Formulas for which Contraction is Admissible

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Abstract

A formula A is said to have the contraction property in a logic L iff whenever $A, A, \Gamma \vdash_L B$ (when Γ is a multiset) also $A, \Gamma \vdash_L B$. In MLL and in $MALL$ without the additive constants a formula has the contraction property iff it is a theorem. Adding the mix rule does not change this fact. In $MALL$ (with or without mix) and in affine logic A has the contraction property iff either A is provable or A is equivalent to the additive constant 0. We present some general proof-theoretical principles from which all these results (and others) easily follow.

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In [1] it is shown, using a somewhat complicated semantic argument, that if A is a formula of the multiplicative fragment of Linear Logic, for which both weakening and contraction on the left-hand side of $\Rightarrow$ are admissible, then A is equivalent to the multiplicative constant 1.[1] This paper extends this result (and its obvious dual for the right-hand side of $\Rightarrow$) in several directions:

(1) We give a simple independent characterization of the formulas for which the contraction rule alone is admissible.

(2) We provide analogous results for extensions of the multiplicative language with the additive operators and for the full multiplicative-additive fragment of LL.

(3) We provide analogous results for the extensions of the various fragments of Linear Logic discussed above with the "mix" rule or the weakening rule.

(4) We provide general proof-theoretical theorems from which all the above results easily follow.

DEFINITION 1.1

(i) MLL is the two-sided Gentzen-type system for the multiplicative fragment of Linear Logic.

(ii) $MALL$ is the two-sided system for the multiplicative-additive fragment of Linear Logic (propositional LL without the exponentials).

(iii) $MALL^-$ is like $MALL$, but without the additive constants T and 0.

DEFINITION 1.2

Let A be a formula, Γ – a multiset of formulas.

[1] We usually follow here the notations of [2], except that linear implication is denoted below by the usual symbol $\rightarrow$, as in all other substructural logics.

(i) A^n is defined inductively as follows: $A^1 = A$, $A^{n+1} = A^n \otimes A$.
(ii) $|A|$ is the length of A (i.e. the number of symbols in A).
(iii) $|\Gamma|$ is the sum of the lengths of the formulas in Γ.

Our main tool in what follows will be theorems of the following type:

THEOREM 1.3
Let G be any of the following systems: MLL, $MLL{+}mix$, $MALL^-$, $MALL^-{+}mix$.[2]
Suppose $n > |\Gamma| + |\Delta|$. Then $\vdash_G \Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ only if $\vdash_G \Rightarrow A_i$ for some i.

PROOF. By a double induction on n and on the length of a cut-free proof of $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$.
Base: $n = 1$. Then $|\Gamma| + |\Delta| = 0$. This means that $\Gamma = \Delta = \emptyset$ and so $\vdash_G \Rightarrow A_1$.
Induction step: Assume the claim for $n-1$. We prove it for n. So suppose that $\vdash_G \Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ and $|\Gamma| + |\Delta| < n$. Then $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ cannot be an axiom $B \Rightarrow B$. We have therefore three cases to consider:

(1) The last inference in the proof of $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ does not involve $A_1 \otimes ... \otimes A_n$. Then $A_1 \otimes ... \otimes A_n$ is present in the r.h.s of at least one of the premises of this last inference. By applying the inner induction hypothesis to that premise (something we can do because of the nature of the rules of these systems) we get $\vdash_G \Rightarrow A_i$ for some i.
(2) $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ is inferred from $\Rightarrow A_n$ and $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_{n-1}$ by $(\Rightarrow \otimes)$: This case is trivial.
(3) $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ is inferred from $\Gamma_1 \Rightarrow \Delta_1, A_n$ and $\Gamma_2 \Rightarrow \Delta_2, A_1 \otimes ... \otimes A_{n-1}$ by $(\Rightarrow \otimes)$ and $\Gamma_1 \cup \Delta_1 \neq \emptyset$. In this case $|\Gamma_2| + |\Delta_2| < |\Gamma| + |\Delta| < n$, and so $|\Gamma_2| + |\Delta_2| < n - 1$. We can therefore apply the main induction hypothesis to $\Gamma_2 \Rightarrow \Delta_2, A_1 \otimes ... \otimes A_{n-1}$.

■

NOTE 1.4
The validity of the cut-elimination theorem (which obtains, by [2], for all these systems) is crucial here: the argument in Case 1 fails in case cut is used, since the cut formula may be of arbitrary length.

DEFINITION 1.5 ([3])
Given a Gentzen-type system G with the permutation rule, a formula A of its language is called:

(i) $(W-L)$ formula if $\vdash_G A, \Gamma \Rightarrow \Delta$ whenever $\vdash_G \Gamma \Rightarrow \Delta$.
(ii) $(W-R)$ formula if $\vdash_G \Gamma \Rightarrow \Delta, A$ whenever $\vdash_G \Gamma \Rightarrow \Delta$.
(iii) $(C-L)$ formula if $\vdash_G A, \Gamma \Rightarrow \Delta$ whenever $\vdash_G A, A, \Gamma \Rightarrow \Delta$.
(iv) $(C-R)$ formula if $\vdash_G \Gamma \Rightarrow \Delta, A$ whenever $\vdash_G \Gamma \Rightarrow \Delta, A, A$.

LEMMA 1.6
If A is a $(C-L)$ formula of some extension G of MLL then $\vdash_G A \Rightarrow A^n$ for all $n \geq 1$.

PROOF. By induction on n, using the fact that $A, A \Rightarrow A^{n+1}$ follows from $A \Rightarrow A$ and $A \Rightarrow A^n$ by $(\Rightarrow \otimes)$. ■

[2] "mix" is the name given in [2] to the rule which allows to infer $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ from $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$.

THEOREM 1.7
Let G be any of the following systems: MLL, $MLL+mix$, $MALL^-$, $MALL^-+mix$. Then A is a $(C-L)$ formula of G iff $\vdash_G A$ (i.e. if $\vdash_G \Rightarrow A$).

PROOF. The "if" part is trivial by a cut. The "only if" is an immediate consequence of lemma 1.6 and theorem 1.3. ∎

COROLLARY 1.8
A is both a $(W-L)$ and a $(C-L)$ formula of G (where $G \in \{MLL, MALL^-, MLL+mix, MALL^- + mix\}$) iff A is equivalent in G to 1.

PROOF. Obviously, $\vdash_G \Rightarrow A$ iff $\vdash_G 1 \Rightarrow A$. Hence A is a $(C-L)$ formula of G iff $\vdash_G 1 \Rightarrow A$ (by theorem 1.7). It is easy to see, on the other hand ([1]) that A is a $(W-L)$ formula of G iff $\vdash_G A \Rightarrow 1$ (indeed, if $\vdash_G \Gamma \Rightarrow \Delta$ then $\vdash_G 1, \Gamma \Rightarrow \Delta$. By applying a cut to this and to $\vdash_G A \Rightarrow 1$ we get $\vdash_G A, \Gamma \Rightarrow \Delta$. The converse is immediate from the the fact that $\vdash_G \Rightarrow 1$). ∎

NOTE 1.9
As noted above, the case $G = MLL$ of the last corollary was first (Γ) stated and proved in [1]. It was, however, implicit already in proposition 2.5 of [3].

COROLLARY 1.10
The rule: from $A \to A \otimes A$ infer A is admissible in all the four systems above.

PROOF. It is obvious that if $\vdash A \to A \otimes A$ then A is a $(C-L)$ formula. ∎

The importance of the last corollary is that it provides an example of an admissible rule of MLL (say) which is neither classically valid nor admissible in LL. To see the point, note that the rule which allows to infer A from $A \otimes B$ is also an admissible rule of MLL (and LL!) that is not a valid rule of MLL (trivial, by cut-elimination), but this rule is at least classically valid. The logic of admissible rules of MLL (and of other fragments of LL) seems to be rather strange indeed, and worth further investigations.

A generalization of corollary 1.10 is the following

COROLLARY 1.11
Let G be as in theorem 1.3. $\vdash_G A$ iff there is $k > 1$ such that $\vdash_G A \to A^k$.

PROOF. By induction on n it is easy to prove that if $\vdash_G A \Rightarrow A^k$ then $\vdash_G A \Rightarrow A^{(k^n)}$ for all $n \geq 1$. Indeed: from k copies of $A \Rightarrow A^{(k^n)}$ one can infer $A^k \Rightarrow A^{k^{n+1}}$, and since $\vdash_G A \Rightarrow A^k$, a cut gives $A \Rightarrow A^{(k^{n+1})}$. Now in the proof of theorem 1.7 all we really need is that $\vdash_G A \Rightarrow A^n$ for arbitrarily large n, and this is the case if $\vdash_G A \Rightarrow A^k$ $(k > 1)$ by the last observation. The converse is trivial, since $\vdash_G A \Rightarrow A^2$ whenever $\vdash_G A$. ∎

COROLLARY 1.12
Let G be as in theorem 1.3. A is equivalent in G to $A \otimes A$ iff there is a formula B such that A is equivalent in G to $B \to B$.

PROOF. It is easy to see that $B \to B$ is equivalent in G to $(B \to B) \otimes (B \to B)$. Hence the "if" part. For the converse, if A is equivalent to $A \otimes A$ then $\vdash_G A$ by corollary 1.10. Since $\vdash_G A \to \big((A \to A) \to A\big)$, it follows that $\vdash_G (A \to A) \to A$. That $\vdash_G A \to (A \to A)$, on the other hand, follows from $\vdash_G A \otimes A \to A$. Hence A is equivalent in G to $A \to A$, and we can take $B = A$. ∎

EXAMPLE 1.13

$A = ((p \to p) \to q) \to q$ is a theorem of G such that $A \otimes A \to A$ is not a theorem. Hence the condition in the last corollary is strictly stronger than that in theorem 1.7. On the other hand, unlike in the case of $A \to A \otimes A$, there are sentences A such that $\vdash_{MLL} A \otimes A \to A$ but $\not\vdash_{MLL} A$. In fact, if B has the property that $\vdash_G B \otimes B \to B$, then so does $C \otimes (C \to B)$ for arbitrary C, since

$$\vdash_G B \otimes B \to B, \big(C \otimes (C \to B)\big) \otimes \big(C \otimes (C \to B)\big) \Rightarrow C \otimes (C \to B).$$

NOTE 1.14

Obviously, duals to the last theorem and its corollaries are also valid. Thus A is a $(C-R)$ formula of G as above iff $\vdash_G A \Rightarrow$, and A is both a $(W-R)$ and a $(C-R)$ formula iff A is equivalent to $-$. In what follows we shall continue to formulate and prove our results only for the left-hand case, leaving the duals in the right-hand case to the reader.

THEOREM 1.15

Assume $G \in \{MALL, MALL + mix\}$. If $|\Gamma| + |\Delta| < n$ and $\vdash_G \Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ then either $\vdash_G \Rightarrow A_i$ for some i or $\vdash_G \Gamma \Rightarrow \Delta, 0$.

PROOF. Again, by a double induction on n and on the length of a cut-free proof of $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$. The base case is as in the proof of theorem 1.3. Assume next the claim for $n-1$. We show it for n.

Case 1: $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ is an axiom. This can happens only if either $0 \in \Gamma$ or $T \in \Delta$. In both cases $\Gamma \Rightarrow \Delta, 0$ is also an axiom.

Case 2: $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ is inferred by a mix, a one-premise additive rule, or a multiplicative rule in which $A_1 \otimes ... \otimes A_n$ is not the main formula. Then one of the premises is of the form $\Gamma^* \Rightarrow \Delta^*, A_1 \otimes ... \otimes A_n$ with $|\Gamma^*| + |\Delta^*| < |\Gamma| + |\Delta| < n$. By applying the inner I.H. (Induction Hypothesis) to this premise we get that either $\vdash_G \Rightarrow A_i$ for some i or $\vdash_G \Gamma^* \Rightarrow \Delta^*, 0$. The first case is what we want. In the second we infer $\Gamma \Rightarrow \Delta, 0$ from $\Gamma^* \Rightarrow \Delta^*, 0$ and the other premises of the last inference of the proof of $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ by exactly the same rule which is applied there.

Case 3: $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ is inferred from $\Gamma_1 \Rightarrow \Delta_1, A_1 \otimes ... \otimes A_n$ and $\Gamma_2 \Rightarrow \Delta_2, A_1 \otimes ... \otimes A_n$ by a two-premises additive rule. Then $|\Gamma_i| + |\Delta_i| < |\Gamma| + |\Delta|$ $(i = 1, 2)$, and so we can apply the inner I.H. to both premises and get that either $\vdash_G \Rightarrow A_j$ for some j or $\vdash_G \Gamma_i \Rightarrow \Delta_i, 0$ $(i = 1, 2)$. In the second case we infer $\Gamma \Rightarrow \Delta, 0$ by exactly the same additive rule.

Case 4: $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ is inferred from $\Rightarrow A_n$ and $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_{n-1}$. Then $\vdash_G \Rightarrow A_n$.

Case 5: $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ is inferred from $\Gamma_1 \Rightarrow \Delta_1, A_1 \otimes ... \otimes A_{n-1}$ and $\Gamma_2 \Rightarrow \Delta_2, A_n$ (where $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$), and $\Gamma_2 \cup \Delta_2 \neq \emptyset$. Then $|\Gamma_1| + |\Delta_1| < n - 1$ and we can apply the main I.H. to $\Gamma_1 \Rightarrow \Delta_1, A_1 \otimes ... \otimes A_{n-1}$. We get that either $\vdash_G \Rightarrow A_i$ for some $i < n$ or $\vdash_G \Gamma_1 \Rightarrow \Delta_1, 0$. Since $\vdash_G 0, \Gamma_2 \Rightarrow \Delta_2, 0$ we get in the second case that $\vdash_G \Gamma \Rightarrow \Delta, 0$ by a cut. ∎

THEOREM 1.16

A is a $(C-L)$ formula of $MALL$ or of $MALL + mix$ iff either $\vdash_G A$ or A is equivalent in G to the additive constant 0 $(G = MALL, MALL + mix)$.

PROOF. Since $\vdash_G 0, \Gamma \Rightarrow \Delta$ for all Γ, Δ, 0 is trivially both a $(C-L)$ and a $(W-L)$ formula. Hence the "if" part. For the converse take $n > |A|$ in lemma 1.6, and infer, using theorem 1.15, that either $\vdash_G \Rightarrow A$ or $\vdash_G A \Rightarrow 0$. Since $\vdash_G 0 \Rightarrow A$ for all A, the theorem follows. ∎

COROLLARY 1.17

(1) A formula is both $(W-L)$ and $(C-L)$ in $MALL$ (or $MALL + mix$) iff it is equivalent there to either the multiplicative 1 or to the additive 0.

(2) A formula is equivalent in $MALL$ ($MALL+mix$) to $A \otimes A$ if either A is equivalent to 0 or there exists B such that A is equivalent to $B \rightarrow B$.

(3) There is $k > 1$ s.t. $\vdash_G A \rightarrow A^k$ ($G = MALL, MALL + mix$) iff $\vdash_G A$ or $\vdash_G A \leftrightarrow 0$.

NOTE 1.18

The first part of corollary 1.17 has first been proved (in the case of $MALL$) in [3] [3].

We now turn to affine logic (also known as BCK logic). This is $MALL$ together with the weakening rule. In this logic every formula is, of course, a $(W-L)$ formula. Other relevant facts are that there is no difference in it between 1 and $\top$ or between 0 and $-$, and that A is a theorem iff it is equivalent to 1 while $A \Rightarrow$ is a theorem iff A is equivalent to 0. In view of these facts the next theorem just states that theorem 1.16 and corollary 1.17 are also valid for affine logic.

THEOREM 1.19

A is a $(C-L)$ formula of affine logic (without exponentials) or any of its fragments iff either $\Rightarrow A$ or $A \Rightarrow$ is provable in that fragment.

PROOF. Because of the weakening rule, if $\vdash A \Rightarrow$ then A is trivially a $(C-L)$ formula ($\vdash A, \Gamma \Rightarrow \Delta$ for all Γ, Δ). Hence the "if" part (the case where $\vdash \Rightarrow A$ again follows by using a cut).

The converse follows easily from lemma 1.6 and the next theorem. ∎

THEOREM 1.20

In affine logic, if $\vdash \Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ and $|\Gamma| + |\Delta| < n$ then either $\vdash \Rightarrow A_i$ for some i or $\vdash \Gamma \Rightarrow \Delta$.

PROOF. The proof is very similar to that of theorem 1.15, with the following modifications:

(1) Case 1 is not really needed.

(2) In Case 2, the subcase of mix should be replaced by the case of a weakening by a formula in Γ or Δ.

(3) In Case 5, $\Gamma \Rightarrow \Delta$ can be obtained from $\Gamma_1 \Rightarrow \Delta_1$ (in case it is the one which is provable) by weakenings.

(4) There is one more case to consider, in which $\Gamma \Rightarrow \Delta, A_1 \otimes ... \otimes A_n$ is inferred from $\Gamma \Rightarrow \Delta$ by a weakening. This case is, of course, trivial, since it means that $\vdash \Gamma \Rightarrow \Delta$.

∎

[3] This fact was brought to my knowledge after the first version of this paper had been written. Schellinx has also used a proof-theoretical argument, though different from the present one. Despite the fact that he has explicitly proved only this corollary, his method can be used to show most of the results of this paper. The general principles (theorems 1.3, 1.15 and 1.20 below) seem to be exceptions.

References

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