A King in every two consecutive tournaments

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Abstract

We think of a tournament $T = ([n], E)$ as a communication network where in each round of communication processor P_i sends its information to P_j, for every directed edge $ij \in E(T)$. By Landau's theorem (1953) there is a King in T, i.e., a processor whose initial input reaches every other processor in two rounds or less. Namely, a processor P_ν such that after two rounds of communication along T's edges, the initial information of P_ν reaches all other processors. Here we consider a more general scenario where an adversary selects an arbitrary series of tournaments $T_1, T_2, \ldots$, so that in each round $s = 1, 2, \ldots$, communication is governed by the corresponding tournament T_s. We prove that for every series of tournaments that the adversary selects, it is still true that after two rounds of communication, the initial input of at least one processor reaches everyone.

Concretely, we show that for every two tournaments T_1, T_2 there is a vertex in $[n]$ that can reach all vertices via (i) A step in T_1, or (ii) A step in T_2 or (iii) A step in T_1 followed by a step in T_2.

1 Introduction

In a study of computational models in distributed systems [1] the following problem concerning n communicating processors was posed. Initially each processor P_i has its own input data item, m_i. Nature selects a series of tournaments $T_1, T_2, \ldots$ on vertex set $[n]$, and communication proceeds in rounds. For every $s = 1, 2, \ldots$ each processor

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P_i communicates in round s every data item that has reached him so far to every processor P_j with $ij \in E(T_s)$. By an old theorem of Landau [2], if $T_1 \equiv T_2$ then after two rounds a *King* emerges. Namely, a processor P_ν, such that m_ν has reached all processors. Here we address the question how many rounds are required for a King to emerge if in each round an arbitrary tournament is selected. Surprisingly the answer is still 2.

2 Two rounds suffice

Let $T_1 = ([n], E_1), T_2 = ([n], E_2)$ be two tournaments. The condition that data item m_i reaches processor P_j after two rounds of communication is denoted by $i \Rightarrow j$. Clearly, this is equivalent to

$$i = j \ \vee \ (i,j) \in E_1 \ \vee \ (i,j) \in E_2 \ \vee \ \exists k \ s.t. \ (i,k) \in E_1 \ \text{ and } (k,j) \in E_2$$

It is useful to note that the negation of this condition $i \not\Rightarrow j$ is equivalent to

$$i \neq j \quad \wedge \quad (j,i) \in E_1 \quad \wedge \quad (j,i) \in E_2 \quad \wedge \quad \Gamma_2(j) \supset \Gamma_1(i) \tag{1}$$

where $\Gamma_\delta(x)$ is the set of out-neighbors of x in tournament T_δ.

Theorem 1. *Let $T_1 = ([n], E_1), T_2 = ([n], E_2)$ be two tournaments. Then there is $\nu \in [n]$ such that $\nu \Rightarrow j$ for every j.*

Proof. By induction on n. The statement is easily verified for $n = 3$. Let n be the smallest integer for which the theorem does not hold and let $T_1 = ([n], E_1), T_2 = ([n], E_2)$ be a counterexample. By minimality of n, for every $n \geq j \geq 1$ there is some $n \geq i \geq 1$ such that $i \Rightarrow_j k$ for every $k \neq i, j$, where $\Rightarrow_j$ indicates that the relation is defined with respect to tournaments $T_1 \setminus \{j\}$ and $T_2 \setminus \{j\}$. When this happens we say that i is *singled out* by j. Clearly $i \not\Rightarrow j$, or else the theorem holds with $\nu = i$, since $i \Rightarrow_j k$ clearly implies $i \Rightarrow k$. Consequently, no vertex is singled out more than once. We denote $\pi(j) = i$ and conclude that π is a permutation on $[n]$, since $\pi(j)$ is defined for every $n \geq j \geq 1$ and π is an injective mapping.

However, this is impossible. By Condition (1), every j satisfies $|\Gamma_2(j)| > |\Gamma_1(\pi(j))|$. But $\sum_j |\Gamma_2(j)| = \sum_j |\Gamma_1(\pi(j))| = \binom{n}{2}$, since π is a permutation. This contradiction completes the proof. $\square$

The same proof yields as well

Corollary 2. *Let $T_1 = ([n], E_1), T_2 = ([n], E_2)$ be two tournaments. Then there is $\mu \in [n]$, such that $i \Rightarrow \mu$ for every i.*

Proof. The same proof works. Just switch between T_1, T_2 and reverse all edges in the two tournaments.

$\square$

References

[1] Y. Afek and E. Gafni, *Asynchrony from Synchrony.* Arxiv, http://arxiv.org/abs/1203.6096 , Janurary 2012

[2] H. Landau. On dominance relations and the structure of animal societies, III: The condition for score structure. *Bulletin of Mathematical Biophysics*, 15(2):143148, 1953.