A King in every two consecutive tournaments

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Abstract

We think of a tournament \( T = ([n], E) \) as a communication network where in each round of communication processor \( P_i \) sends its information to \( P_j \), for every directed edge \( ij \in E(T) \). By Landau’s theorem (1953) there is a King in \( T \), i.e., a processor whose initial input reaches every other processor in two rounds or less. Namely, a processor \( P_v \) such that after two rounds of communication along \( T \)’s edges, the initial information of \( P_v \) reaches all other processors. Here we consider a more general scenario where an adversary selects an arbitrary series of tournaments \( T_1, T_2, \ldots \), so that in each round \( s = 1, 2, \ldots \), communication is governed by the corresponding tournament \( T_s \). We prove that for every series of tournaments that the adversary selects, it is still true that after two rounds of communication, the initial input of at least one processor reaches everyone.

Concretely, we show that for every two tournaments \( T_1, T_2 \) there is a vertex in \([n]\) that can reach all vertices via (i) A step in \( T_1 \), or (ii) A step in \( T_2 \) or (iii) A step in \( T_1 \) followed by a step in \( T_2 \).

1 Introduction

In a study of computational models in distributed systems [1] the following problem concerning \( n \) communicating processors was posed. Initially each processor \( P_i \) has its own input data item, \( m_i \). Nature selects a series of tournaments \( T_1, T_2, \ldots \) on vertex set \([n]\), and communication proceeds in rounds. For every \( s = 1, 2, \ldots \) each processor

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$P_i$ communicates in round $s$ every data item that has reached him so far to every processor $P_j$ with $ij \in E(T_s)$. By an old theorem of Landau [2], if $T_1 \equiv T_2$ then after two rounds a King emerges. Namely, a processor $P_{\nu}$, such that $m_{\nu}$ has reached all processors. Here we address the question how many rounds are required for a King to emerge if in each round an arbitrary tournament is selected. Surprisingly the answer is still 2.

2 Two rounds suffice

Let $T_1 = ([n], E_1), T_2 = ([n], E_2)$ be two tournaments. The condition that data item $m_i$ reaches processor $P_j$ after two rounds of communication is denoted by $i \Rightarrow j$. Clearly, this is equivalent to

$$i = j \lor (i, j) \in E_1 \lor (i, j) \in E_2 \lor \exists k \text{ s.t. } (i, k) \in E_1 \text{ and } (k, j) \in E_2$$

It is useful to note that the negation of this condition $i \not\Rightarrow j$ is equivalent to

$$i \neq j \land (j, i) \in E_1 \land (j, i) \in E_2 \land \Gamma_2(j) \supset \Gamma_1(i) \quad (1)$$

where $\Gamma_\delta(x)$ is the set of out-neighbors of $x$ in tournament $T_\delta$.

**Theorem 1.** Let $T_1 = ([n], E_1), T_2 = ([n], E_2)$ be two tournaments. Then there is $\nu \in [n]$ such that $\nu \Rightarrow j$ for every $j$.

*Proof.* By induction on $n$. The statement is easily verified for $n = 3$. Let $n$ be the smallest integer for which the theorem does not hold and let $T_1 = ([n], E_1), T_2 = ([n], E_2)$ be a counterexample. By minimality of $n$, for every $n \geq j \geq 1$ there is some $n \geq i \geq 1$ such that $i \Rightarrow j k$ for every $k \neq i, j$, where $\Rightarrow j$ indicates that the relation is defined with respect to tournaments $T_1 \setminus \{j\}$ and $T_2 \setminus \{j\}$. When this happens we say that $i$ is singed out by $j$. Clearly $i \not\Rightarrow j$, or else the theorem holds with $\nu = i$, since $i \Rightarrow j k$ clearly implies $i \Rightarrow k$. Consequently, no vertex is singed out more than once. We denote $\pi(j) = i$ and conclude that $\pi$ is a permutation on $[n]$, since $\pi(j)$ is defined for every $n \geq j \geq 1$ and $\pi$ is an injective mapping.

However, this is impossible. By Condition (1), every $j$ satisfies $|\Gamma_2(j)| > |\Gamma_1(\pi(j))|$. But $\sum_j |\Gamma_2(j)| = \sum_j |\Gamma_1(\pi(j))| = \binom{n}{2}$, since $\pi$ is a permutation. This contradiction completes the proof.

The same proof yields as well

**Corollary 2.** Let $T_1 = ([n], E_1), T_2 = ([n], E_2)$ be two tournaments. Then there is $\mu \in [n]$, such that $i \Rightarrow \mu$ for every $i$. 

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Proof. The same proof works. Just switch between $T_1, T_2$ and reverse all edges in the two tournaments.

References
