Basis and Spectral Representations

School of Computer Science
Tel-Aviv University

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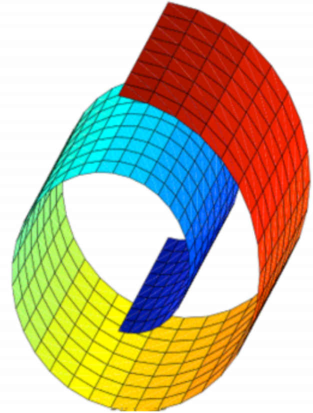
Introduction

The course will focus on modern and advanced methods in scientific computing with applications to mathematics, computer science and engineering. Lectures will be given on the following topics:

- Sparse Representations
- Compressed Sensing
- Fast Multi-pole method & Fast Gauss Transform
- Matrix Perturbation
- Matrix Completion
- Patch-Tensor Embedding
- Diffusion Maps
- Local Diffusion Folders
- Prolate Spheroidal Wave Functions

Motivation

Manifold learning:The goal



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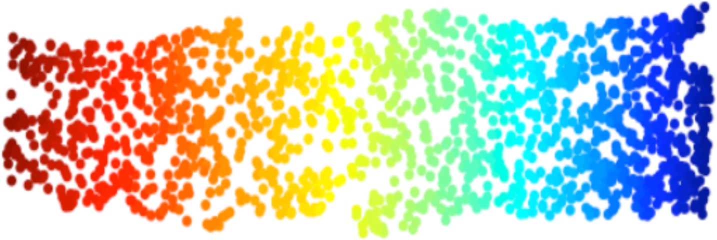
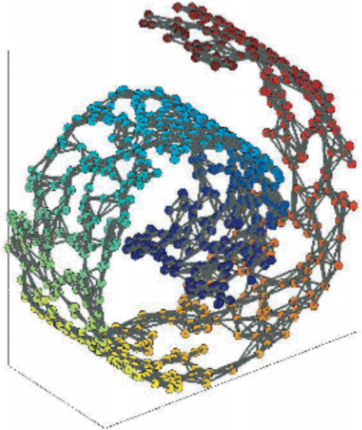
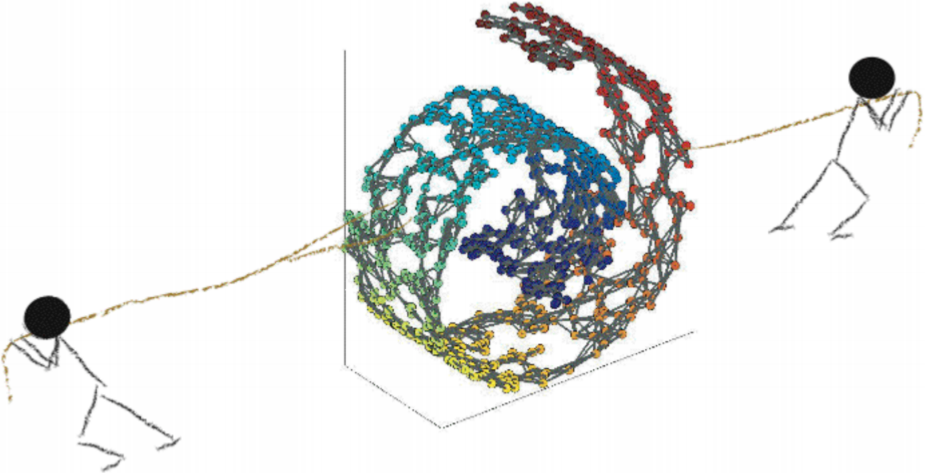


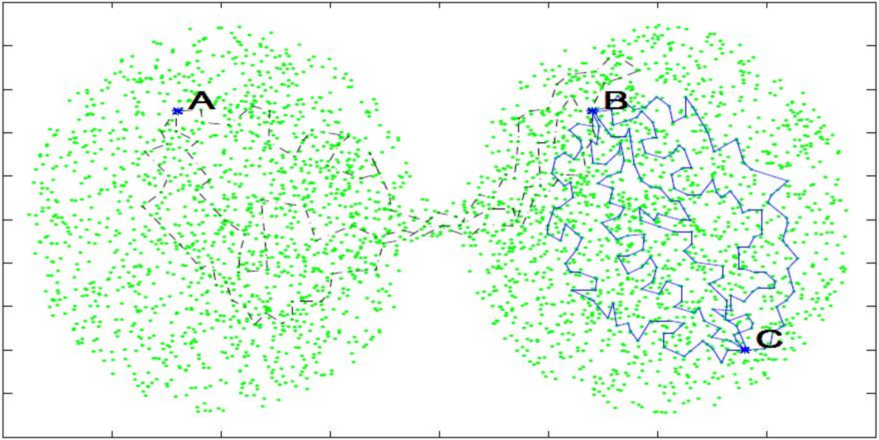
Motivation
Manifold learning:Kernel methods





Motivation

Manifold learning:Diffusion maps



Matrix Approximation and Completion

- Matrix completion is a popular research field with many applications to biology, machine learning, image processing and other scientific fields.
- In particular - Low Rank completions
- Example: The Netflix problem for movie recommendations:
  - Rows correspond to viewers, columns to movies
  - Entry $X_{ij} \in \{1, ..., 5\}$ is the rating
  - $480,000$ viewers $\times 18,000$ movies $\Rightarrow 8.9 \times 10^9$ entries
- Each viewer recommends 200 movies on average, so only 1.2% of the entries contain data
- The task is to predict the ratings that viewers will give to movies they have not yet rated

Matrix Approximation and Completion

Mathematically, the problem is:

$$\begin{aligned} &\text{minimize } \operatorname{rank}(\mathbf{X}) \\ &\text{subject to } X_{ij} = M_{ij}, \quad (i,j) \in \Omega \end{aligned} \tag{1}$$

We are looking for a matrix with minimal rank, such that the given entries do not change. The problem in Eq. 1 is NP-hard Complete by minizing the rank:

$$\mathbf{M} = \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & ? \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & ? \end{bmatrix}.$$

The missing entry should be 6 and the rank will be 1. In this case, the solution is unique.

Matrix Approximation and Completion

In the course we will discuss some matrix completion related topics:

- Least squares approximation of matrices under different spectral regularizations
- Methods for approximating just some of the entries
- Discuss convergence of different methods: local and global convergence
- Discuss today's state of the art methods for matrix completion and their drawbacks

Fast Multi-pole Method

The FMM is a method developed to speed up computations on n-body problem.

- Developed by Greengard and Rokhlin in the late 80s
- Considered by some one of the top ten most important algorithms of the 20th century
- Can be used to accelerate matrix-vector multiplication with certain structure from $\mathcal{O}(n^2)$ to $\mathcal{O}(n)$
- Can be used to accelerate iterative methods for solving linear system of equations that perform matrix-vector multiplication (for example, conjugate gradient method).
- Also can be used for electromagnetic problems
- Can be utilized for Fast Gauss Transform

Prolate Spheroidal Wave Functions

Bandlimited functions are important in many fields of mathematics and engineering.

- PSWFs are the eigenfunctions of the bandlimited Fourier transform:

$$\int_{-1}^{1} e^{ic\,\omega\cdot x}\,\psi_n^c(\omega)\,d\omega = \lambda_n^c\,\psi_n^c(x)$$

- c is a fixed bandlimit. $x \in [-1, 1]$, $\omega \in [-1, 1]$

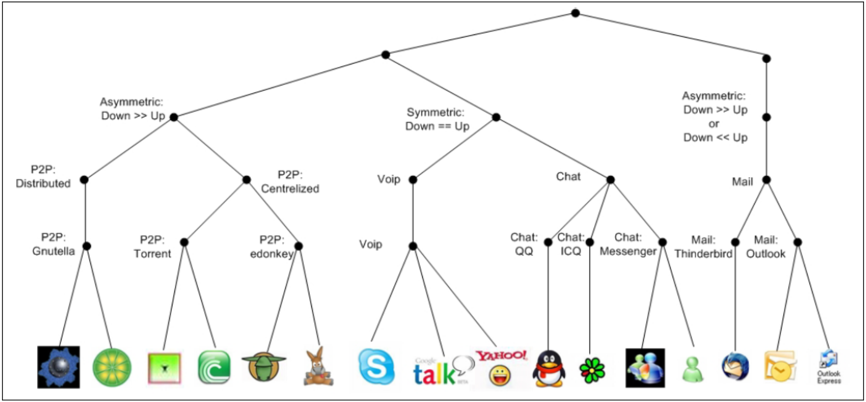
Prolate Spheroidal Wave Functions

The PSWFs developed in the 60s by David Slepian and are used in the following fields:

- Signal Processing: Filter design, signal reconstruction
- Interpolation
- Approximation

We will focus mainly on the properties of the PSWFs and their application to approximation of bandlimited functions (Shkolnisky & Tygert)

Localized diffusion folders



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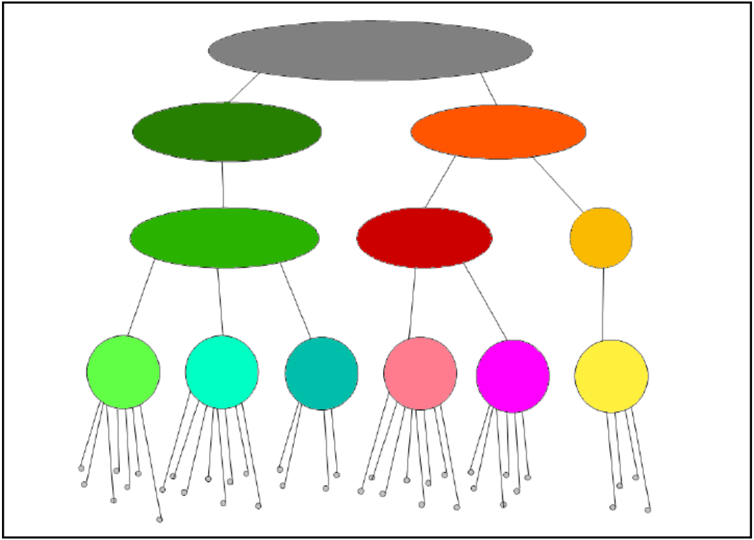

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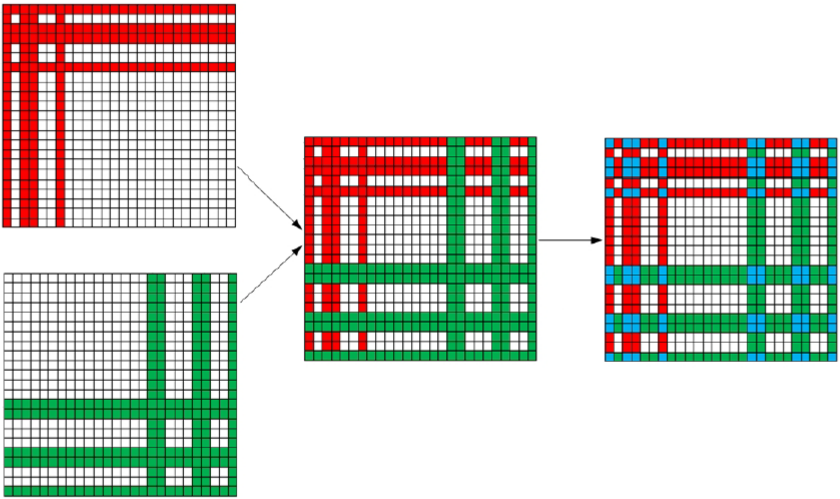

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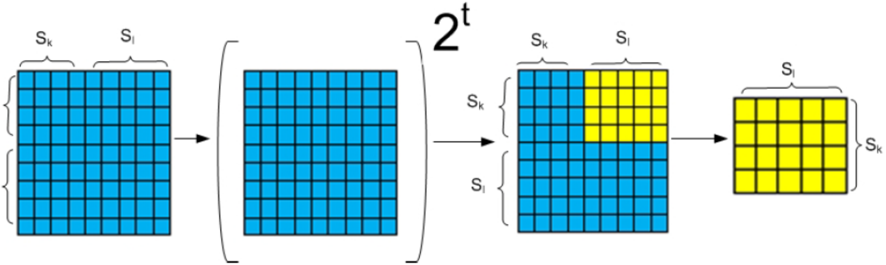
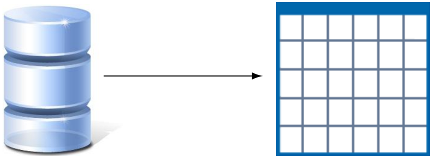


Diffusion Maps

Diffusion Maps



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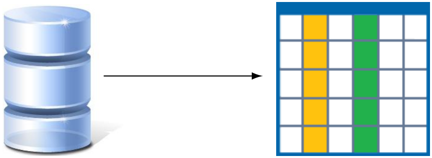

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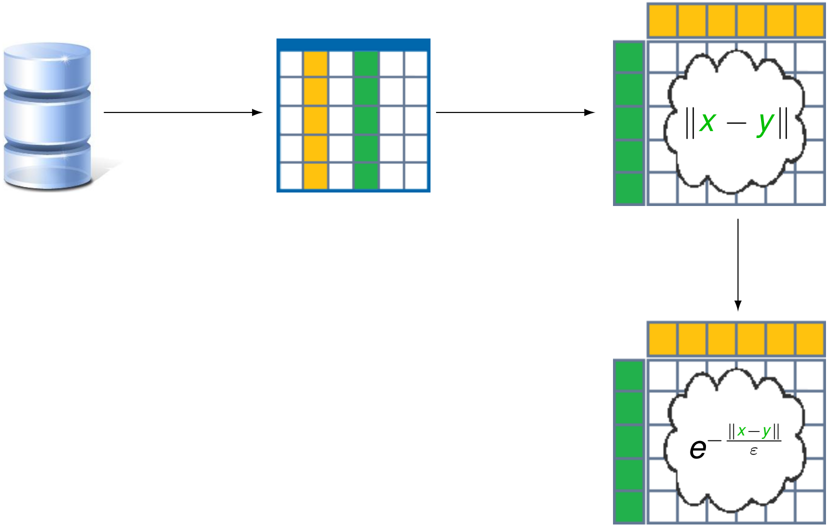

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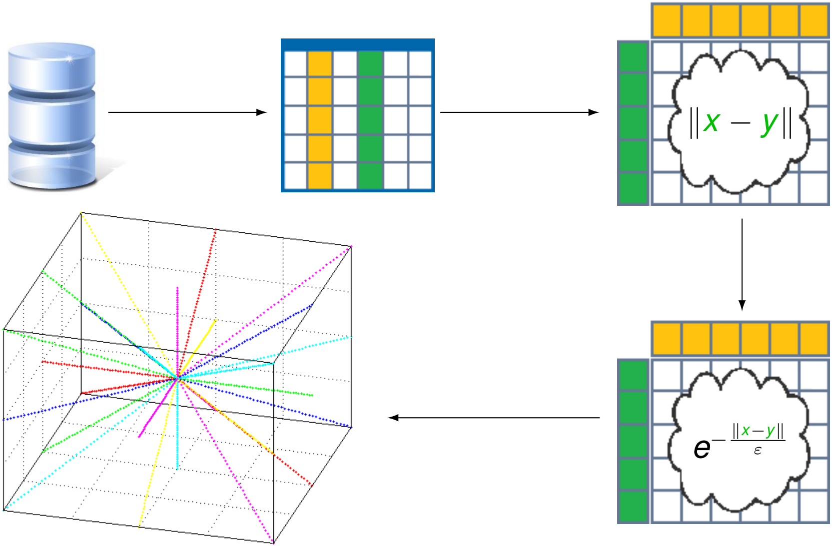
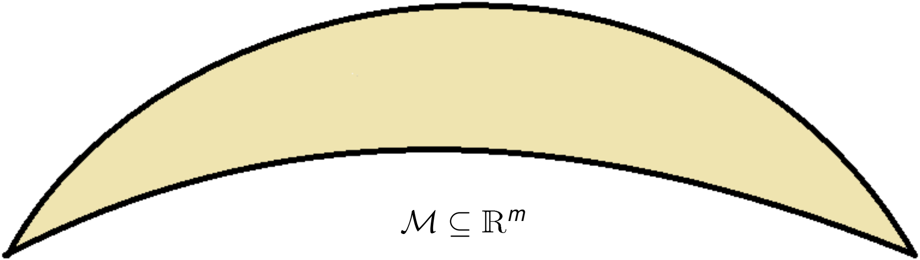


Patch to Tensor Embedding

How can we discriminate between two similar data points that have dissimilar neighborhood?

Patch to Tensor Embedding



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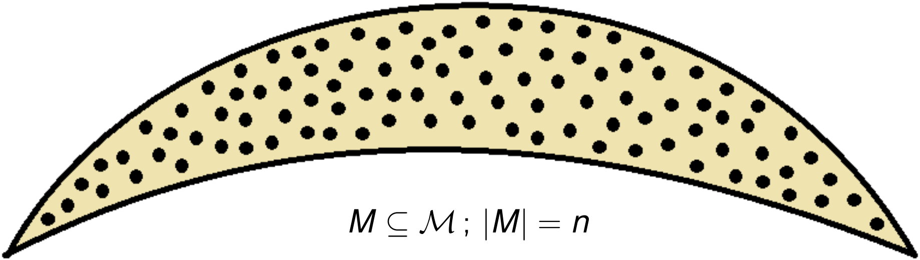
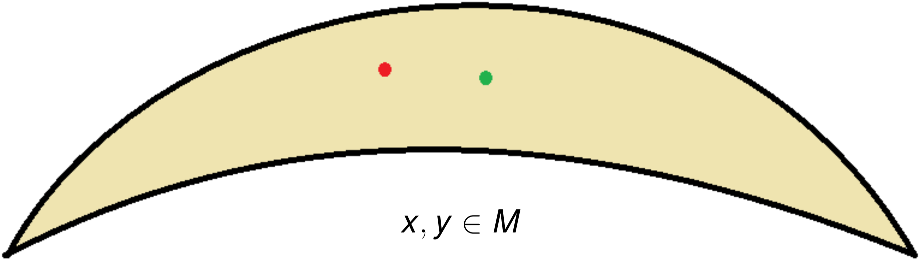


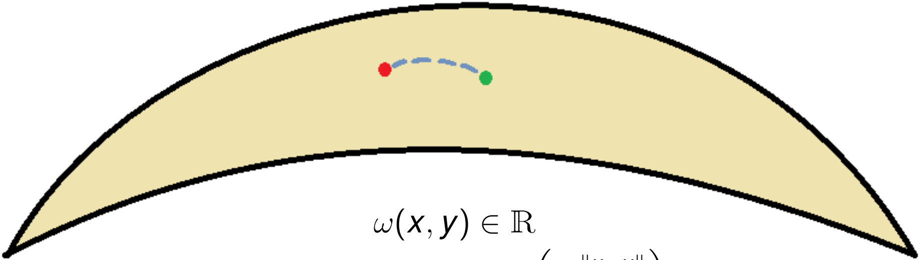
$M \subseteq \mathcal{M}\,;\ |M| = n$

Patch to Tensor Embedding



$x, y \in M$

Patch to Tensor Embedding

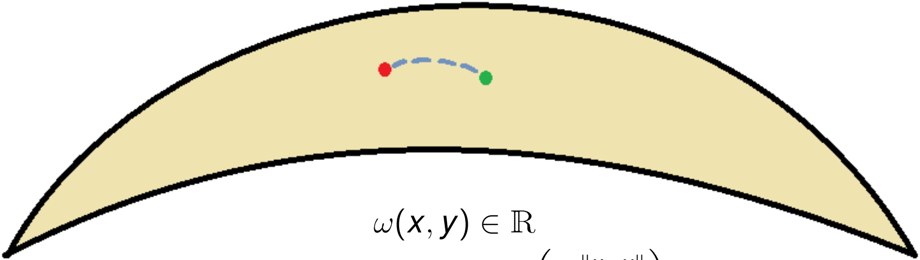


$$\omega(x, y) \in \mathbb{R}$$

$$\text{e.g., } \omega(x, y) = \exp\left(-\frac{\|x-y\|}{\varepsilon}\right)$$

Patch to Tensor Embedding

$$u \in M \mapsto \Psi(u) \in \mathbb{R}^{\ell \ll m}$$



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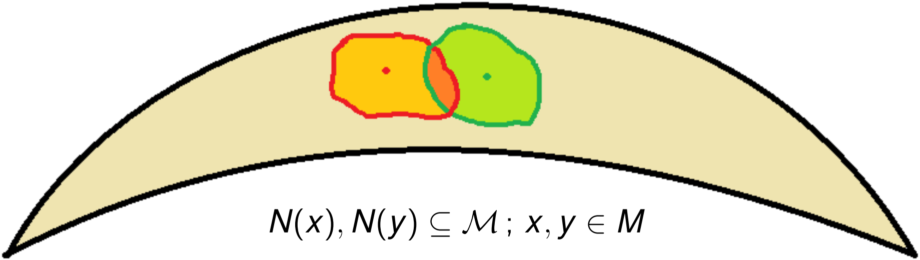
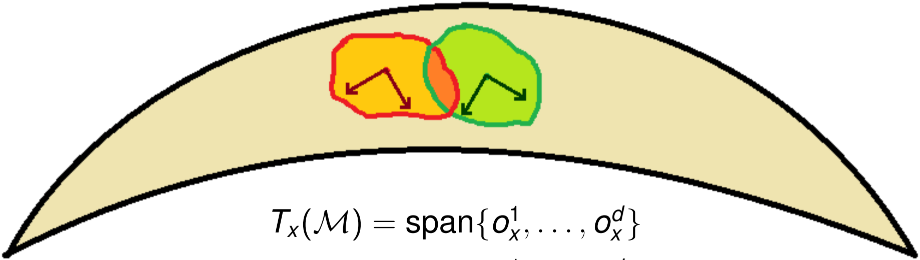


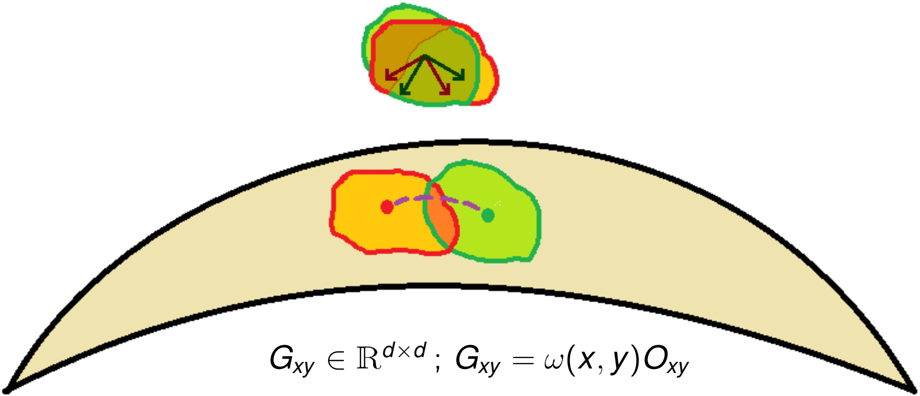
$$N(x), N(y) \subseteq \mathcal{M} \, ; \; x, y \in M$$

Patch to Tensor Embedding



$$T_x(\mathcal{M}) = \text{span}\{o_x^1, \ldots, o_x^d\}$$
$$T_y(\mathcal{M}) = \text{span}\{o_y^1, \ldots, o_y^d\}$$

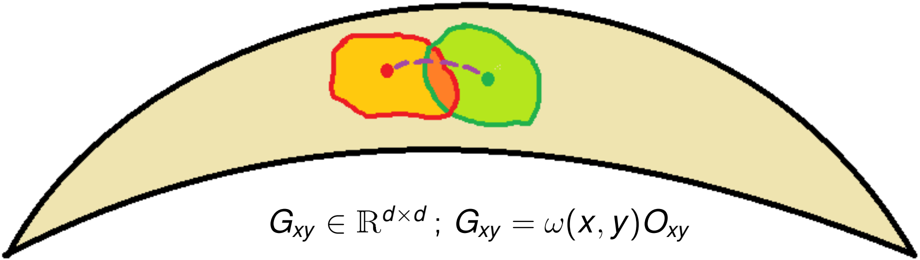
Patch to Tensor Embedding



$$G_{xy} \in \mathbb{R}^{d \times d}\,;\; G_{xy} = \omega(x, y) O_{xy}$$

Patch to Tensor Embedding

$$N(u \in M) \mapsto \mathcal{T}_u \in \mathbb{R}^{\ell} \otimes \mathbb{R}^{d}$$



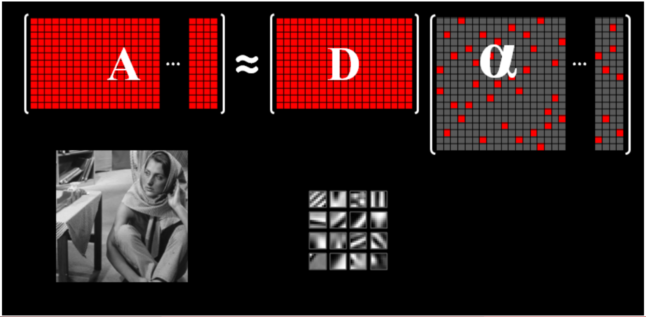
$$G_{xy} \in \mathbb{R}^{d \times d}; \; G_{xy} = \omega(x, y) O_{xy}$$

Sparse Representations and the k-SVD Algorithm

- We will overview different methods finding compact representations for matrices.That is, to present a matrix using sparse, linear combination of a small dictionary.
- Motivation : image compression, de-noising, image completion - you name it!

Sparse Representations and the k-SVD Algorithm

- In mathematical form : Given a matrix A, we would like the find matrices D and α, such that $A \approx D\alpha$ where D a column matrix of a small set of columns in A, and α is sparse.
    - Here, each column of α is a sparse linear combination of the columns of D forming a column of A.



Sparse Representations and the k-SVD Algorithm

- we will go over the k-SVD algorithm which find such representation in an elegant way, using ideas from the Singgular Value Decomposition, and k-Means algorithms.
- Based on papers and lectures of Michael Elad.

Matrix Perturbation Theory and Its Applications

- In a nutshell : how does a small change in the input affects the output?
- Given a matrix A and some function ϕ which operates on A, we are interested in understanding how a perturbation added to the matrix, affects ϕ.
- That is to understand the relation between $\phi(A)$ and $\phi(A+\epsilon)$, where ϵ is a small perturbation (can be noise).
- Interesting ϕ operators can be ones which finds the eigenvalues of A, its eigenvectors, $\|A\|$ and so on.

Matrix Perturbation Theory and Its Applications

- Example application(1) : Computation of the Google page rank algorithm - how to update the ranks without recomputing the entire algorithm.
- Example application(2) : updating a training set profile with small changes to the input set.
- Books
  - Matrix Analysis - R. Bhatia
  - Matrix Perturbation Theory - Stewart and Sun.