Canonical Propositional Gentzen-Type Systems

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Abstract. Canonical propositional Gentzen-type systems are systems which in addition to the standard axioms and structural rules have only pure logical rules which have the subformula property, introduce exactly one occurrence of a connective in their conclusion, and no other occurrence of any connective is mentioned anywhere else in their formulation. We provide a constructive coherence criterion for the non-triviality of such systems, and show that a system of this kind admits cut elimination iff it is coherent. We show also that the semantics of such systems is provided by non-deterministic two-valued matrices (2-Nmatrices). 2-Nmatrices form a natural generalization of the classical two-valued matrix, and every coherent canonical system is sound and complete for one of them. Conversely, with any 2-Nmatrix it is possible to associate a coherent canonical Gentzen-type system which has for each connective at most one introduction rule for each side, and is sound and complete for that 2-Nmatrix. We show also that every coherent canonical Gentzen-type system either defines a fragment of the classical two-valued logic, or a logic which has no finite characteristic matrix.

1 Introduction

There is a long tradition starting from [Gen69] according to which the meaning of a connective is determined by the introduction and elimination rules which are associated with it.[1] The supporters of this thesis usually have in mind Natural Deduction systems of an ideal type. In this type of "canonical" systems each connective has its own introduction and elimination rules, which should meet the following conditions: in a rule for a connective $\diamond$, this connective should be mentioned exactly once, and no other connective should be involved. The rules should also be pure (in the sense of [Avr91]). Unfortunately, already the handling of classical negation requires rules which are not canonical in this sense. This problem was solved by Gentzen himself by moving to what is now known as Gentzen-type systems or sequential calculi. These calculi employ in their classical version multiple-conclusion two-sided sequents, and instead of introduction and elimination rules they use left introduction rules and right introduction rules. The intuitive notions of "canonical form of a rule" and "canonical system" can

[1] See e.g. [Hod86] and [Sun86] for discussions and references.

be adapted to such systems in a straightforward way, and it is well known that the usual classical connectives can indeed be fully characterized by canonical Gentzen-type rules. Moreover: although this can be done in several ways, in all of them the cut-elimination theorem obtains.

In this paper we shall considerably generalize these known facts. We shall define "canonical" Gentzen-type rules and systems in precise terms, and provide a constructive *coherence* criterion for their non-triviality. We then show that a canonical system admits cut-elimination iff it is coherent. Moreover: we show that any coherent set of canonical introduction rules for a connective $\diamond$ completely determines the meaning of $\diamond$. For this we shall need however to generalize the usual semantics of classical logic.

The structure of the rest of this paper is as follows: In section 2 we review some basic concepts related to logics. In section 3 we define canonical Gentzen-type systems, formulate the coherence criterion for their non-triviality, and investigate some special important types of them. In section 4 we introduce non-deterministic two-valued matrices (2-Nmatrices). This is a generalization of the classical two-valued (deterministic) matrices, and it provides the semantics of coherent canonical Gentzen-type systems. In the same section and in section 5 we show how to associate a 2-Nmatrix with every coherent canonical system **G**, so that **G** is sound and complete for that 2-Nmatrix. This allows us to prove that every system of this type admits cut elimination, and it defines either a fragment of the classical two-valued logic, or a logic which has no finite characteristic matrix. In section 6 we show that the connection works also in the other direction: with any 2-Nmatrix it is possible to associate a coherent canonical Gentzen-type system **G** which is sound and complete for that 2-Nmatrix. Moreover: for this we can confine ourselves to systems which have for any connective at most one left introduction rule and at most one right introduction rule, and these rules can be given a particularly concise normal form. We conclude the paper with some remarks and directions for further research.

2 Preliminaries

In what follows $\mathcal{L}$ is a propositional language with a finite set of connectives, $\mathcal{W}$ is its set of wffs, ψ, ϕ, τ denote arbitrary formulas (of $\mathcal{L}$), and Γ, Δ denote sets of formulas.

Definition 1.

1. [Sco74] *A (*Scott*)* consequence relation *(*scr *for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ that satisfies the following conditions:*

s-R	strong reflexivity:	*if* $\Gamma \cap \Delta \neq \emptyset$ *then* $\Gamma \vdash \Delta$.
M	monotonicity:	*if* $\Gamma \vdash \Delta$ *and* $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ *then* $\Gamma' \vdash \Delta'$.
C	cut:	*if* $\Gamma \vdash \psi, \Delta$ *and* $\Gamma', \psi \vdash \Delta'$ *then* $\Gamma, \Gamma' \vdash \Delta, \Delta'$.

2. $\vdash$ *is* finitary *if the following condition holds for all* $\Gamma, \Delta \subseteq \mathcal{W}$: *if* $\Gamma \vdash \Delta$ *then* $\Gamma' \vdash \Delta'$ *for some finite* $\Gamma' \subseteq \Gamma$ *and* $\Delta' \subseteq \Delta$. $\vdash$ *is* uniform *if for every*

uniform substitution σ *and every* Γ *and* Δ*, if* $\Gamma \vdash \Delta$ *then* $\sigma(\Gamma) \vdash \sigma(\Delta)$. $\vdash$ *is* consistent *(or* non-trivial*) if there exist non-empty* Γ *and* Δ *s.t.* $\Gamma \not\vdash \Delta$.
3. *A* propositional logic *is a pair* $\langle \mathcal{L}, \vdash \rangle$*, where* $\mathcal{L}$ *is a propositional language and* $\vdash$ *is a uniform consistent scr for* $\mathcal{L}$.

Note: There are exactly four inconsistent finitary scrs in any given language: the one in which $\Gamma \vdash \Delta$ iff Γ and Δ are non-empty; the one in which $\Gamma \vdash \Delta$ iff Γ is non-empty; the one in which $\Gamma \vdash \Delta$ iff Δ is non-empty; and the one in which $\Gamma \vdash \Delta$ for all Γ and Δ. All of them should be considered trivial, and are excluded from our definition of a *logic*.

3 Canonical Gentzen-type systems

Definition 2. *A Gentzen-type system* **G** *is* standard *if its set of axioms includes the standard axioms* $\Gamma, \psi \Rightarrow \psi, \Delta$ *and it has all the standard structural rules (including cut).* [2]

From now on by a "calculus" we shall mean a standard Gentzen-type calculus, and Γ and Δ will denote *finite* sets of formulas.

In an ideal Gentzen-type system (of which the usual systems for classical logic provide the principal examples) every logical rule should be an introduction rule for one connective, it should introduce exactly one occurrence of that connective in its conclusion, and no other occurrence of that connective or any other connective should be mentioned anywhere else in its formulation. Moreover: the rule should be *pure* (i.e., there should be no side conditions limiting its application), and its side formulas should be immediate subformulas of the principal formula. The next definition formulates this idea in exact terms, and provides a method for describing such rules.

Definition 3.

1. *A* canonical rule *of arity n is an expression of the form* $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m}/C$*, where* $m \geq 0$*,* C *is either* $\diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ *or* $\Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ *for some connective* $\diamond$ *(of arity n), and for all* $1 \leq i \leq m$*,* $\Pi_i \Rightarrow \Sigma_i$ *is a clause such that* $\Pi_i, \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.[3]
2. *An application of a canonical rule* $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} \ / \ \diamond(p_1, \ldots, p_n) \Rightarrow$ *is any inference step of the form:*

$$\frac{\{\Gamma_i, \Pi_i^* \Rightarrow \Delta_i, \Sigma_i^*\}_{1 \leq i \leq m}}{\Gamma, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow \Delta}$$

where Π_i^* *and* Σ_i^* *are obtained from* Π_i *and* Σ_i *(respectively) by substituting* ψ_j *for* p_j *(for all* $1 \leq j \leq n$*),* Γ_i, Δ_i *are any sets of formulas,* $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$*, and* $\Delta = \bigcup_{i=1}^{m} \Delta_i$*. An application of a canonical rule with a conclusion of the form* $\Rightarrow \diamond(p_1, \ldots, p_n)$ *is defined similarly.*

[2] This means that we can take Γ, Δ in a sequent $\Gamma \Rightarrow \Delta$ to be finite *sets* of formulas.

[3] By a clause we mean a sequent which consists of atomic formulas only. When propositional clauses are written in this way, resolution and cut amount to the same thing.

Note: While sequents are written in a metalanguage for $\mathcal{L}$ (which includes the extra symbol $\Rightarrow$), a canonical rule is formulated in a meta-meta language of $\mathcal{L}$ (which includes one further extra symbol: /).

Example 1. The two usual introduction rules for classical conjunction can be formulated as the following canonical rules: $\{p_1, p_2 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow$ and $\{\Rightarrow p_1\ ,\ \ \Rightarrow p_2\} \ / \ \ \Rightarrow p_1 \wedge p_2$. Applications of these rules have the form:

$$\frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \wedge \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma' \Rightarrow \Delta', \phi}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \psi \wedge \phi}$$

Definition 4. *A standard calculus is called* canonical *if in addition to the standard axioms and the standard structural rules it has only canonical logical rules.*

A given canonical calculus may be simplified in various ways. In later sections we'll go deeper into questions concerning simplifications and normalizations of rules and calculi. For our immediate purposes we shall need only very obvious simplifications:

Definition 5. *A canonical rule is called* superfluous *if its set of premises is classically inconsistent (which is the case iff it is possible to obtain the empty clause from it using resolutions (= cuts)). A logical rule in a canonical calculus* **G** *is called* redundant *in* **G** *if its set of premises is subsumed by the set of premises of another rule of* **G** *which has the same conclusion.*

Example 2. A rule with the set of premises $\{p_1, p_2 \Rightarrow\ \ ,\ p_1 \Rightarrow p_2\ ,\ \ \Rightarrow p_1\}$ is superfluous. If a calculus **G** has the two rules $\{\Rightarrow p_1\} / \diamond(p_1, p_2) \Rightarrow$ and $\{\Rightarrow p_1\ ,\ \ \Rightarrow p_2\} / \diamond(p_1, p_2) \Rightarrow$ then the latter rule is redundant in **G**.

Proposition 1. *Let* **G** *be a canonical calculus, and let* **G**$'$ *be the calculus that is obtained from* **G** *by deleting superfluous and redundant rules. Then* **G**$'$ *is equivalent to* **G**. *Moreover, every sequent that has a cut-free proof in* **G**$'$ *also has such a proof in* **G**.

Proof. An application of a superfluous rule in **G** can be simulated in **G**$'$ by using cuts on its premises followed by a single weakening. The rest of the proposition is trivial.

Every canonical Gentzen-type system **G** naturally defines a uniform (and finitary) scr. However, in order to ensure that it defines a *logic*, we need to impose some constraints on its set of rules. The following definition provides a constructive equivalent of the consistency condition:

Definition 6. *A canonical calculus* **G** *is called* coherent, *if for every two rules* $S_1 / \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ *and* $S_2 / \ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ *of* **G**, *the set of clauses* $S_1 \cup S_2$ *is classically inconsistent (and so the empty clause can be derived from it using cuts).*

Example 3. The two classical rules for conjunction described in Example 1 form a coherent set of rules. Here $S_1 = \{p_1, p_2 \Rightarrow \}$, $S_2 = \{ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}$ and so $S_1 \cup S_2$ is the classically inconsistent set $\{p_1, p_2 \Rightarrow\ ,\ \ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}$.

Example 4. Let T be the famous "Tonk" connective of Prior ([Pri60]). It is defined in our framework by the following pair of rules: $\{p_1 \Rightarrow \}\ /\ p_1 T p_2 \Rightarrow$ and $\{ \Rightarrow p_2\}\ /\ \ \Rightarrow p_1 T p_2$. This set is not coherent, since the set $\{p_1 \Rightarrow\ ,\ \ \Rightarrow p_2\}$ is classically consistent. The resulting calculus is of course inconsistent.

It is not difficult to show that a consistent canonical calculus should necessarily be coherent. The converse is an immediate corollary of the general cut-elimination theorem for coherent canonical calculi that we shall prove below. Note that coherence is also a necessary condition for cut elimination:

Proposition 2. *A canonical calculus which admits cut elimination is coherent.*

Proof. In canonical systems, *clauses* which are not axioms can be proved only by using cuts on non-atomic formulas. Thus, if a canonical calculus admits cut elimination it must be consistent and hence coherent.

As a first step for proving our general cut-elimination theorem we shall reduce the problem to some special type of coherent canonical calculi.

Definition 7.

1. *A canonical rule is called* separated *if all its premises are unit clauses.*
2. *A separated rule* $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m}/C$ *of arity* n *is called* full *if* $m = n$ *and for every* $1 \leq i \leq n$, $\Pi_i \cup \Sigma_i = \{p_i\}$

Example 5. $\{p_1, p_2 \Rightarrow \}\ /\ p_1 \wedge p_2 \Rightarrow$ is not separated. $\{p_1 \Rightarrow \}\ /\ p_1 \wedge p_2 \Rightarrow$ is separated, but not full. $\{p_1 \Rightarrow\ ,\ \ \Rightarrow p_2\}\ /\ p_1 \wedge p_2 \Rightarrow$ is full.

Definition 8. *A canonical calculus is called* separated *(*full*) if all its logical rules are separated (full), and none of them is superfluous or redundant.*

Note that a full canonical calculus **G** is coherent simply iff no two rules of **G** for the same connective have the same set of premises but different conclusions.

Proposition 3. *Every canonical calculus* **G** *has an equivalent full canonical calculus* **G**$'$*. Moreover: if* **G** *is coherent then so is* **G**$'$*, and every sequent that has a cut-free proof in* **G**$'$ *also has such a proof in* **G***.*

We shall first explain the process of transforming a canonical calculus to an equivalent full canonical calculus by using an example. The transition is made in two stages: first, every canonical rule is split into separated rules, and then each of these rules is split into full rules.

Example 6. Take the classical introduction rules for conjunction:

$$[\wedge\Rightarrow] \quad \{p_1, p_2 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow \qquad \{\ \Rightarrow p_1\ ,\ \Rightarrow p_2\} \ / \ \Rightarrow p_1 \wedge p_2 \quad [\Rightarrow\wedge]$$

The second rule is already full. The first is neither full nor separated. We can replace it by the following pair of separated rules:

$$\{p_1 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow \qquad \{p_2 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow$$

This pair is equivalent to the original rule. Indeed, given the two rules, by applying the first to $\Gamma, \psi_1, \psi_2 \Rightarrow \Delta$ we obtain $\Gamma, \psi_1 \wedge \psi_2, \psi_2 \Rightarrow \Delta$. By applying the second to this sequent we obtain $\Gamma, \psi_1 \wedge \psi_2 \Rightarrow \Delta$. The other direction of the equivalence is obvious in view of weakening. Moreover: a given cut-free proof that uses the new rules can trivially be transformed into a cut-free proof that uses the original rule.

Next, the first of the two new rules can again be replaced by the following pair of full rules:

$$\{p_1 \Rightarrow\ ,\ p_2 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow \qquad \{p_1 \Rightarrow\ ,\ \Rightarrow p_2\} \ / \ p_1 \wedge p_2 \Rightarrow$$

This pair is equivalent to the original (separated) rule. Indeed, using the two new rules, the original one can be simulated as follows. Given $\Gamma, \psi_1 \Rightarrow \Delta$, an application of the first of the two new rules to this sequent and to the standard axiom $\psi_2 \Rightarrow \psi_2$ yields $\Gamma, \psi_1 \wedge \psi_2 \Rightarrow \Delta, \psi_2$. By applying the second of the two new rules to this sequent and to $\Gamma, \psi_1 \Rightarrow \Delta$ we obtain $\Gamma, \psi_1 \wedge \psi_2 \Rightarrow \Delta$. The other direction of the equivalence is obvious, and again does not use the cut rule.

The second separated rule above is similarly split into the following full rules:

$$\{p_2 \Rightarrow\ ,\ p_1 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow \qquad \{p_2 \Rightarrow\ ,\ \Rightarrow p_1\} \ / \ p_1 \wedge p_2 \Rightarrow$$

To conclude, from the first original classical two rules for conjunction, we obtain the following equivalent set of four full rules:

$$\{p_1 \Rightarrow\ ,\ p_2 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow \qquad \{p_1 \Rightarrow\ ,\ \Rightarrow p_2\} \ / \ p_1 \wedge p_2 \Rightarrow$$
$$\{\ \Rightarrow\ p_1,\ p_2 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow \qquad \{\ \Rightarrow p_1\ ,\ \Rightarrow p_2\} \ / \ \Rightarrow p_1 \wedge p_2$$

The general procedure for replacing a non-separated rule R by an equivalent set of separated ones is to put first in this set every separated rule which can be obtained by selecting exactly one formula from each premise of R (preserving its side). We then use Proposition 1 to remove any superfluous or redundant rule. The general procedure for splitting a given separated rule R of arity n into an equivalent set of full rules is to put in this set every full rule which has the same conclusion as R, and whose set of premises is an extension of that of R (thus if R has $m < n$ premises, it will be split into 2^{n-m} full rules). It is not difficult to see that these transformations preserve coherence of a system as well as its set of provable sequents, and that any cut-free proof in the resulting system can be simulated by a cut free proof in the original one (the converse is also true, but will be proved later).

Example 7. Suppose we have a canonical rule for a ternary connective:

$$\{p_1, p_2 \Rightarrow \ , \ p_1 \Rightarrow p_2 \ , \ p_3 \Rightarrow p_2\} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow$$

The first stage of the above process produces the following rules:

$$\begin{array}{rl}
(1) & \{p_1 \Rightarrow \ , \ p_3 \Rightarrow \ \} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow \\
(2),(4) & \{p_1 \Rightarrow \ , \ \Rightarrow p_2\} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow \\
(3) & \{p_1 \Rightarrow \ , \ \Rightarrow p_2 \ , \ p_3 \Rightarrow \ \} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow \\
(5) & \{p_2 \Rightarrow \ , \ p_1 \Rightarrow \ , \ p_3 \Rightarrow \ \} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow \\
(6) & \{p_2 \Rightarrow \ , \ p_1 \Rightarrow \ , \ \Rightarrow p_2\} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow \\
(7) & \{p_2 \Rightarrow \ , \ \Rightarrow p_2 \ , \ p_3 \Rightarrow \ \} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow \\
(8) & \{p_2 \Rightarrow \ , \ \Rightarrow p_2\} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow
\end{array}$$

(6),(7),(8) are superfluous and we discard them. (3),(5) are redundant because of rules (1),(2), and we discard them as well. The next stage is to extend (1),(2) into full rules:

$$\begin{array}{l}
\{p_1 \Rightarrow \ , \ p_2 \Rightarrow \ , \ p_3 \Rightarrow \ \} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow \\
\{p_1 \Rightarrow \ , \ \Rightarrow p_2 \ , \ p_3 \Rightarrow \ \} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow \\
\{p_1 \Rightarrow \ , \ \Rightarrow p_2 \ , \ \Rightarrow p_3\} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow
\end{array}$$

Notation 1 Let $\mathbf{G}$ be a canonical calculus. By $\mathbf{G}^F$ we shall denote the equivalent full calculus that is obtained using the process of Proposition 3.

We shall later show that if $\mathbf{G}$ is coherent then $\mathbf{G}^F$ is the unique full canonical calculus that is equivalent to $\mathbf{G}$.

Corollary 1. *If cut elimination obtains for every coherent full canonical calculus then it obtains for every coherent canonical calculus.*

It remains therefore to show that cut elimination obtains for every coherent full canonical calculus. For that, we shall need to use some semantic arguments. The corresponding semantics will be described next.

4 Semantics: Two-Valued Non-Deterministic Matrices

For the semantics of coherent canonical calculi we need structures which generalize the ordinary concept of a multi-valued matrix. The idea behind this generalization is to allow non-deterministic computations of truth-values. Thus the value that a valuation assigns in these structures to a complex formula is not always uniquely determined by the values that it assigns to its subformulas, but can be chosen non-deterministically from a certain nonempty set of options. The precise definition is as follows:

Definition 9. [AL00] *A* non-deterministic matrix *(*Nmatrix *for short) for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{T}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{T}$ (its designated values), and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{\diamond}$ from $\mathcal{T}^n$ to $2^{\mathcal{T}} - \{\emptyset\}$. A* valuation *in $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{T}$ that satisfies the condition: if $\diamond$ is an n-ary connective, and $\psi_1, \ldots, \psi_n \in \mathcal{W}$, then $v(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$. v* satisfies *a formula ψ in $\mathcal{M}$ ($v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a* model of *Γ in $\mathcal{M}$ ($v \models^{\mathcal{M}} \Gamma$) if it satisfies every formula in Γ. Δ* follows *from Γ in $\mathcal{M}$ ($\Gamma \vdash_{\mathcal{M}} \Delta$) if for every model v of Γ in $\mathcal{M}$, $v \models^{\mathcal{M}} \phi$ for some $\phi \in \Delta$.*

Notes:

1. Every (deterministic) matrix[4] can be identified with an Nmatrix whose functions in $\mathcal{O}$ always return singletons.
2. It is easy to verify that if $\mathcal{M}$ is an Nmatrix for $\mathcal{L}$ then $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a logic. In [AL00] it is proved that if $\mathcal{M}$ is finite then this logic is necessarily finitary (i.e., the compactness theorem obtains for it).

In this paper we shall use a special type of Nmatrices: those with exactly two truth values, which may be identified with the classical truth values. We shall show in fact that there is a strong connection between such Nmatrices and coherent canonical Gentzen-type calculi. [5]

Notation 2 A two-valued Nmatrix in which $\mathcal{T} = \{t, f\}$ and $\mathcal{D} = \{t\}$ will be called a *2-Nmatrix.*

Notation 3 For $x \in \{t, f\}$, denote: $-x = f$ if $x = t$, and $-x = t$ if $x = f$.

Notation 4 The expression Φ, a^x will denote $\Phi \cup \{a\}$ if $x = t$ and Φ if $x = f$ (note that Φ might be empty here).

Notation 5 The full canonical rule $\{p_i^{-x_i} \Rightarrow p_i^{x_i}\}_{1 \leq i \leq n} \;/\; \diamond(p_1, \ldots, p_n) \Rightarrow$ where $x_1, \ldots, x_n \in \{t, f\}$ will be denoted below by either $[\diamond(x_1, \ldots, x_n) : f]$ or by $[\diamond(x_1, \ldots, x_n) \Rightarrow]$. The rule with the same premises but with the complementary conclusion will be denoted by $[\diamond(x_1, \ldots, x_n) : t]$ or by $[\Rightarrow \diamond(x_1, \ldots, x_n)]$.

Definition 10.

1. *Let* **G** *be a coherent full canonical calculus. The 2-Nmatrix that is* defined by **G** *is the following: For each n-ary connective $\diamond$ and every $x_1, \ldots, x_n \in \{t, f\}$ we define*

$$\tilde{\diamond}(x_1, \ldots, x_n) = \begin{cases} \{y\} & \textit{if } [\diamond(x_1, \ldots, x_n) : y] \textit{ is a rule of } \mathbf{G} \ (\textit{for } y \in \{t, f\}) \\ \{t, f\} & \textit{otherwise} \end{cases}$$

   (This is well-defined since we assume that **G** is coherent.)

[4] See e.g. [Urq86].

[5] Valuations in two-valued Nmatrices form a special type of what are called *bivaluations* in [Béz99]. Another related idea is Meyer's *metavaluations* (see e.g. [Dun86]).

2. *The 2-Nmatrix that is defined by a coherent canonical calculus* **G** *is the 2-Nmatrix that is defined by* $\mathbf{G}^F$.

Example 8. Suppose **G** has only one rule for the ternary connective $\diamond$: the one given at the beginning of Example 7. Then the three final rules obtained in that example determine the following interpretation of $\diamond$: $\tilde{\diamond}(f,f,f) = \tilde{\diamond}(f,t,f) = \tilde{\diamond}(f,t,t) = \{f\}$ (and $\tilde{\diamond}(x_1,x_2,x_3) = \{t,f\}$ for all other x_1, x_2, x_3).

Proposition 4. *Every coherent canonical calculus* **G** *is sound for the 2-Nmatrix that it defines.*

Proof. By Proposition 3 and Definition 10, we may assume w.l.o.g. that **G** is full. Let $\mathcal{M}$ be the 2-Nmatrix that is defined by **G**. We say that a valuation v in $\mathcal{M}$ *satisfies* a sequent $\Gamma \Rightarrow \Delta$ if $v(\psi) = t$ for some $\psi \in \Delta$, or $v(\psi) = f$ for some $\psi \in \Gamma$. It is easy to verify that:
(1) A valuation v satisfies a sequent $\Gamma, \psi^{-x} \Rightarrow \Delta, \psi^x$ iff either v satisfies $\Gamma \Rightarrow \Delta$, or $v(\psi) = x$.
(2) $\Gamma \vdash_{\mathcal{M}} \Delta$ iff every valuation v in $\mathcal{M}$ satisfies $\Gamma \Rightarrow \Delta$.

Consider now an application of the rule $[\diamond(x_1,\ldots,x_n) : y]$, and assume that v is a valuation which satisfies all the premises $\{\Gamma_i, \psi_i^{-x_i} \Rightarrow \Delta_i, \psi_i^{x_i}\}_{1\leq i\leq n}$ of this application. Then v satisfies also its conclusion, $\Gamma, \psi^{-y} \Rightarrow \Delta, \psi^y$. Indeed, either v satisfies $\Gamma_i \Rightarrow \Delta_i$ for some i, and hence also $\Gamma \Rightarrow \Delta$, or else $v(\psi_i) = x_i$ for all $1 \leq i \leq n$, and since $\tilde{\diamond}(x_1,\ldots,x_n) = \{y\}$, necessarily $v(\psi) = y$. In both cases (1) entails that v satisfies $\Gamma, \psi^{-y} \Rightarrow \Delta, \psi^y$. It follows by (2) that $[\diamond(x_1,\ldots,x_n) : y]$ is sound for $\mathcal{M}$.

Corollary 2. *A canonical calculus is consistent iff it is coherent.*

Proof. It is easy to show that if a calculus is consistent then it is coherent. The converse follows from Proposition 4, and the fact that every Nmatrix induces a consistent logic.

Corollary 3. *The consistency of a canonical calculus is decidable.*

5 Completeness and Cut-elimination

Notation 6 For $x \in \{t,f\}$, denote: $ite(x,A,B) =$ if x then A else B. Note: $ite(x,A,B) = ite(-x,B,A)$.

Theorem 7. *Every coherent full canonical calculus admits cut elimination, and it is complete for the 2-Nmatrix that it defines.*

Proof. By Proposition 4, we can prove completeness and cut elimination together by showing that if $\Gamma \Rightarrow \Delta$ does not have a cut-free proof in a coherent full canonical calculus **G**, then $\Gamma \not\vdash_{\mathcal{M}} \Delta$, where $\mathcal{M}$ is the 2-Nmatrix defined by **G**. For this extend first $\Gamma \Rightarrow \Delta$ to a sequent $\Gamma^* \Rightarrow \Delta^*$ with the following properties:

1. $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$.

2. $\Gamma^* \Rightarrow \Delta^*$ does not have a cut-free proof in **G**.
3. For every rule $[\diamond(x_1, \ldots, x_n) : y]$ in **G**, if $\diamond(\psi_1, \ldots, \psi_n) \in ite(y, \Delta^*, \Gamma^*)$ then for some $1 \leq i \leq n$, $\psi_i \in ite(x_i, \Delta^*, \Gamma^*)$.

This extension is possible, because if a sequent $\Gamma' \Rightarrow \Delta'$ does not have a cut-free proof and $\diamond(\psi_1, \ldots, \psi_n) \in ite(y, \Delta', \Gamma')$ then for some $1 \leq i \leq n$, $\Gamma', \psi_i^{-x_i} \Rightarrow \Delta', \psi_i^{x_i}$ does not have a cut-free proof (because otherwise by adding an application of $[\diamond(x_1, \ldots, x_n) : y]$ to the proofs of these sequents we obtain a cut-free proof for $\Gamma', \psi^{-y} \Rightarrow \Delta', \psi^y$, which is exactly $\Gamma' \Rightarrow \Delta'$).

The refuting valuation is now defined as follows:
For atomic q, $v(q) = t$ iff $q \in \Gamma^*$.

$$v(\diamond(\psi_1, \ldots, \psi_n)) = \begin{cases} t & \text{if } \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n)) = \{t\} \text{ or} \\ & \quad [\tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n)) = \{t, f\} \text{ and } \diamond(\psi_1, \ldots, \psi_n) \in \Gamma^*] \\ f & \text{otherwise} \end{cases}$$

v is obviously a legal $\mathcal{M}$-valuation. We now show by induction on the complexity of a formula $\psi \in \Gamma^* \cup \Delta^*$ that if $\psi \in \Gamma^*$ then $v(\psi) = t$, and if $\psi \in \Delta^*$ then $v(\psi) = f$.

- Assume ψ is atomic. If $\psi \in \Gamma^*$ then $v(\psi) = t$ by definition. If $\psi \in \Delta^*$ then $\psi \notin \Gamma^*$ by property 2 of $\Gamma^* \Rightarrow \Delta^*$. Hence $v(\psi) = f$.
- Let $\psi = \diamond(\psi_1, \ldots, \psi_n)$ and let $x_i = v(\psi_i)$ for $1 \leq i \leq n$.
  Assume $\psi \in \Gamma^*$, but $v(\psi) = f$. According to the definition of v, this can happen only if $\tilde{\diamond}(x_1, \ldots, x_n) = \{f\}$. It follows that $\psi \in ite(v(\psi), \Delta^*, \Gamma^*)$ in this case. Hence $\psi_i \in ite(x_i, \Delta^*, \Gamma^*)$ for some $1 \leq i \leq n$ by property 3 of $\Gamma^* \Rightarrow \Delta^*$ and the fact that by Definition 10, $\tilde{\diamond}(x_1, \ldots, x_n) = \{f\}$ in $\mathcal{M}$ only if $[\diamond(x_1, \ldots, x_n) : f]$ is a rule of **G**. On the other hand $\psi_i \in ite(x_i, \Gamma^*, \Delta^*)$ for all $1 \leq i \leq n$ by the induction hypothesis. This contradicts property 2 of $\Gamma^* \Rightarrow \Delta^*$.
  Now assume $\psi \in \Delta^*$, but $v(\psi) = t$. According to the definition of v, there are two possibilities here:
  1. $\tilde{\diamond}(x_1, \ldots, x_n) = \{t\} = \{v(\psi)\}$. We get from this a contradiction like in the previous case.
  2. $\tilde{\diamond}(x_1, \ldots, x_n) = \{t, f\}$ and $\psi \in \Gamma^*$. Since $\psi \in \Delta^*$ as well, this contradicts property 2 of $\Gamma^* \Rightarrow \Delta^*$.

By property 1 of $\Gamma^* \Rightarrow \Delta^*$ and what we have just proved, v is a model of Γ in $\mathcal{M}$ which does not satisfy any element of Δ. Hence $\Gamma \not\vdash_{\mathcal{M}} \Delta$.

Theorem 8. *A canonical calculus admits cut elimination iff it is coherent.*

Proof. The "only if" part is just Proposition 2. For the "if" part, suppose $\Gamma \Rightarrow \Delta$ has a proof in a coherent canonical calculus **G**. Let $\mathcal{M}$ be the 2-Nmatrix that is defined by **G** (and $\mathbf{G}^F$). Then $\Gamma \vdash_{\mathcal{M}} \Delta$ by Proposition 4. Theorem 7 implies therefore that $\Gamma \Rightarrow \Delta$ has a cut-free proof in $\mathbf{G}^F$. Hence, by Proposition 3, it also has a cut-free proof in **G**.

Theorem 9. *Every coherent canonical calculus is sound and complete for the 2-Nmatrix that it defines.*

Proof. This immediately follows from Theorem 7 and Proposition 4.

Theorem 10. *Let* **G** *be a canonical calculus. Then either* **G** *is inconsistent, or it defines a logic which is a fragment of classical logic, or it has no finite characteristic matrix.*

Proof. Assume that **G** is consistent. Then it defines a logic $L(\mathbf{G})$. By Corollary 2 and Theorem 9, $L(\mathbf{G})$ is induced by some 2-Nmatrix. If this 2-Nmatrix includes only deterministic connectives (i.e.: connectives which return singletons for every combination of truth values), then **G** has a characteristic two-valued *matrix*, and so it is a fragment of classical logic. Otherwise it has no finite characteristic matrix by Theorem 4.10 of [AL00].

We turn now to some more corollaries of the completeness theorem. First, a result that was promised at the end of section 3:

Corollary 4. *If* **G** *is a coherent canonical calculus then it has a unique equivalent full canonical calculus.*

Proof. Let **G** be a coherent canonical calculus, and let $\mathbf{G}'$ be a full canonical calculus that is equivalent to **G**. We need to show that $\mathbf{G}' = \mathbf{G}^F$. Suppose this is not the case. Let $\mathcal{M}^F$ and $\mathcal{M}'$ be the 2-Nmatrices that are defined by $\mathbf{G}^F$ and $\mathbf{G}'$ respectively. Since the two calculi are different then according to Definition 10 there is some n-ary connective $\diamond$ and some $x_1, \ldots, x_n \in \{t, f\}$ such that the interpretation of $\diamond$ on $x_1, \ldots, x_n$ is different in $\mathcal{M}^F$ and in $\mathcal{M}'$. Suppose w.l.o.g. that in $\mathcal{M}^F$, $\tilde{\diamond}(x_1, \ldots, x_n)$ is $\{f\}$, whereas in $\mathcal{M}'$ it is either $\{t\}$ or $\{t, f\}$. It is easy to see that $\{p_i \mid x_i = t\} \cup \{\diamond(p_1, \ldots, p_n)\} \vdash_{\mathcal{M}^F} \{p_i \mid x_i = f\}$, while this is not the case in $\vdash_{\mathcal{M}'}$. Hence, by Theorem 7, $\mathbf{G}^F$ and $\mathbf{G}'$ are not equivalent. This contradicts the fact that they are both equivalent to **G**.

Our next results compare strength of rules and introduce a normal form.

Definition 11. *Let R_1 and R_2 be two canonical rules (in the same language). We say that R_1 is* at least as strong *as R_2 if any application of R_2 can be simulated using R_1 and the standard axioms and structural rules (including cut). We say that R_1 and R_2 are* equivalent *if each of them is at least as strong as the other.*

The characterization below of the strength of a rule can be summarized as follows: a rule is stronger when its set of premises is weaker!

Proposition 5. *A canonical rule S_1/C_1 is at least as strong as the canonical rule S_2/C_2 iff $C_1 = C_2$, and every clause in S_1 classically follows from S_2 (this is equivalent to saying that every clause in S_1 is subsumed by some clause that can be derived from the clauses of S_2 using resolutions).*

Proof. The "if" part can easily be proved directly. For the converse we need the completeness theorem. We omit the details.

Corollary 5. *Two canonical rules S_1/C and S_2/C are equivalent if S_1 and S_2 are classically equivalent (as sets of clauses).*

This corollary naturally leads to the following economical normal form for canonical rules:

Definition 12. *A canonical rule is in* Resolution Normal Form *(RNF) if its set of premises S does not include a standard axiom, and any resolvent of two elements of S is subsumed by some other element of S.*

Corollary 6. *Every canonical rule has an equivalent canonical rule in RNF.*

An example of transforming a rule to *RNF* will be given in the next section.

6 Calculi for 2-Nmatrices

In the previous section we associate with every coherent canonical calculus a 2-Nmatrix, for which it is sound and complete. In this section we go in the other direction, and associate with a given 2-Nmatrix coherent canonical calculi which are sound and complete for it. One way of doing so is rather obvious:

Definition 13. *Let $\mathcal{M}$ be a 2-Nmatrix. The full calculus that is defined by $\mathcal{M}$ is the canonical calculus* **G** *that has the rule $[\diamond(x_1, \ldots, x_n) : y]$ for each n-ary connective $\diamond$ and for every $x_1, \ldots, x_n, y \in \{t, f\}$ such that $\widetilde{\diamond}(x_1, \ldots, x_n) = \{y\}$.*

Note: The full calculus that is defined by a 2-Nmatrix is obviously coherent.

Proposition 6. *The full calculus that is defined by a 2-Nmatrix is sound and complete for it.*

Proof. The proof is similar to that of Proposition 4 and Theorem 7.

We introduce now for any given 2-Nmatrix a calculus of a more regular form.

Theorem 11. *Every 2-Nmatrix $\mathcal{M}$ has a sound and complete coherent canonical calculus which for every connective has at most one introduction rule on the left, and at most one introduction rule on the right.*

Proof. Let $\mathbf{G}(\mathcal{M})$ be the canonical calculus which for any n-ary connective $\diamond$ has the following rules (where $\widetilde{\diamond}$ is the interpretation of $\diamond$ in $\mathcal{M}$):

$$[\diamond \Rightarrow] \quad \{\{p_i \mid x_i = t\} \Rightarrow \{p_i \mid x_i = f\}\}_{t \in \widetilde{\diamond}(x_1,\ldots,x_n)} \; / \; \diamond(p_1, \ldots, p_n) \Rightarrow$$

$$[\Rightarrow \diamond] \quad \{\{p_i \mid x_i = t\} \Rightarrow \{p_i \mid x_i = f\}\}_{f \in \widetilde{\diamond}(x_1,\ldots,x_n)} \; / \; \Rightarrow \diamond(p_1, \ldots, p_n)$$

Note that if $t \in \widetilde{\diamond}(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n$, then the first rule is superfluous and can be discarded, while if $\widetilde{\diamond}(x_1, \ldots, x_n) = \{f\}$ for all $x_1, \ldots, x_n$ then that rule does not have any premises, i.e. it is a nonstandard axiom (this type of axioms is permitted in canonical systems!). Similarly, if $f \in \widetilde{\diamond}(x_1, \ldots, x_n)$ for all

$x_1, \ldots, x_n$ then the second rule can be discarded, while if $\widetilde{\diamond}(x_1, \ldots, x_n) = \{t\}$ for all $x_1, \ldots, x_n$ then that rule does not have any premises.

The soundness of $\mathbf{G}(\mathcal{M})$ is easy to verify. Take for example $[\diamond \Rightarrow]$. To show its soundness, assume that $\Gamma, \psi_1^{x_1}, \ldots, \psi_n^{x_n} \vdash_{\mathcal{M}} \Delta, \psi_1^{-x_1}, \ldots, \psi_n^{-x_n}$ for all $x_1, \ldots, x_n$ such that $t \in \widetilde{\diamond}(x_1, \ldots, x_n)$, and let v be a model of $\Gamma \cup \{\diamond(\psi_1, \ldots, \psi_n)\}$ in $\mathcal{M}$. Then there are $y_1, \ldots, y_n \in \{t, f\}$ such that $t \in \widetilde{\diamond}(y_1, \ldots, y_n)$ and $v(\psi_i) = y_i$ for all i. Since $\Gamma, \psi_1^{y_1}, \ldots, \psi_n^{y_n} \vdash_{\mathcal{M}} \Delta, \psi_1^{-y_1}, \ldots, \psi_n^{-y_n}$ by assumption, it follows that v is a model of one of the elements of Δ. The soundness of $[\Rightarrow \diamond]$ is proved similarly, while the proof of completeness is similar to that of Theorem 7.

Corollary 7. *Every coherent canonical calculus has an equivalent canonical calculus in which every connective has at most one introduction rule for each side.*

Example 9. Suppose we have the following interpretation for a binary connective $\diamond$, which makes it a very close relative of the classical conjunction:

$$\widetilde{\diamond}(t,t) = \{t\}, \quad \widetilde{\diamond}(t,f) = \{t,f\}, \quad \widetilde{\diamond}(f,t) = \widetilde{\diamond}(f,f) = \{f\}$$

The corresponding two rules as given in the proof of the last theorem are:

$$[\diamond \Rightarrow] \quad \{p_1, p_2 \Rightarrow \ , \ p_1 \Rightarrow p_2\} \ / \ \diamond(p_1, p_2) \Rightarrow$$

$$[\Rightarrow \diamond] \quad \{p_1 \Rightarrow p_2 \ , \ p_2 \Rightarrow p_1 \ , \ \Rightarrow p_1, p_2\} \ / \ \Rightarrow \diamond(p_1, p_2)$$

We next transform these two rules into rules in *RNF* as follows. Consider the set of premises of the second rule. Its closure under cut is:

$$\{p_1 \Rightarrow p_2 \ , \ p_2 \Rightarrow p_1 \ , \ \Rightarrow p_1, p_2 \ , \ \Rightarrow p_1 \ , \ \Rightarrow p_2 \ , \ p_1 \Rightarrow p_1 \ , \ p_2 \Rightarrow p_2\}$$

We now discard the last two standard axioms, and remove also the original three clauses since they are subsumed by $\Rightarrow p_1$ and $\Rightarrow p_2$. A similar process can be applied to the first rule. We are left with the simpler rules:

$$[\diamond \Rightarrow]' \quad \{p_1 \Rightarrow \ \} \ / \ \diamond(p_1, p_2) \Rightarrow$$

$$[\Rightarrow \diamond]' \quad \{\Rightarrow p_1 \ , \ \Rightarrow p_2\} \ / \ \Rightarrow \diamond(p_1, p_2)$$

Note that both rules are frequently used in the literature as introduction rules for classical conjunction.

Note: Given a 2-Nmatrix $\mathcal{M}$, the system $\mathbf{G}(\mathcal{M})$ which is constructed in the proof of Theorem 11 is a natural basis for a tableaux proof system for validity in $\mathcal{M}$. In fact, an application of a rule in $\mathbf{G}(\mathcal{M})$ backwards is like a step in a tableaux. For example: $[\diamond \Rightarrow]$ says that if $v(\diamond(\psi_1, \ldots, \psi_n)) = t$ then there are some $x_1, \ldots, x_n \in \{t, f\}$ such that $t \in \widetilde{\diamond}(x_1, \ldots, x_n)$.

7 Conclusion

We defined canonical calculi which are the most natural type of multiple conclusion Gentzen-type systems, and showed that such calculi are non-trivial iff they satisfy a certain constructive coherence condition. We introduced the semantics of two-valued non-deterministic matrices (2-Nmatrices) for such calculi, and proved that the following are equivalent for any given logic L:

1. L is defined by some coherent, canonical Gentzen-type system.
2. L is defined by some cut-free, canonical Gentzen-type system.
3. L is the logic of some 2-Nmatrix.

One of the by-products of our work is a strong evidence for the thesis according to which the meaning of a connective is given by its introduction (and "elimination") rules (in some appropriate deduction system). We have shown that at least in the framework of multiple-conclusion consequence relations, any reasonable set of canonical introduction rules completely determine the semantics of a connective. For this it is not even necessary that the left introduction rules and the right introduction rules for a given connective precisely "match" (in the sense of [Bel62] and [Sun86]). It suffices that there would be no conflict between them (where this condition is defined in precise terms).

Obvious directions for further research are the following:

1. To extend the ideas and results to first order languages.
2. To develop an analogous framework and theory for single-conclusion consequence relations and Natural Deduction systems.
3. To generalize the framework to arbitrary finite n-valued Nmatrices, possibly using sequents with n components like e.g. in [BFZ94][6] (see also the survey papers [BFS00] and [Häh99] for more references and further details).

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