensional points

$$u_1(x_1, y_1, z_1), u_2(x_2, y_2, z_2) : \rho(u_1, u_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Here u_1, u_2 are points on the unit sphere:

$$u_i(\alpha_i, \beta_i) = (\sin \alpha_i \cos \beta_i, \sin \alpha_i \sin \beta_i, \cos \alpha_i), \;\; 0 \leq \alpha_i < \pi, \;\; 0 \leq \beta_i < 2\pi.$$

Tables 1-3 describe the percent of coefficients above a certain threshold in the LFB decomposition; the computation is for one window only. Table 1 compares the sparsities of matrices for the interaction of a window with itself ("diagonal" blocks), interaction of close windows (blocks of strong interaction) and of distant windows.

Table 3 describes the dependence of the percent of significant coefficients on the smoothness k of the bell [5] for two distant windows 32×32.

k	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
2	0.61	2.35	8.36	22.4	51.14	81.88
4	0.13	0.39	1.35	3.14	7.20	17.11
6	0.084	0.24	0.49	1.12	2.75	5.39
8	0.056	0.15	0.35	0.69	1.24	2.37
10	0.063	0.12	0.26	0.54	1.00	1.64
11	0.070	0.13	0.25	0.50	0.92	1.56

Table 3
Percent of coefficients above a threshold in a diagonal block 32^4, with 16 points of overlapping, $\nu = 2048$ for various steepness k of the bell

We observe that with a smoother bell function the number of significant coefficients is less. This matches the theoretical results since the representation of the kernel in LFB can be reduced to the Fourier transform of the residual (which is smooth) multiplied by bells.

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