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Finite-valued Logics for Information Processing

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Abstract. We examine the issue of collecting and processing information from various sources, which involves handling incomplete and inconsistent information. Inspired by the framework first proposed by Belnap, we consider structures consisting of information sources which provide information about the values of formulas of classical propositional logic, and a processor which collects that information and extends it by deriving conclusions following from it according to the truth tables of classical logic, applied forward and backward. Our model extends Belnap's in allowing the sources to provide information also about complex formulas. As that framework cannot be captured by finite ordinary logical matrices, we use Nmatrices for that purpose. In opposition to the approach proposed in our earlier work, we assume that the information sources are reasonable, i.e. that they provide information consistent with certain coherence rules.

We provide sound and complete sequent calculi admitting strong cut elimination for the logic of a single information source, and (several variants of) the logic generated by the source and processor structures described above. In doing this, we also provide new characterizations for some known logics. We prove that, in opposition to the variant with unconstrained information sources considered earlier, the latter logic cannot be generated by structures with any bounded number of sources.

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1. Introduction

The goal of this paper is to present some novel applications of logics with finite-valued semantics (especially three-valued and four-valued ones) to information processing, and their relationships with some well-known previous approaches in that area. Our practical motivations stem from the necessity to deal on a daily basis with vast amounts of information provided by a multitude of various sources. That information is often incomplete or inconsistent — but we nevertheless need to draw reasonable conclusions from it, and use them a basis for our decisions and actions.

The first researcher to propose a logical framework for dealing with the above issue was Belnap. His famous model, presented in [10, 11], consisted of *information sources* which provided information about the values of *atomic* formulas of some logical language, and a *processor* which collected and processed information from the sources. Belnap's model was later extended by Carnielli and Lima-Marques [13], but none of the above proposals went beyond the simple case of sources providing information solely about *atomic* formulas.

As this is insufficient for dealing with many situations commonly encountered in various information environments, in this paper we consider a more general approach, where the sources may also provide information about *complex formulae*. Moreover, we assume that the sources are *reasonable* in the sense of providing coherent and consistent information.

The rest of the paper is organized as follows. In Section 2, we introduce logical preliminaries, and present the basic finite-valued matrices used for reasoning about information processing. In Section 3, we define the information processing framework discussed in the paper. We start with presenting Belnap's basic information processing model, and proving its correspondence to Dunn-Belnap's four-valued matrix, and in case of complete information — to Priest's three-valued matrix. Next we define general information processing structures for an arbitrary language $\mathcal{L}$, and the logics (consequence relations) introduced by classes of such structures. Section 4 is devoted to information processing structures for the language L_C of propositional classical logic. We begin with defining the standard information processor for L_C, and giving an effective way to compute the final valuation generated by such a processor. Then we define the notions of an information source and a standard existential information processing (EIP) structure for L_C. After recalling the notion of non-deterministic matrices (Nmatrices), we prove that the class of information sources is identical with the class of legal valuations in a certain three-valued Nmatrix, and the class of all processor valuations generated by EIP structures — with the class of legal valuations in a certain four-valued Nmatrix (and in case of structures providing complete information — in another three-valued Nmatrix). We also show that the corresponding consequence relations cannot be generated by ordinary finite-valued matrices. Finally, we prove that there are EIP processor valuations which cannot be generated by a finite number of source valuations, and that the logics generated by classes of all EIP structures with at most n sources are different for different values of n, as well as different from the logic generated by the class of all EIP structures. In Section 5, we present sequent proof systems for all the six considered logics, and prove their strong soundness, completeness and cut-elimination. Section 6 presents an outline of future work.

2. Basic Finite-valued Logics of Information

2.1. Logical Preliminaries

In the sequel, $\mathcal{L}$ denotes a propositional language with a set $\mathcal{A}$ of atomic formulas and a set $\mathcal{F}_\mathcal{L}$ of well-formed formulas. We denote the elements of $\mathcal{A}$ by p, q, r (possibly with subscripted indexes), and the elements of $\mathcal{F}_\mathcal{L}$ by ψ, φ. Sets of formulas in $\mathcal{F}_\mathcal{L}$ (theories) are denoted by $\mathcal{T}$, and finite sets of such formulas are denoted by Γ or Δ. Following the usual convention, we shall abbreviate $\Gamma \cup \{\psi\}$ by Γ, ψ. More generally, we shall write Γ, Δ instead of $\Gamma \cup \Delta$.

Definition 2.1. A (Tarskian) *consequence relation* for a language $\mathcal{L}$ is a binary relation $\vdash$ between theories in $\mathcal{F}_\mathcal{L}$ and formulas in $\mathcal{F}_\mathcal{L}$ satisfying the following three conditions:

Reflexivity: if $\psi \in \mathcal{T}$, then $\mathcal{T} \vdash \psi$.
Monotonicity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.
Transitivity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T}, \psi \vdash \varphi$, then $\mathcal{T} \vdash \varphi$.

Let $\vdash$ be a consequence relation for $\mathcal{L}$.

- We say that $\vdash$ is *structural* if, for every uniform $\mathcal{L}$-substitution θ and every $\mathcal{T}$ and ψ, if $\mathcal{T} \vdash \psi$ then $\theta(\mathcal{T}) \vdash \theta(\psi)$ (where $\theta(\mathcal{T}) = \{\theta(\varphi) \mid \varphi \in \mathcal{T}\}$).
- We say that $\vdash$ is *consistent* (or *non-trivial*) if there exist some non-empty theory $\mathcal{T}$ and some formula ψ such that $\mathcal{T} \not\vdash \psi$.
- We say that $\vdash$ is *finitary* if, for every theory $\mathcal{T}$ and every formula ψ such that $\mathcal{T} \vdash \psi$, there is a *finite* theory $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \vdash \psi$.

Definition 2.2.

1. A (propositional) *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$ such that $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural, consistent, and finitary consequence relation for $\mathcal{L}$.[1]
2. A logic $\langle \mathcal{L}, \vdash \rangle$ is *decidable* if, given a finite $\Gamma \subseteq \mathcal{F}_\mathcal{L}$ and $\psi \in \mathcal{F}_\mathcal{L}$, it is decidable whether $\Gamma \vdash \psi$ or not.

The most standard semantic (model-theoretical) way of defining a consequence relation (and so a logic) is by using the so-called logical matrices:

Definition 2.3. A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values,
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, called the *designated* elements of $\mathcal{V}$, and
- $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_\mathcal{M} : \mathcal{V}^n \to \mathcal{V}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

[1]The condition of being consistent, and even more the condition of being finitary, are not always included in the definition of a logic, but they should be satisfied by any *applied logic*. This is why we have included them here.

Definition 2.4. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- A *(legal) valuation* in $\mathcal{M}$ is a function $v : \mathcal{F}_\mathcal{L} \to \mathcal{V}$ such that
$$v(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$
for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{F}_\mathcal{L}$.
- A valuation v in a matrix $\mathcal{M}$ is a *model* of:
  - a formula ψ ($v \models^\mathcal{M} \psi$) if $v(\psi) \in \mathcal{D}$.
  - a theory $T \subseteq \mathcal{F}_\mathcal{L}$ ($v \models^\mathcal{M} T$) if $v \models^\mathcal{M} \psi$ for all $\psi \in T$.
- The *formula consequence relation* induced by $\mathcal{M}$ is the relation $\vdash_\mathcal{M}$ on $\mathcal{P}(\mathcal{F}_\mathcal{L}) \times \mathcal{F}_\mathcal{L}$ such that $T \vdash_\mathcal{M} \varphi$ if every model of T in $\mathcal{M}$ is also a model of φ.

It has been shown in [22] that if $\mathcal{M}$ is a *finite* matrix for $\mathcal{L}$ (i.e., if its set of truth-values is finite) then $\langle \mathcal{L}, \vdash_\mathcal{M} \rangle$ is a decidable logic according to Definition 2.2 (in particular, $\vdash_\mathcal{M}$ is finitary). Below we refer to $\langle \mathcal{L}, \vdash_\mathcal{M} \rangle$ as *the logic induced by the matrix* $\mathcal{M}$.

2.2. Basic Matrices for Information Processing

Let L_C be the propositional language based on the connectives $\neg$, $\wedge$, and $\vee$.

1. Classical logic $\mathcal{CL}$ is induced by the matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{0, 1\}$, $\mathcal{D} = \{1\}$, and $\widetilde{\neg}$, $\widetilde{\vee}$, and $\widetilde{\wedge}$ are given by the standard two-valued truth tables.
2. The logic $\mathcal{KL}$ of Kleene is induced by the 3-valued Kleene matrix $\mathcal{M}_K^3 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where
$$\mathcal{V} = \{0, 1, \mathrm{I}\}, \quad \mathcal{D} = \{1\}, \quad \mathcal{O} = \{\widetilde{\neg}, \widetilde{\vee}, \widetilde{\wedge}\}$$
and the interpretations of the connectives are given by the following tables:

$\widetilde{\neg}$	
0	1
1	0
I	I

$\widetilde{\vee}$	0	1	I
0	0	1	I
1	1	1	1
I	I	1	I

$\widetilde{\wedge}$	0	1	I
0	0	0	0
1	0	1	I
I	0	I	I

3. The logic $\mathcal{LP}$ of Priest ([21]) is induced by the 3-valued Priest matrix $\mathcal{M}_P^3 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{0, 1, \mathrm{I}\}$, $\mathcal{D} = \{1, \mathrm{I}\}$, and the interpretations of the connectives are as in $\mathcal{M}_K^3$.
4. The logic $\mathcal{DB}$ is induced by the Dunn-Belnap matrix ([15, 11, 10]) $\mathcal{M}_B^4 = \langle \mathcal{V}_4, \mathcal{D}_4, \mathcal{O}_4 \rangle$, where $\mathcal{V}_4 = \{\mathbf{f}, \bot, \top, \mathbf{t}\}$, $\mathcal{D}_4 = \{\top, \mathbf{t}\}$, and the interpretations of the connectives are as follows:

$\widetilde{\neg}$	
f	**t**
$\bot$	$\bot$
$\top$	$\top$
t	**f**

$\widetilde{\vee}$	**f**	$\bot$	$\top$	**t**
f	**f**	$\bot$	$\top$	**t**
$\bot$	$\bot$	$\bot$	**t**	**t**
$\top$	$\top$	**t**	$\top$	**t**
t	**t**	**t**	**t**	**t**

$\widetilde{\wedge}$	**f**	$\bot$	$\top$	**t**
f	**f**	**f**	**f**	**f**
$\bot$	**f**	$\bot$	**f**	$\bot$
$\top$	**f**	**f**	$\top$	$\top$
t	**f**	$\bot$	$\top$	**t**

Some explanatory comments:

- We have used here 1 ("true") and 0 ("false") to denote the classical truth values. The interpretation of the third value I is usually taken as "unknown" in case of $\mathcal{KL}$, and "inconsistent" in case of $\mathcal{LP}$. Thus, in the two three-valued matrices mentioned in 2. and 3. above, the "truth-values" are actually *information values*, representing *knowledge* about the truth/falsity of a proposition. Accordingly, in those matrices 1 actually means "known to be true", while 0 means "known to be false". In turn, the meaning of I in $\mathcal{LP}$ is "known to be true and also known to be false", while in $\mathcal{KL}$ I represents lack of knowledge.

- The basic idea behind $\mathcal{M}_B^4$ is to have *both* the "unknown" value of $\mathcal{KL}$ and the "inconsistent" value of $\mathcal{LP}$ in the same matrix. The first of those values is denoted here by $\bot$, while the second by $\top$. Further, **t** and **f** roughly correspond to the classical truth-values. The set of elements of $\mathcal{M}_B^4$ is often identified with $\{1,0\} \times \{1,0\}$. Given a tuple $\langle a, b\rangle$, the first component a represents knowledge about the truth of a formula, while b represents knowledge about its falsity (whereby the two pieces of knowledge may be independent). According to this interpretation, the meanings of the four values are as follows:

$\mathbf{t} = \langle 1, 0\rangle$	- known to be true but not known to be false
$\mathbf{f} = \langle 0, 1\rangle$	- known to be false but not known to be true
$\top = \langle 1, 1\rangle$	- known to be true and known to be false
$\bot = \langle 0, 0\rangle$	- not known to be true and not known to be false

This representation leads to two natural partial orders on $\mathcal{V}_4$: the *"knowledge"* order $\leq_k$ defined by $\langle a_1, b_1\rangle \leq_k \langle a_2, b_2\rangle$ iff $a_1 \leq a_2$ and $b_1 \leq b_2$, and the *"truth"* order $\leq_t$ defined by $\langle a_1, b_1\rangle \leq_t \langle a_2, b_2\rangle$ iff $a_1 \leq a_2$ and $b_1 \geq b_2$. Each of these relations induces a lattice structure on $\mathcal{V}_4$, and together they induce what is known as the *bilattice* $\mathcal{FOUR}$ (see Figure 1 below). It should be noted that the operations $\widetilde{\vee}$ and $\widetilde{\wedge}$ on $\mathcal{V}_4$ correspond (respectively) to the lattice operations $sup_{\leq_t}$ and $inf_{\leq_t}$ induced by $\leq_t$, while $\widetilde{\neg}$, $\widetilde{\vee}$, and $\widetilde{\wedge}$ are all *monotonic* with respect to $\leq_k$.

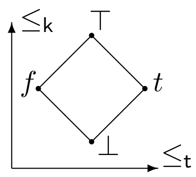

Figure 1. $\mathcal{FOUR}$

Note 2.5. $\mathcal{FOUR}$ has been generalized to a family of structures called *bilattices*, which are very useful for handling knowledge and information. See e.g. [18, 17, 3] for further details.

Note 2.6. It should be noted that the upper and lower parts of the bilattice shown in the above figure (containing **f**, $\top$, **t** and **f**, $\bot$, **t**, respectively) are closed under the interpretations of the L_C connectives in $\mathcal{FOUR}$. What is more, the matrix corresponding to the upper sublattice is isomorphic to Priest's matrix

$\mathcal{M}_P^3$ under the isomorphism $h(\mathbf{f}) = 0, h(\top) = \mathrm{I}, h(\mathbf{t}) = 1$, while that corresponding to the lower one is isomorphic to Kleene matrix $\mathcal{M}_K^3$ under the isomorphism $g(\mathbf{f}) = 0, g(\bot) = \mathrm{I}, g(\mathbf{t}) = 1$. Hence from now on both $\mathcal{M}_P^3$ and $\mathcal{M}_K^3$ will be viewed as submatrices of $\mathcal{M}_B^4$.

3. Information Processing Framework

3.1. Belnap's Model

The development of new network technologies allows several agents to access and update large knowledge bases, sometimes simultaneously. The process of combining information originating from different sources and drawing conclusions from it is very complex, especially in case of contradictory (inconsistent) information.

A framework for dealing with this situation based on the 4-valued logic $\mathcal{DB}$ described above was first developed by Belnap in [10, 11]. Its main idea is to interpret the elements of $\mathcal{V}_4$ as subsets of $\{0, 1\}$ which reflect the information on the truth/falsity of a formula derived by a processor from the information obtained by it from some set of independent sources. In Belnap's model, a source may provide the processor with information about *atomic* formulas of L_C. To each such formula p, it either assigns 1 to say that p is true, 0 to say it is false, or I if it has no information about p. The processor then assigns to an atom p a subset $d(p)$ of $\{0, 1\}$ according to the following two simple principles:

(dbA) $1 \in d(p)$ iff some source has assigned 1 to p

(dbB) $0 \in d(p)$ iff some source has assigned 0 to p

Accordingly, the values in $\mathcal{V}_4$ are now interpreted as follows:

$\mathbf{t} = \{1\}$	- said to be true but not said to be false
$\mathbf{f} = \{0\}$	- said to be false but not said to be true
$\top = \{0, 1\}$	- said to be true and said to be false
$\bot = \emptyset$	- not said to be true and not said to be false

Next, the processor assigns a subset $d(\varphi)$ of $\{0, 1\}$ also to non-atomic formulas of L_C using the basic classical principles. In other words:

(db1) $0 \in d(\neg\varphi)$ iff $1 \in d(\varphi)$;

(db2) $1 \in d(\neg\varphi)$ iff $0 \in d(\varphi)$;

(db3) $1 \in d(\varphi \vee \psi)$ iff $1 \in d(\varphi)$ or $1 \in d(\psi)$;

(db4) $0 \in d(\varphi \vee \psi)$ iff $0 \in d(\varphi)$ and $0 \in d(\psi)$;

(db5) $1 \in d(\varphi \wedge \psi)$ iff $1 \in d(\varphi)$ and $1 \in d(\psi)$;

(db6) $0 \in d(\varphi \wedge \psi)$ iff $0 \in d(\varphi)$ or $0 \in d(\psi)$.

Belnap's model yields the following characterizations of Dunn-Belnap and Priest matrices in terms of processor valuations:

Theorem 3.1.

1. Any processor valuation d obtained in accordance with (dbA)–(dbB) and (db1)–(db6) above is a legal valuation in $\mathcal{M}_B^4$. Conversely, every legal valuation in $\mathcal{M}_B^4$ can be obtained as a processor valuation d in a source-processor framework with at most two sources.

2. Call a source *classical* if it does not use the value I. If all sources are classical, then the resulting processor valuation is a legal valuation in the Priest Matrix $\mathcal{M}_P^3$ (see Note 2.6). Conversely, every legal valuation in $\mathcal{M}_P^3$ can be obtained as a processor valuation d in a source-processor framework with (at most two) classical sources.

Proof:
Part 1. $(\Rightarrow)$ Assume d is a processor valuation obtained out of a set of sources S according to rules (dbA-B), (db1-6). Then it is very easy to check, basing on the above rules, that d is compliant with $\mathcal{M}_B^4$. For example, assume that $v(\varphi) = \perp, v(\psi) = \top$. As $\perp = \emptyset$ and $\top = \{0, 1\}$, by the assumption $0 \notin v(\varphi)$ and $1 \in v(\psi)$. Hence $0 \notin v(\varphi \vee \psi)$ by (db4) and $1 \in v(\varphi \vee \psi)$ by (db3). This yields $v(\varphi \vee \psi) = \{1\} =$ **t** — which is exactly the value assigned to the disjunction of $\perp$ and $\top$ in the matrix $\mathcal{M}_B^4$.
$(\Leftarrow)$ Let v be a legal valuation in $\mathcal{M}_B^4$, and define the source valuations s^0, s^1 as follows: for any $p \in \mathcal{A}$,

$$s^0(p) = \begin{cases} 0 & \text{if } 0 \in v(p) \\ \mathrm{I} & \text{otherwise} \end{cases} \qquad s^1(p) = \begin{cases} 1 & \text{if } 1 \in v(p) \\ \mathrm{I} & \text{otherwise} \end{cases}$$

Now take $S = \{s^0, s^1\}$, and let d be the processor valuation obtained from S using (dbA–B), (db1–6). Then by (dbA-B) we clearly have $d(p) = v(p)$ for every $p \in \mathcal{A}$. Since by $(\Rightarrow)$ proved above d is a legal valuation in $\mathcal{M}_B^4$ and each such valuation is uniquely defined by its values for atoms, then $d = v$.
Part 2. $(\Rightarrow)$ According to Note 2.6, we will treat $\mathcal{M}_P^3$ as a submatrix of $\mathcal{M}_B^4$, with I replaced by $\top$. By Part 1, a processor valuation obtained from classical sources is a legal valuation in $\mathcal{M}_B^4$. Since by induction such a valuation does not take the value I, then it is a legal valuation in $\mathcal{M}_P^3$, too.
$(\Leftarrow)$ Let v be a legal valuation in $\mathcal{M}_P^3$, and define the source valuations s^0, s^1 as follows: for any $p \in \mathcal{A}$,

if $v(p) =$ **f**, then $s^0(p) = s^1(p) = 0$

if $v(p) =$ **t**, then $s^0(p) = s^1(p) = 1$

if $v(p) = \top$, then $s^0(p) = 0, s^1(p) = 1$

Then s^0, s^1 are classical sources. Taking $S = \{s^0, s^1\}$, we can easily prove — in a way quite similar to the proof of $(\Leftarrow)$ in Part 1 — that the processor valuation obtained from S coincides with v. □

3.2. General information processing structures

Belnap's model is adequate for the case in which the sources are simple relational databases. However, it does not capture all the situations encountered in practice. In particular, knowledge bases and disjunctive databases can provide information also about *complex* formulas. A more general framework, suitable for handling such a situation, where a source may provide the processor with information (in the form of a truth value from $\{0, 1, \mathrm{I}\}$) about *arbitrary* formulas of L_C, was considered in [7]. In that framework, the assignment of subsets of $\{0, 1\}$ to formulas of L_C is carried out in two stages. In the first stage, the processor collects the information from the sources according to some strategy. The most basic strategy (used also in Belnap's model) is the existential one, in which the processor *initially* includes a value $x \in \{0,1\}$ in subset of $\{0,1\}$ being the value of a formula ψ iff some source assigns x to ψ. In the second stage, the processor expands the information collected at the previous stage by adding to it new information that can be derived from the initial one using certain rules based on the truth tables of classical logic. Here the crucial assumption made in [7] is that the final assignment v developed by the processor should include everything that can be derived from the classical truth tables without assuming consistency or full knowledge. In practice, this means that conditions (db3) and (db6) above should be weakened, so that only the "if" part is retained (but not the "only if").

Now the point of introducing this general framework is to allow situations in which a source may know (for example) that a certain disjunction holds, without knowing which of the disjuncts is the true one. This does not mean, of course, that a reasonable source may arbitrarily assign truth-values to formulas[2]. In the adopted framework, we expect it to be able to derive from the information it has everything that can be derived from it using the classical truth tables without assuming full knowledge. This means, e.g., that if either φ or ψ is assigned 1, then so should be $\varphi \vee \psi$, and also that $\varphi \vee \psi$ may be assigned 1 only if either φ or ψ is assigned 1, or if both of them are assigned I. Moreover, a reasonable source needs to be consistent — which means that it cannot assign 1 to φ and at the same time 0 to $\varphi \vee \psi$.

Obviously, the framework considered in [7] can be generalized by considering provision of information also about values of formulas of languages other than L_C, e.g. first-order or modal one. Accordingly, below we start by formulating a more general notion of an information structure connected with an arbitrary language $\mathcal{L}$ of some logic. To better capture the multifarious combinations of different information collecting and processing mechanisms, we introduce a separate notion of an *information collector* which carries out the former process, while a *processor* is, like in [7], responsible for expanding the collected information.

Definition 3.1. Consider an arbitrary language $\mathcal{L}$, and let $\mathcal{A}_{\mathcal{L}}$ and $\mathcal{F}_{\mathcal{L}}$ be the set of all atomic formulas and the set of all formulas of $\mathcal{L}$, respectively.

- By a *low-level valuation* for $\mathcal{L}$ we mean a function $v : \mathcal{F}_{\mathcal{L}} \to \{0, 1, \mathrm{I}\}$. The set of all such valuations will be denoted by $\mathbf{V}_{\mathcal{L}}$.

- By a *source valuation*, or *information source*, for $\mathcal{L}$ we mean a low-level valuation $s \in \mathbf{V}_{\mathcal{L}}$ which satisfies certain conditions specific for the semantics of $\mathcal{L}$. The set of all source valuations for $\mathcal{L}$ will be denoted by $\mathbf{S}_{\mathcal{L}}$.

- By a *high-level valuation* for $\mathcal{L}$ we mean a function $v : \mathcal{F}_{\mathcal{L}} \to \mathcal{P}(\{0,1\})$. The set of all high-level valuations for $\mathcal{L}$ will be denoted by $\mathbf{H}_{\mathcal{L}}$.

- By an *information collector* we mean a mapping $C : \mathcal{P}_m(\mathbf{S}_{\mathcal{L}}) \to \mathbf{H}_{\mathcal{L}}$, where $\mathcal{P}_m(\mathbf{S}_L)$ denotes the set of all multisubsets of $\mathbf{V}_{\mathcal{L}}$, i.e. the set of all multisets whose elements are valuations in $\mathbf{S}_{\mathcal{L}}$.

- By an *information processor* we mean a mapping $P : \mathbf{H}_{\mathcal{L}} \to \mathbf{H}_{\mathcal{L}}$.

All the notions above are defined relative to a language $\mathcal{L}$, but once $\mathcal{L}$ is fixed, the subscript $\mathcal{L}$ in the notation introduced above will be dropped.

[2]Ignoring this crucial issue was the main deficiency of [7].

The exact shape of the information processor will depend on the semantics of the concrete language $\mathcal{L}$. On the other hand, information collectors may implement various information collecting strategies. The two basic collectors represent the existential and universal strategies (considered already in [7]), and are defined as follows:

Definition 3.2.

- The *existential information collector* is the mapping $C_e : \mathcal{P}_m(\mathbf{S}_\mathcal{L}) \to \mathbf{H}_\mathcal{L}$ such that, for any $S \in \mathcal{P}_m(\mathbf{S}_\mathcal{L})$, and any $\varphi \in \mathcal{F}_\mathcal{L}$,

$$\forall x \in \{0,1\}[\, x \in (C_e(S))(\varphi) \text{ iff } \exists s \in S.\ s(\varphi) = x]$$

- The *universal information collector* is the mapping $C_u : \mathcal{P}_m(\mathbf{S}_\mathcal{L}) \to \mathbf{H}_\mathcal{L}$ such that, for any $S \in \mathcal{P}_m(\mathbf{S}_\mathcal{L})$, and any $\varphi \in \mathcal{F}_\mathcal{L}$,

$$\forall x \in \{0,1\}[\, x \in (C_u(S))(\varphi) \text{ iff } \forall s \in S.\ s(\varphi) = x]$$

Definition 3.3. Let C be an information collector, and P — an information processor for a language $\mathcal{L}$. By an *information processing* $\langle C, P\rangle$*-structure* for $\mathcal{L}$ we mean a tuple $\mathcal{S} = \langle S, g, d\rangle$, where:

1. $S \in \mathcal{P}_m(\mathbf{S}_\mathcal{L})$ is a multiset of information sources for $\mathcal{L}$;
2. $g = C(S)$ (g is called the *global* $\langle C, P\rangle$*-valuation generated by* $\mathcal{S}$);
3. $d = P(g_S) = P(C(S))$ (d is called the the *processor* $\langle C, P\rangle$*-valuation generated by* $\mathcal{S}$).

The prefix $\langle C, P\rangle$ in the above definition will be omitted if C and P are understood or immaterial.

Note that the use of multisets in the above definition allows us to capture the situation when different information sources provide the same information. As this is quite common in practice, identifying sources that provide the same information would lead to incorrect results in various information collecting strategies based on the "majority opinion", or the percentage share of sources providing certain information.

3.3. Induced consequence relations

Let $\mathcal{L}$ be a language. Each information processing structure for $\mathcal{L}$ generates in a natural way a satisfaction relation on $\mathcal{F}_\mathcal{L}$ determined by its processor valuation. Accordingly, each class of information processing structures for $\mathcal{L}$ induces a corresponding consequence relation:

Definition 3.4.

- Let $\mathcal{S} = \langle S, g, d\rangle$ be an information processing structure. Then $\mathcal{S}$ *satisfies* (is a *model* of) a formula $\varphi \in \mathcal{F}$ (or φ is *satisfied in* $\mathcal{S}$), in symbols $\models_\mathcal{S} \varphi$, iff $1 \in d(\varphi)$.
- Let $\mathcal{J}$ be a class of information processing structures. The *formula consequence relation* induced by $\mathcal{J}$ is the relation $\vdash_\mathcal{J}$ on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$ such that $T \vdash_\mathcal{J} \varphi$ if every $\mathcal{S} \in \mathcal{J}$ which is a model of T is also a model of φ.

Note 3.5. It is easy to see that $\vdash_{\mathcal{J}}$ is a structural consequence relation. In Corollaries 4.4, 4.5, and 4.3 below we show that in the most important cases it is also *finitary*, i.e. $\langle \mathcal{L}, \vdash_{\mathcal{J}} \rangle$ is a *logic* (according to Definition 2.2).

In the particular case of L_C, it can be seen that *formulas* of this language do not have sufficient expressive power for describing some important facts regarding the information processing structures. First, there is no way to express the fact that a certain formula φ is *not true* — i.e., $1 \notin d(\varphi)$ — for, unlike classical logic, this is not equivalent to the truth of $\neg\varphi$. Similarly, we cannot express *disjunctive knowledge* of the form "one of the formulas φ and ψ is known to be true" — i.e., either $1 \in d(\varphi)$ or $1 \in d(\psi)$ — for this is *not* equivalent to $1 \in d(\varphi \vee \psi)$, as the latter can hold without either $1 \in d(\varphi)$ or $1 \in d(\psi)$ holding.

These problems can be overcome by using Gentzen-type *sequents*, which provide the means for expressing the above two types of knowledge. Namely, given a information processing structure $\langle S, g, d \rangle$, a sequent $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k$ expresses the information that either $1 \notin d(\varphi_1)$, or $1 \notin d(\varphi_2)$, or ... or $1 \notin d(\varphi_n)$, or $1 \in d(\psi_1)$, or ... or $1 \in d(\psi_k)$. Note again that this type of information cannot be expressed by any formula of the language, for the usual translation of sequents to formulae known from classical logic is not adequate here.

Another shortcoming of the classical language L_C is that it does not possess any implication connective that corresponds to the consequence relations generated by the classes of information processing structures that we investigate below. Again this problem is (essentially) overcome by using sequents, since sequents provide a non-nestable version of implication.

For the above reasons, the proof mechanisms we will use for reasoning about information processing structures in what follows are Gentzen-type sequent calculi which manipulate sequents. To enable this, we need to introduce sequential counterparts of the relations defined in Definition 3.4.

Definition 3.6. Let $\mathcal{L}$ be a language.

- By a *sequent* of $\mathcal{L}$ we mean a structure of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite *sets* of formulas of $\mathcal{L}$. The set of all sequents in the language $\mathcal{L}$ is denoted by $Seq_{\mathcal{L}}$.

- Let $\mathcal{S} = \langle S, g, d \rangle$ be an information processing structure. Then $\mathcal{S}$ *satisfies* (is a *model* of) a sequent $\Sigma = \Gamma \Rightarrow \Delta$, in symbols $\models_{\mathcal{S}} \Sigma$, iff either $\mathcal{S}$ is a model of some formula in Δ, or it is not a model of some formula in Γ.

- Let $\mathcal{J}$ be a class of information processing structures for $\mathcal{L}$. The *sequent consequence relation* induced by $\mathcal{J}$ is the relation $\vdash_{\mathcal{J}}$ on $\mathcal{P}(Seq_{\mathcal{L}}) \times Seq_{\mathcal{L}}$ such that $Q \vdash_{\mathcal{J}} \Sigma$ if every $\mathcal{S} \in \mathcal{J}$ which is a model of Q is also a model of Σ.

In the above definition, for the sake of simplicity, we use the same symbol for the formula consequence relation and for the sequent consequence relation. However, this will not cause any misunderstanding, for it will be always clear from the context which relation we actually have in mind.

It is important to recall here two relationships which always hold between a formula consequence relation and the standard sequent consequence relation which is derived from it like in Definition 3.6.

Fact 3.1. For any set of formulas $\Gamma \subseteq \mathcal{F}$ and any formula $\varphi \in \mathcal{F}$, we have:

- If Γ is finite, then $\Gamma \vdash_{\mathcal{J}} \varphi$ iff $\vdash_{\mathcal{J}} (\Gamma \Rightarrow \varphi)$.
- $\Gamma \vdash_{\mathcal{J}} \varphi$ iff $\{\Rightarrow \psi \mid \psi \in \Gamma\} \vdash_{\mathcal{J}} (\Rightarrow \varphi)$.

4. Information processing structures for L_C

The general framework introduced in the preceding section allows for considering many different variants of information processing structures. Out of them, in this paper we will examine in detail a class of structures for L_C being a direct extension of Belnap's model — namely, structures employing the existential information collector C_e from Definition 3.2, and the so-called standard information processor P_s that extends the information collected by C_e using the rules based on the truth tables of classical logic defined in [7].

Let $\mathcal{F}$ be the set of formulas of L_C, $\mathcal{A}$ its set of atomic formulas, $\mathbf{V} = \mathbf{V}_{L_C}$, and $\mathbf{H} = \mathbf{H}_{L_C}$.

Definition 4.1. Let $g, d \in \mathbf{H}$. We say that $g \preceq d$ if $g(\varphi) \subseteq d(\varphi)$ for every formula φ.

Obviously, $\preceq$ is a partial order on $\mathbf{H}$.

Definition 4.2.

The *standard information processor for* L_C is the information processor $P_s : \mathbf{H} \rightarrow \mathbf{H}$ such that, for any $g \in \mathbf{H},\ P_s(g) = d$, where d is the $\preceq$-minimal high-level valuation in $\mathbf{H}$ satisfying the following conditions:

(d0) $g \preceq d$;

(d1) $0 \in d(\neg\varphi)$ iff $1 \in d(\varphi)$;

(d2) $1 \in d(\neg\varphi)$ iff $0 \in d(\varphi)$;

(d3) $1 \in d(\varphi \vee \psi)$ *if* $1 \in d(\varphi)$ or $1 \in d(\psi)$;

(d4) $0 \in d(\varphi \vee \psi)$ iff $0 \in d(\varphi)$ and $0 \in d(\psi)$;

(d5) $1 \in d(\varphi \wedge \psi)$ iff $1 \in d(\varphi)$ and $1 \in d(\psi)$;

(d6) $0 \in d(\varphi \wedge \psi)$ *if* $0 \in d(\varphi)$ or $0 \in d(\psi)$.

In the definition of the valuation d derived by the standard information processor P_s from a high-level valuation g, rules (d1)–(d6) go both down and up the formula structure, which does not explicitly provide an effective way of computing d out of g. We will now prove that d can be obtained from g by a single "downward" pass, followed by a single "upward" pass.

Definition 4.3. Let $h \in \mathbf{H}$ be any high-level valuation.

(A) $D(h)$, the *downward closure* of h, is the $\preceq$-minimal valuation $\overline{h} \in \mathbf{H}$ such that for any $x \in \{0, 1\}$, and any $\varphi \in \mathcal{F}$,

(0) $h \preceq \overline{h}$

(1) $x \in \overline{h}(\neg\varphi) \Rightarrow 1 - x \in \overline{h}(\varphi)$

(2) $0 \in \overline{h}(\varphi \vee \psi) \Rightarrow 0 \in \overline{h}(\varphi)$ and $0 \in \overline{h}(\psi)$

(3) $1 \in \overline{h}(\varphi \wedge \psi) \Rightarrow 1 \in \overline{h}(\varphi)$ and $1 \in \overline{h}(\psi)$

(B) $U(h)$, the *upward closure* of h, is the $\preceq$-minimal valuation $h^* \in \mathbf{H}$ such that for any $x \in \{0,1\}$, any $p \in \mathcal{A}$, and any $\varphi \in \mathcal{F}$,

(i) $x \in h(p) \Rightarrow x \in h^*(p)$ for $p \in \mathcal{A}$

(ii) $x \in h^*(\varphi) \Rightarrow 1 - x \in h^*(\neg\varphi)$

(iii) $1 \in h^*(\varphi)$ or $1 \in h^*(\psi)$ or $1 \in h(\varphi \vee \psi) \Rightarrow 1 \in h^*(\varphi \vee \psi)$

(iv) $0 \in h^*(\varphi)$ and $0 \in h^*(\psi) \Rightarrow 0 \in h^*(\varphi \vee \psi)$

(v) $1 \in h^*(\varphi)$ and $1 \in h^*(\psi) \Rightarrow 1 \in h^*(\varphi \wedge \psi)$

(vi) $0 \in h^*(\varphi)$ or $0 \in h^*(\psi)$ or $0 \in h(\varphi \wedge \psi) \Rightarrow 0 \in h^*(\varphi \wedge \psi)$

Proposition 4.1. The standard information processor P_s is the composition of operations D and U, i.e., for any $g \in \mathbf{H}$, $P_s(g) = U(D(g))$.

Proof:
For convenience, denote $d = P_s(g), \overline{g} = D(g), \overline{g}^* = U(D(g))$.

It can easily be seen that if $h \in \mathbf{H}$ satisfies conditions (d0)–(d6) from Definition 4.1, then $\overline{g}^* \preceq h$. It follows that in order to prove that $\overline{g}^*$ is the minimal high-level valuation satisfying conditions (d0)–(d6) in Definition 4.1, it suffices to prove that $\overline{g}^*$ satisfies these conditions.

We start with proving (d0). Since $g \preceq \overline{g}$, we prove instead that

$$\overline{g}(\varphi) \subseteq \overline{g}^*(\varphi) \tag{1}$$

for any $\varphi \in \mathcal{F}$. We argue by induction on the structure of φ.

First, for atomic φ, i.e. $\varphi = p \in \mathcal{A}$, (1) follows from (i) in the definition of U.

Now we assume that (1) holds for φ, ψ, and prove that it also holds for $\neg\varphi, \varphi \vee \psi, \varphi \wedge \psi$:

$\xi = \neg\varphi$**:** By (1) in the definition of D, for any $x \in \{0,1\}$, we have $x \in \overline{g}(\neg\varphi) \to 1 - x \in \overline{g}(\varphi)$. By the inductive assumption for φ, the latter implies $1 - x \in \overline{g}^*(\varphi)$, whence by (ii) in the definition of U we get $1 - (1 - x) = x \in \overline{g}^*(\neg\varphi)$. Accordingly, (1) holds for $\neg\varphi$.

$\xi = \varphi \vee \psi$**:** If $0 \in \overline{g}(\varphi \vee \psi)$, then by (2) in the definition of D we get $0 \in \overline{g}(\varphi)$ and $0 \in \overline{g}(\psi)$, whence $0 \in \overline{g}^*(\varphi)$ and $0 \in \overline{g}^*(\psi)$ by the inductive assumption for φ, ψ. By (iv) in in the definition of U, this implies $0 \in \overline{g}^*(\varphi \vee \psi)$. Finally, if $1 \in \overline{g}(\varphi \vee \psi)$, then $1 \in \overline{g}^*(\varphi \vee \psi)$ by the last clause in the antecedent of (iii) in the definition of U. Thus (1) holds for $\varphi \vee \psi$ too.

$\xi = \varphi \wedge \psi$**:** The proof is dual to that given in the previous case.

Accordingly, (d0) holds. To prove satisfaction of (d1)–(d6), we first observe that (d3) and (d6) follow from Points (iii) and (vi) in the definition of U, respectively, while the backward implications in (d1)-(d2) and (d4)–(d5) follow from Points (ii) and (iv), (v) in that definition, respectively. To see that the converse implications also hold for $\overline{g}^*$, it is enough to note that (due to the "upward" construction of $\overline{g}^*$

from $\overline{g}$) the only way to assign a value $x \in \{0, 1\}$ to $\overline{g}^*(\neg\varphi)$ is by using (ii), the only way to assign 0 to $\overline{g}^*(\varphi \vee \psi)$ is by using (iv), and the only way to assign 1 to $\overline{g}^*(\varphi \wedge \psi)$ is by using (v). Hence the single-way implications in those conditions can in fact be replaced by equivalences, and consequently $\overline{g}^*$ satisfies (d1)–(d6) too. □

An obvious consequence of the above theorem, which will prove useful later on, is the possibility to reduce the above method for computing g to a single upward pass if the high-level valuation g is closed under rules (A)(1)–(3) of Definition 4.2:

Corollary 4.1. If a valuation $g \in \mathbf{H}$ is closed under Conditions (A)(1)–(3) of Definition 4.2, then $D(g) = g$ and $P_s(g) = U(g)$.

4.1. Information sources and standard existential information processing structures

As mentioned in the introduction, in this paper we are concerned with information sources which meet certain coherence conditions stemming from the truth tables of the underlying logic — here, classical propositional logic.

Definition 4.4.

A *source valuation*, or *information source*, for L_C is a low-level valuation $s \in \mathbf{V}$ such that:

(s1) $s(\neg\varphi) = 0$ iff $s(\varphi) = 1$;

(s2) $s(\neg\varphi) = 1$ iff $s(\varphi) = 0$;

(s3) If $s(\varphi) = 1$ or $s(\psi) = 1$ then $s(\varphi \vee \psi) = 1$;

(s4) If $s(\varphi \vee \psi) = 1$ and $s(\varphi) = 0$ then $s(\psi) = 1$;

(s5) If $s(\varphi \vee \psi) = 1$ and $s(\psi) = 0$ then $s(\varphi) = 1$;

(s6) $s(\varphi \vee \psi) = 0$ iff $s(\varphi) = 0$ and $s(\psi) = 0$;

(s7) $s(\varphi \wedge \psi) = 1$ iff $s(\varphi) = 1$ and $s(\psi) = 1$;

(s8) If $s(\varphi) = 0$ or $s(\psi) = 0$ then $s(\varphi \wedge \psi) = 0$;

(s9) If $s(\varphi \wedge \psi) = 0$ and $s(\psi) = 1$ then $s(\varphi) = 0$;

(s10 If $s(\varphi \wedge \psi) = 0$ and $s(\varphi) = 1$ then $s(\psi) = 0$.

The set of all source valuations for L_C is denoted by $\mathbf{S}$.

Note 4.5. Condition (s4), for example, means that a reasonable source cannot say that it knows that either φ is true or ψ is true, but it does not know which of them, even though it knows that φ is false.

After these preliminaries, we can now define the type of information processing structures that we will be dealing with from now on:

Definition 4.6.

1. By a *standard existential information processing structure*, shortly: *EIP structure*, for L_C we mean any information processing $\langle C_e, P_s\rangle$-structure for L_C (see Definitions 3.3, 3.2, and 4.1). The class of all EIP structures will be denoted by $\mathcal{EIP}$.

2. Let $S \in \mathcal{P}_m(\mathbf{S})$ be a multiset of source valuations. If $\mathcal{S} = \langle S, g, d\rangle$ is an EIP structure (i.e. $d = P_s(C_e(S))$) then d is called the *EIP processor valuation generated by* S.

3. A high-level valuation $v \in \mathbf{H}$ is called an *EIP processor valuation* if there exists $S \in \mathcal{P}_m(\mathbf{S})$ such that v is the EIP processor valuation generated by S.

Definition 4.4 implies that if $\langle S, g, d\rangle \in \mathcal{EIP}$, then d satisfies conditions (d0)–(d6) in Definition 4.1, while, by Definition 4.2, g is defined by the following formula: for any $\varphi \in \mathcal{F}$,

$$\forall x \in \{0,1\}\ (x \in g(\varphi) \text{ iff } \exists s \in S.\ s(\varphi) = x) \tag{2}$$

It is easy to check that the processor valuation d of an EIP structure can be computed in a simple way, by applying the upper closure operator from Definition 4.2.

Corollary 4.2. For any EIP structure $\mathcal{S} = \langle S, g, d\rangle$, we have $d = U(g)$, where U is defined as in Definition 4.2. In other words, d can be inductively defined as follows:

(i) $x \in g(p) \Rightarrow x \in d(p)$ for any $x \in \{0,1\}$ and any $p \in \mathcal{A}$

(ii) $x \in d(\varphi) \Rightarrow 1 - x \in d(\neg\varphi)$ for any $x \in \{0,1\}$

(iii) $1 \in d(\varphi)$ or $1 \in d(\psi)$ or $1 \in g(\varphi \vee \psi) \Rightarrow 1 \in d(\varphi \vee \psi)$

(iv) $0 \in d(\varphi)$ and $0 \in d(\psi) \Rightarrow 0 \in d(\varphi \vee \psi)$

(v) $1 \in d(\varphi)$ and $1 \in d(\psi) \Rightarrow 1 \in d(\varphi \wedge \psi)$

(vi) $0 \in d(\varphi)$ or $0 \in d(\psi)$ or $0 \in g(\varphi \wedge \psi) \Rightarrow 0 \in d(\varphi \wedge \psi)$

Proof:
By Definition 4.4, we have $d = P_s(g)$, where g is defined according to Equation (2). Since each source valuation $s \in S$ satisfies Conditions (s1)–(s10) in Definition 4.3, from Equation (2) we can easily deduce that g satisfies Conditions (A) (1)–(3) of Definition 4.2. Indeed: (A)(1) follows immediately from Eq. (2) and (s1)–s(2), (A)(2) — from Eq. (2) and (s6), and finally (A)(3) from Eq. (2) and (s7). Thus the claim follows from Corollary 4.1. □

4.2. Non-deterministic Semantics

It is not difficult to verify that any valuation in the Kleene Matrix $\mathcal{M}^3_K$ is a legal source valuation for L_C, and that we have the following alternatives for the characterizations given in Theorem 3.1:

Theorem 4.1.

1. If $\mathcal{S} = \langle S, g, d\rangle$ is an EIP structure in which all elements of S are legal valuations in $\mathcal{M}^3_K$, then the processor valuation d is a legal valuation in $\mathcal{M}^4_B$. Conversely, every valuation of $\mathcal{M}^4_B$ coincides with the processor valuation in some EIP structure having exactly two sources, both of which are $\mathcal{M}^3_K$-valuations.

2. If $\mathcal{S} = \langle S, g, d\rangle$ is an EIP structure in which all elements of S are classical valuations then the processor valuation d is a legal valuation in $\mathcal{M}^3_P$. Conversely, every legal valuation in $\mathcal{M}^3_P$ coincides with the processor valuation in some EIP structure having exactly two sources, both of which are classical.

Proof:
Part 1.

($\Rightarrow$) It can be easily checked that all legal valuations in $\mathcal{M}^3_K$ satisfy Conditions (s1)–(s10) of Definition 4.3, and hence are legal information sources in **S**. Further, if all elements of S are legal valuations in $\mathcal{M}^3_K$, then from the properties of $\mathcal{M}^3_K$ it follows that, for any $s \in S$, $1 \in s(\varphi \vee \psi)$ iff $1 \in s(\varphi)$ or $1 \in s(\psi)$, and analogously $0 \in s(\varphi \wedge \psi)$ iff $0 \in s(\varphi)$ or $0 \in s(\psi)$. From this we can easily conclude that $1 \in g(\varphi \vee \psi)$ iff $1 \in g(\varphi)$ or $1 \in g(\psi)$, and $0 \in g(\varphi \wedge \psi)$ iff $0 \in g(\varphi)$ or $0 \in g(\psi)$. Since $g \preceq d$ by Definition 4.1, by the foregoing the components $1 \in g(\varphi \vee \psi)$ and $0 \in g(\varphi \wedge \psi)$ can be respectively deleted from Conditions (iii) and (vi) in Corollary 4.2. As d is the $\preceq$-minimal valuation in **H** satisfying all Conditions of Corollary 4.2, this clearly implies that the implications in (d3) and (d6) of Definition 4.1 can be replaced by equivalences. Consequently, d satisfies conditions (db1)–(db6) preceding Theorem 3.1, whence by that Theorem d is a legal valuation in $\mathcal{M}^4_B$.

($\Leftarrow$) Assume now v is a legal valuation in $\mathcal{M}^4_B$. Define two valuations $s^0, s^1 \in \mathbf{V}$ as follows: for any $p \in \mathcal{A}$, we take:

$$s^0(p) = \begin{cases} 0 & \text{if } 0 \in v(p) \\ \mathrm{I} & \text{otherwise} \end{cases} \qquad s^1(p) = \begin{cases} 1 & \text{if } 1 \in v(p) \\ \mathrm{I} & \text{otherwise} \end{cases}$$

and extend the above partial valuations to total valuations of formulas in $\mathcal{F}$ using the truth tables of Kleene logic. Take $S = \{s^0, s^1\}$, and consider the EIP structure $\langle S, g, d\rangle$, where $g = G_e(S), d = P_s(g)$. Then, by the definition of s^0, s^1, both of these valuations are legal in $\mathcal{M}^3_K$. Moreover, $g(p) = v(p)$ by Equation 2 in Definition 4.4. Accordingly, from (i) of Corollary 4.2 it follows that $v(p) \subseteq d(p)$ for any $p \in \mathcal{A}$. As no values are added to $d(p)$ by Conditions (ii)-(vi) of that Corollary, and d is the $\preceq$-minimal valuation satisfying (i)-(vi), this implies $d(p) = v(p)$ for any $p \in \mathcal{A}$. Since v is a legal valuation in $\mathcal{M}^4_B$, and by ($\Rightarrow$) proved above so is d, we must have $d = v$, for any such valuation is uniquely defined by it values for atomic formulas.

Part 2. In accordance with Note 2.6, Priest matrix $\mathcal{M}^3_P$ will be treated as a submatrix of $\mathcal{M}^4_B$, with I replaced by $\top$, 1 by **t**, and 0 by **f**.

($\Rightarrow$) As each classic valuation is a legal valuation in $\mathcal{M}_K^3$, then by Part 1 of the theorem d is a legal valuation in $\mathcal{M}_B^4$. Since none of the source valuations in S take the value I, then $g(\varphi) \neq \perp$ for any $\varphi \in \mathcal{F}$ by Equation (2) following Definition 4.4, and hence the same holds for d in view of Corollary 4.2. As $\mathcal{M}_P^3$ is a submatrix of $\mathcal{M}_B^4$ not containing $\perp$, we conclude that d is a legal valuation in $\mathcal{M}_P^3$.

($\Leftarrow$) Assume now v is a legal valuation in $\mathcal{M}_P^3$. Define valuations $s^0, s^1 \in \mathbf{V}$ as follows: for any $p \in \mathcal{A}$, we put:

If $v(p) = \mathbf{f}$, then $s^0(p) = s^1(p) = 0$

If $v(p) = \mathbf{t}$, then $s^0(p) = s^1(p) = 1$

If $v(p) = \top$, then $s^0(p) = 0, s^1(p) = 1$

and extend the above partial valuations to total valuations of formulas in $\mathcal{F}$ using the truth tables of Priest logic. The rest of the proof is analogous to that of ($\Leftarrow$) in Part 1, and is left to the reader. □

Corollary 4.3. Let $\mathcal{PKL}$ be the class of EIP structures in which all information sources are legal valuations in $\mathcal{M}_K^3$, and let $\mathcal{PCL}$ be the class of EIP structures in which all information sources are classical. Then both $\langle L_C, \vdash \mathcal{PKL}\rangle$ and $\langle L_C, \vdash \mathcal{PCL}\rangle$ are logics.

Proof:
Immediate by Theorem 4.1. □

On the other hand, neither a general source valuation nor a general EIP processor valuation can be characterized by finite ordinary matrices (see Theorem 4.4). The reason is that such valuations do not respect the principle of truth-functionality, according to which the truth-value of a complex formula is uniquely determined by the truth-values of its immediate subformulas. Hence instead of ordinary matrices we need here their generalization introduced in [8], in which the truth-value of a formula is chosen non-deterministically from some nonempty set of options:

Definition 4.7.

1. A *non-deterministic matrix* (Nmatrix) for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$, where
   - $\mathcal{V}$ is a non-empty set (of truth values).
   - $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.
   - $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.
2. Let $\mathcal{M}$ be an Nmatrix for $\mathcal{L}$. An $\mathcal{M}$*-valuation* ν is a function $\nu : \mathcal{F}_{\mathcal{L}} \to \mathcal{V}$ such that, for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$,
$$\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)).$$
3. Given an Nmatrix $\mathcal{M}$, the notions of a model of a formula or a theory in $\mathcal{M}$, and the consequence relation associated with $\mathcal{M}$ are defined exactly as in the deterministic case (Definition 2.4).

Clearly, ordinary matrices (Definition 2.3) can be identified with Nmatrices in which the operations always return a singleton. Hence the semantic framework of Nmatrices is indeed a generalization of the semantic framework of matrices. This generalization enjoys all the important properties of the narrower framework. In particular:

Fact 4.1. ([8]) Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language $\mathcal{L}$.

1. Let $v : \mathcal{W} \to \mathcal{V}$ be a partial valuation whose domain $\mathcal{W}$ is a set of formulas of $\mathcal{L}$ which is closed under subformulas. Assume that v is consistent with $\mathcal{M}$. Then v can be extended to a full legal valuation of formulas in $\mathcal{M}$.

2. If $\mathcal{M}$ is finite (i.e., $\mathcal{V}$ is finite) then $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a decidable propositional logic according to Definition 2.2 (In particular : $\vdash_{\mathcal{M}}$ is finitary if $\mathcal{M}$ is finite).

On the other hand, the use of Nmatrices allows us in many cases to provide a finite-valued semantics to logics that cannot be characterized using finite deterministic matrices. Moreover: this semantics can frequently be constructed in a *modular* way — something that cannot be done in the framework of deterministic matrices (see [5, 6] for details).[3] Accordingly, Nmatrices have found important applications in reasoning under uncertainty, proof theory, and other subjects (see [9] for a comprehensive survey). In the present context, their importance is due to the following basic representation theorem:

Theorem 4.2.

1. A function $s : \mathcal{F} \to \{0, 1, \mathrm{I}\}$ is a source valuation iff it is an $\mathcal{M}^3_r$-valuation, where the Nmatrix $\mathcal{M}^3_r = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined as follows:

$$\mathcal{V} = \{0, 1, \mathrm{I}\}, \quad \mathcal{D} = \{1\}, \quad \mathcal{O} = \{\tilde{\neg}, \tilde{\vee}, \tilde{\wedge}\}$$

and the interpretations of the connectives are given by the following tables:

$\tilde{\neg}$	
0	$\{1\}$
1	$\{0\}$
I	$\{\mathrm{I}\}$

$\tilde{\vee}$	0	1	I
0	$\{0\}$	$\{1\}$	$\{\mathrm{I}\}$
1	$\{1\}$	$\{1\}$	$\{1\}$
I	$\{\mathrm{I}\}$	$\{1\}$	$\{\mathrm{I}, 1\}$

$\tilde{\wedge}$	0	1	I
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{\mathrm{I}\}$
I	$\{0\}$	$\{\mathrm{I}\}$	$\{0, \mathrm{I}\}$

2. A function $d : \mathcal{F} \to \{\mathbf{f}, \bot, \top, \mathbf{t}\}$ is an EIP processor valuation iff it is an $\mathcal{M}^4_I$-valuation, where the Nmatrix $\mathcal{M}^4_I = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined as follows:

$$\mathcal{V} = \{\mathbf{f}, \bot, \top, \mathbf{t}\}, \mathcal{D} = \{\top, \mathbf{t}\}, \mathcal{O} = \{\tilde{\neg}, \tilde{\vee}, \tilde{\wedge}\}$$

and the interpretations of the connectives are given by the following tables:

$\tilde{\neg}$	
$\mathbf{f}$	$\{\mathbf{t}\}$
$\bot$	$\{\bot\}$
$\top$	$\{\top\}$
$\mathbf{t}$	$\{\mathbf{f}\}$

$\tilde{\vee}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}, \top\}$	$\{\mathbf{t}, \bot\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\bot$	$\{\mathbf{t}, \bot\}$	$\{\mathbf{t}, \bot\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\top$	$\{\top\}$	$\{\mathbf{t}\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

$\tilde{\wedge}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\bot$	$\{\mathbf{f}\}$	$\{\mathbf{f}, \bot\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}, \bot\}$
$\top$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\top\}$	$\{\top\}$
$\mathbf{t}$	$\{\mathbf{f}\}$	$\{\mathbf{f}, \bot\}$	$\{\top\}$	$\{\mathbf{t}, \top\}$

[3] The converse is also true: Given a finite Nmatrix, a corresponding proof system for it can often be derived in a modular way.

Proof:
Part 1. The proof is by a simple check that a valuation $s : \mathcal{F} \to \{0, 1, \mathrm{I}\}$ satisfies conditions (s1)–(s10) in Definition 4.3 iff s is a legal valuation in $\mathcal{M}_r^3$. Indeed, it is easy to see that the group of conditions among (s1)–(s10) containing a particular connective determines precisely the interpretation of that connective in $\mathcal{M}_r^3$. For example, in view of Condition (s6), $\mathrm{I}\tilde{\vee}\mathrm{I}$, cannot equal 0, but both I and 1 are possible, for (s3) is a one-sided implication, and (s4), (s5) do not apply. Hence $\mathrm{I}\tilde{\vee}\mathrm{I} = \{\mathrm{I}, 1\}$.
Part 2. The forward implication is again by a simple check. Namely, we prove that if a valuation d satisfies Conditions (d1)–(d6) of Definition 4.1, then d respects the Nmatrix $\mathcal{M}_I^4$.

For the backward implication, let v be a legal valuation in $\mathcal{M}_I^4$. We must construct $S \in \mathcal{P}_m(\mathbf{S})$ such that v is the EIP processor valuation generated by S. To do this, we split v into a family of low-level valuations S_v defined as follows:

1. For any atomic formula $p \in \mathcal{A}$, we include in S_v source valuations s_p^0, s_p^1 such that:

$$s_p^0(p) = \begin{cases} 0 & \text{if } 0 \in v(p) \\ \mathrm{I} & \text{otherwise} \end{cases} \qquad s_p^1(p) = \begin{cases} 1 & \text{if } 1 \in v(p) \\ \mathrm{I} & \text{otherwise} \end{cases}$$

2. For any formula of the form $\varphi \vee \psi$ such that $1 \in v(\varphi \vee \psi)$, we include in S_v a source valuation $s_{\varphi\vee\psi}$ such that:

$$s_{\varphi\vee\psi}(\varphi \vee \psi) = 1, \text{ and } s_{\varphi\vee\psi}(\chi) = \mathrm{I} \text{ for every proper subformula } \chi \text{ of } \varphi \vee \psi$$

3. For any formula of the form $\varphi \wedge \psi$ such that $0 \in v(\varphi \wedge \psi)$, we include in S_v a source valuation $s_{\varphi\wedge\psi}$ such that:

$$s_{\varphi\wedge\psi}(\varphi \wedge \psi) = 0, \text{ and } s_{\varphi\wedge\psi}(\chi) = \mathrm{I} \text{ for every proper subformula } \chi \text{ of } \varphi \wedge \psi$$

4. The partial source valuations given in Items 1–3 above are extended to all formulas in $\mathcal{F}$ by assigning the value I to all atoms for which the values of those valuations have not been defined in Items 1,2, 3, and then extending the resulting new partial valuations to all formulas according to the truth tables of the Kleene matrix $\mathcal{M}_K^3$.

Next, we prove that the family S_v defined in this way has the following properties:

(i) Each $s \in S_v$ represents a legal source valuation in $\mathbf{S}$.

(ii) The EIP processor valuation generated by S_v is v.

We begin with Point (i). It is easy to see that all the partial valuations $s_p^0, s_p^1, s_{\varphi\vee\psi}, s_{\varphi\wedge\psi}$ defined in Items 1–3 above observe the Nmatrix $\mathcal{M}_r^3$, and so do their extensions defined by assigning I to all atoms whose value has not been defined in Items 1–3. As every valuation in $\mathcal{M}_K^3$ is a legal valuation in $\mathcal{M}_r^3$, then all total valuations in S_v obtained out of the above-mentioned partial valuations according to Item 4 also observe $\mathcal{M}_r^3$, and so are legal information sources in $\mathbf{S}$.

To prove Point (ii), denote $g = C_e(S), d = P_s(g)$. We have to prove that

$$d(\varphi) = v(\varphi) \tag{3}$$

for any $\varphi \in \mathcal{F}$.

We argue by induction on the structure of formulas.

For any $x \in \{0,1\}$, $x \in g(p)$ iff $x = s(p)$ for some $s \in S_v$. However, by the definition of the valuations in S_v, $x = s(p)$ for $s \in S_v$ iff $s = s^x(p)$ and $x \in v(p)$. Thus $g(p) = v(p)$. As by Corollary 4.2 we have $d(p) = g(p)$, this yields $d(p) = v(p)$.

Assume now (3) holds for formulas φ_1, φ_2. We shall prove it holds for $\neg\varphi_1, \varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2$, too.

$\neg\varphi_1$: Substituting $1-y$ for x in (ii) of Corollary 4.2, we conclude that, for any $y \in \{0,1\}$, $y \in d(\neg\varphi_1)$ iff $1-y \in d(\varphi_1)$. Since by the inductive hypothesis $d(\varphi_1) = v(\varphi_1)$, this yields $y \in d(\neg\varphi_1)$ iff $1-y \in v(\varphi_1)$. As v is a legal valuation in $\mathcal{M}_I^4$, the latter holds iff $y = 1-(1-y) \in v(\neg\varphi_1)$, whence (3) holds for $\neg\varphi_1$.

$\varphi_1 \vee \varphi_2$: By (iv) of Corollary 4.2, we have $0 \in d(\varphi_1 \vee \varphi_2)$ iff $0 \in d(\varphi_1)$ and $0 \in d(\varphi_2)$, which, by the inductive hypothesis, holds iff $0 \in v(\varphi_1)$ and $0 \in v(\varphi_2)$. As v is a legal valuation in $\mathcal{M}_I^4$, the latter in turn holds iff $0 \in v(\varphi_1 \vee \varphi_2)$.

Assume now $1 \in d(\varphi_1 \vee \varphi_2)$. Then, by (iii) of Corollary 4.2, either $1 \in d(\varphi_1)$ or $1 \in d(\varphi_2)$ or $1 \in g(\varphi_1 \vee \varphi_2)$. By the inductive hypothesis, the first two cases are equivalent to $1 \in v(\varphi_1)$ and $1 \in v(\varphi_2)$, respectively, and both the latter options yield $1 \in v(\varphi_1 \vee \varphi_2)$ by the disjunction truth table of $\mathcal{M}_I^4$. If finally $1 \in g(\varphi_1 \vee \varphi_2)$, then $s(\varphi_1 \vee \varphi_2) = 1$ for some $s \in S_v$. Then one of the following must hold:

1. $s(\varphi_i) = 1$ for some i,
2. $s(\varphi_1) = s(\varphi_2) =$ I and $s(\varphi_1 \vee \varphi_2) = 1$.

In Case 1, we have $1 \in g(\varphi_i)$, whence $1 \in d(\varphi_i)$ by (i) of Corollary 4.2, and $1 \in v(\varphi_i)$ by the inductive hypothesis. Consequently, $1 \in v(\varphi_1 \vee \varphi_2)$ by the disjunction truth table of $\mathcal{M}_I^4$.

In Case 2, we must clearly have $s = s_{\varphi_1 \vee \varphi_2}$, where the latter source is defined as in Item 2 above (with φ, ψ replaced by φ_1, φ_2). Hence $1 \in v(\varphi_1 \vee \varphi_2)$ by the quoted Item 2.

For the opposite direction, suppose $1 \in v(\varphi_1 \vee \varphi_2)$. Then, by the definition of S_v, S_v contains a source $s = s_{\varphi_1 \vee \varphi_2}$ such that $s(\varphi_1 \vee \varphi_2) = 1$. Hence $1 \in g(\varphi_1 \vee \varphi_2)$, whence $1 \in d(\varphi_1 \vee \varphi_2)$ by (i) of Corollary 4.2.

$\varphi_1 \wedge \varphi_2$: The proof is again analogous to that for disjunction, with $s_{\varphi \vee \psi}$ of Item 2 replaced by $s_{\varphi \wedge \psi}$ of Item 3, (iv) of Corollary 4.2 replaced by (v), and (iii) of that Corollary replaced by (vi).

Thus (3) holds for all formulae, and hence $v = d$. □

Corollary 4.4. $\langle L_C, \vdash_{\mathcal{EIP}} \rangle$ is a (decidable) logic.

Proof:
Immediate from the last theorem and Fact 4.1. □

Theorem 4.3. A function d is an EIP processor valuation generated by a multiset of sources which provide jointly complete information about all atomic formulas iff it is an $\mathcal{M}^3_I$-valuation, where the Nmatrix $\mathcal{M}^3_I = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined as follows: $\mathcal{V} = \{\mathbf{f}, \top, \mathbf{t}\}, \mathcal{D} = \{\top, \mathbf{t}\}, \mathcal{O} = \{\tilde{\neg}, \tilde{\vee}, \tilde{\wedge}\}$, and the interpretations of the connectives are given by the following tables:

$\tilde{\neg}$	
$\mathbf{f}$	$\{\mathbf{t}\}$
$\top$	$\{\top\}$
$\mathbf{t}$	$\{\mathbf{f}\}$

$\tilde{\vee}$	$\mathbf{f}$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}, \top\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\top$	$\{\top\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

$\tilde{\wedge}$	$\mathbf{f}$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\top$	$\{\mathbf{f}\}$	$\{\top\}$	$\{\top\}$
$\mathbf{t}$	$\{\mathbf{f}\}$	$\{\top\}$	$\{\mathbf{t}, \top\}$

Proof:
($\Rightarrow$) Suppose that $\mathcal{S} = \langle S, g, d \rangle$ is an EIP structure, where the sources in S taken together provide complete information about all atomic formulas in $\mathcal{A}$. Then, for any $p \in \mathcal{A}$, there exists $s \in S$ such that $s(p) \neq$ I. By Equation (2) following Definition 4.4, this implies $g(p) \neq \bot$ for any $p \in \mathcal{A}$. In turn, it is easy to see that by Corollary 4.2 the latter yields $d(\varphi) \neq \bot$ for any $\varphi \in \mathcal{F}$. Hence, in view of Theorem 4.2, d is legal valuation in $\mathcal{M}^4_I$ which never takes the value $\bot$. However, comparing the truth tables of $\mathcal{M}^4_I$ and $\mathcal{M}^3_I$, we can easily see that any such valuation must observe $\mathcal{M}^3_I$ — whence so does d.
($\Leftarrow$) The backward implication is proved analogously as in Part 2 of Theorem 4.2, with the additional observation that if $v(\varphi) \neq \bot$ for each $\varphi \in \mathcal{F}$, then the sources in $\{s^0_p, s^1_p \mid p \in \mathcal{A}\}$ defined as in the above-mentioned proof taken together provide complete information about all atomic formulas. □

Corollary 4.5. Let $\mathcal{PJC}$ be the class of EIP structures where the sources provide jointly complete information about all atomic formulas. Then $\langle L_C, \vdash_{\mathcal{PJC}} \rangle$ is a decidable logic.

Proof:
Immediate from the last theorem and Fact 4.1. □

Next we prove that the use of non-deterministic matrices is indeed unavoidable for capturing the logics generated by the valuations considered in Theorems 4.2 and 4.3.

Theorem 4.4. There is no ordinary, finite-valued matrix which generates the logic of a single source. The same applies to the logic $\langle L_C, \vdash_{\mathcal{EIP}} \rangle$ of EIP structures, and to the logic $\langle L_C, \vdash_{\mathcal{PJC}} \rangle$.

Proof:
By Theorems 4.2 and 4.3, we have to prove that no ordinary matrix generates the logic induced by $\mathcal{M}^3_r$, or the logic induced by $\mathcal{M}^4_B$, or the logic induced by $\mathcal{M}^3_I$.

Suppose $\mathcal{M}$ is an ordinary n-valued matrix. For $p \in \mathcal{A}$, define $p^1 = p$ and $p^{k+1} = p \vee p^k$ for $k = 1, 2, \ldots$. Now let v be any valuation legal in $\mathcal{M}$. Since $\mathcal{M}$ is n-valued *matrix*, it can easily be seen that $v(p^{n+1}) \in \{v(p^1), \ldots, v(p^n)\}$. As a result, for any $q \in \mathcal{A}$, $v(q \vee p^{n+1}) \in \{v(q \vee p^1), \ldots, v(q \vee p^n)\}$. Since v is an arbitrary valuation in $\mathcal{M}$, this yields $q \vee p^1, \ldots, q \vee p^n \vdash_{\mathcal{M}} q \vee p^{n+1}$. However, the above does not hold for either $\mathcal{M}^3_r$, $\mathcal{M}^3_I$, or $\mathcal{M}^4_B$. Indeed: the valuation v defined by $v(q) =$ I, $v(p^i) =$ I for $1 \leq i \leq n+1$, $v(q \vee p^i) = \mathbf{t}$ for $1 \leq i \leq n$, and $v(q \vee p^{n+1}) =$ I is legal in $\mathcal{M}^3_r$, $v \models q \vee p^i$ for $1 \leq i \leq n$ and $v \not\models q \vee p^{n+1}$. The counter-example for $\mathcal{M}^4_B$ and $\mathcal{M}^3_I$ is the valuation u defined by: $u(q) = \mathbf{f}$, $v(p^i) = \mathbf{f}$ for $1 \leq i \leq n+1$, $v(q \vee p^i) = \top$ for $1 \leq i \leq n$, and $v(q \vee p^{n+1}) = \mathbf{f}$. □

Next we show that, unlike the case of Belnap's model (see Theorem 3.1) or of the too general model considered in [7] (where no conditions whatever were imposed on the behavior of sources), the logic of EIP structures cannot be generated by any subclass of $\mathcal{EIP}$ in which there is a finite upper bound on the number of sources.

Lemma 4.1. There are EIP processor valuations which cannot be obtained from any finite number of source valuations.

Proof:
To give an example of such a valuation, take any infinite sequence of atomic formulae $p_0, p_1, p_2, \ldots$, and define sequences $B_1, B_2, \ldots$ and $C_1, C_2, \ldots$ of formulae recursively as follows:

$$B_1 = C_1 = p_0 \vee p_1$$
$$B_{n+1} = \neg C_n \vee p_{n+1} \quad (n \geq 1)$$
$$C_{n+1} = C_n \vee B_{n+1} \quad (n \geq 1)$$

Now let us define a partial valuation v_0 by taking:

$$v_0(p_n) = \perp \text{ for all } n \geq 0$$
$$v_0(B_n) = \mathbf{t}, \quad v(C_n) = \mathbf{t}, \quad v(\neg C_n) = \mathbf{f} \text{ for all } n \geq 1$$

As the set of formulas occurring in the above definition is closed under subformulas and v_0 is compliant with the matrix $\mathcal{M}_I^4$, then by Fact 4.1 v_0 can be extended to a legal total valuation v in $\mathcal{M}_I^4$, which is a valid EIP processor valuation.

Suppose S is a set of sources in **S** generating the valuation v, and let g be the global valuation generated by S defined as in Equation (2) following Definition 4.4. Then $g \preceq v$ by condition (d0) of Definition 4.1 for v. Hence as $v(p_k) = \perp$ for all $k \geq 0$, then also $g(p_k) = \perp$ for all $k \geq 0$. Since by Equation 2 for any $x \in \{0, 1\}$ we have $x \in g(p_k)$ iff $x = s(p_k)$ for some $s \in S$, then $g(p_k) = \perp$ for all $k \geq 0$ implies $s(p_k) = \mathrm{I}$ for all $k \geq 0$ and each $s \in S$.

Now suppose that for a source $s \in S$ we have $s(B_k) = 1$ for some $k > 1$. As by Theorem 4.2 s is a legal valuation in the Nmatrix $\mathcal{M}_r^3$ and $C_k = C_{k-1} \vee B_k$, then from the truth table of $\mathcal{M}_r^3$ for disjunction we immediately get $s(C_k) = 1$. Since $C_{n+1} = C_n \vee B_{n+1}$ for each $n \geq 1$, then by simple induction we obtain $s(C_n) = 1$ for any $n \geq k$, whence $s(\neg C_k) = 0$ for any $n \geq k$ by the truth table of $\mathcal{M}_r^3$ for negation. As $B_n = \neg C_{n-1} \vee p_n$ for $n > 1$, then from the foregoing we conclude — using again the truth table for disjunction — that $s(B_n) = s(\neg C_{n-1} \vee p_n) = \mathrm{I}$ for any $n > k$.

Consequently, $s(B_k) = 1$ must also imply that $s(B_n) = \mathrm{I}$ for all $1 \leq n < k$. Indeed, since $v(B_n) = \mathbf{t} = \{1\}$ and $g(B_n) \subseteq v(B_n)$, then $0 \notin g(B_n)$, whence $s(B_n) \in \{1, \mathrm{I}\}$. However, $s(B_n) = 1$ would imply by the reasoning given above that $s(B_l) = \mathrm{I}$ for each $l > n$, contradicting the fact that $k > n$ and $s(B_k) = 1$.

This implies that for every $k > 1$ we need a different source $s \in S$ to assign 1 to B_k, thus ensuring that $1 \in g(B_k)$. As by Corollary 4.2 we have $v = U(g)$, then, by the definition of U in Definition 4.2, $g(B_k)$ indeed needs to contain 1 in order to ensure that $v(B_k) = \mathbf{t}$. Hence S must be infinite. □

Theorem 4.5. Let $\mathcal{EIP}_{\leq n}$ be the class of EIP structures in which the number of sources is at most n. Then $\vdash_{\mathcal{EIP}} \neq \vdash_{\mathcal{EIP}_{\leq n}}$ and $\vdash_{\mathcal{EIP}_{\leq n}} \neq \vdash_{\mathcal{EIP}_{\leq k}}$ for $n \neq k$. Moreover: for every n, there is a finite set Γ_n of formulas and a formula ψ_n such that $\Gamma_n \vdash_{\mathcal{EIP}_{\leq n}} \psi_n$, but neither $\Gamma_n \vdash_{\mathcal{EIP}} \psi_n$ nor $\Gamma_n \vdash_{\mathcal{EIP}_{\leq k}} \psi_n$ for any $k > n$.

Proof:
Let $\Gamma_n = \{B_i \mid 1 \leq i \leq n+1\} \cup \{C_i \mid 1 \leq i \leq n+1\}$, where B_i and C_i are defined as in the proof of the Lemma 4.1, and let

$$\psi_n = (p_0 \vee \neg p_0) \vee \ldots \vee (p_{n+1} \vee \neg p_{n+1}) \vee \neg B_1 \vee \ldots \neg B_{n+1} \vee \neg C_1 \vee \ldots \neg C_{n+1}$$

It can be easily checked that an EIP processor valuation v is a model of Γ_n that is not a model of ψ_n iff $v(p_i) = \perp$ and $v(B_i) = v(C_i) = \mathbf{t}$ for every $1 \leq i \leq n+1$. Now from the proof of Lemma 4.1 it easily follows that such an EIP processor valuation v exists, but any set of sources that generates it must have at least $n+1$ elements. Hence Γ_n and ψ_n have the required properties. $\square$

5. Proof systems

In this section we provide cut-free Gentzen-type systems for all the logics considered above (with one exception, all of them have been known before).

Definition 5.1.

1. The system G^4 is given in Figure 2 below:

Axioms: $\Gamma, \psi \Rightarrow \Delta, \psi$

Rules: Cut and the following logical rules:

$$[\neg\neg\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta} \qquad [\Rightarrow\neg\neg]\ \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \neg\neg\psi}$$

$$[\wedge\Rightarrow]\ \frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad [\Rightarrow\wedge]\ \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

$$[\neg\wedge\Rightarrow]\ \frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \varphi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\wedge]\ \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \varphi)}$$

$$[\vee\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad [\Rightarrow\vee]\ \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

$$[\neg\vee\Rightarrow]\ \frac{\Gamma, \neg\psi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \vee \varphi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\vee]\ \frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \vee \varphi)}$$

Figure 2. The proof system G^4

2. The system $G_{\mathcal{KL}}$ is obtained from G^4 by adding to it $\neg\psi, \psi, \Gamma \Rightarrow \Delta$ as axioms.

3. The system $G_{\mathcal{LP}}$ is obtained from G^4 by adding to it $\Gamma \Rightarrow \Delta, \neg\psi, \psi$ as axioms.

4. The system G_I^4 is obtained from G^4 by deleting the rules $[\vee \Rightarrow]$ and $[\neg\wedge \Rightarrow]$.

5. The system G_I^3 is obtained from G_I^4 by adding to it $\Gamma \Rightarrow \Delta, \neg\psi, \psi$ as axioms.

6. The system G_r^3 is obtained from $G_{\mathcal{KL}}$ by replacing $[\vee \Rightarrow]$ and $[\neg\wedge \Rightarrow]$ with the following rules:

$$[\vee\Rightarrow_A] \quad \frac{\Gamma \Rightarrow \Delta, \neg\psi \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad [\vee\Rightarrow_B] \quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}$$

$$[\neg\wedge\Rightarrow_A] \quad \frac{\Gamma, \neg\psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg(\psi \wedge \varphi) \Rightarrow \Delta} \qquad [\neg\wedge\Rightarrow_B] \quad \frac{\Gamma, \neg\varphi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\psi \wedge \varphi) \Rightarrow \Delta}$$

Theorem 5.1.

1. **Soundness and completeness:** For any set of sequents $S \subseteq Seq_{L_C}$ and any sequent $\Sigma \in Seq_{L_C}$:

   (a) $S \vdash_{G^4} \Sigma$ iff $S \vdash_{\mathcal{M}_B^4} \Sigma$ (and so $\vdash_{G^4} \Gamma \Rightarrow \psi$ iff $\Gamma \vdash_{\mathcal{M}_B^4} \psi$).
   (b) $S \vdash_{G_{\mathcal{KL}}} \Sigma$ iff $S \vdash_{\mathcal{M}_K^3} \Sigma$ (and so $\vdash_{G_{\mathcal{KL}}} \Gamma \Rightarrow \psi$ iff $\Gamma \vdash_{\mathcal{M}_K^3} \psi$).
   (c) $S \vdash_{G_{\mathcal{LP}}} \Sigma$ iff $S \vdash_{\mathcal{M}_P^3} \Sigma$ (and so $\vdash_{G_{\mathcal{LP}}} \Gamma \Rightarrow \psi$ iff $\Gamma \vdash_{\mathcal{M}_P^3} \psi$).
   (d) $S \vdash_{G_I^4} \Sigma$ iff $S \vdash_{\mathcal{M}_I^4} \Sigma$ (and so $\vdash_{G_I^4} \Gamma \Rightarrow \psi$ iff $\Gamma \vdash_{\mathcal{M}_I^4} \psi$).
   (e) $S \vdash_{G_I^3} \Sigma$ iff $S \vdash_{\mathcal{M}_I^r} \Sigma$ (and so $\vdash_{G_I^3} \Gamma \Rightarrow \psi$ iff $\Gamma \vdash_{\mathcal{M}_I^3} \psi$).
   (f) $S \vdash_{G_r^3} \Sigma$ iff $S \vdash_{\mathcal{M}_r^3} \Sigma$ (and so $\vdash_{G_r^3} \Gamma \Rightarrow \psi$ iff $\Gamma \vdash_{\mathcal{M}_r^3} \psi$).

2. **Cut-elimination:** Let $G \in \{G^4, G_{\mathcal{KL}}, G_{\mathcal{LP}}, G_I^4, G_I^3, G_r^3\}$. Then G admits strong cut elimination: If $S = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ then $S \vdash_G \Sigma$ iff there is a proof of Σ from S in G where all cuts are made on formulas in $\bigcup_{i=1}^n (\Gamma_i \cup \Delta_i)$ (in particular, $\vdash_G \Sigma$ iff Σ has a cut-free proof in G).

Proof:
The results about G_I^4 and G_I^3 were already proved in [7]. The soundness and completeness of $G^4, G_{\mathcal{KL}}$, and $G_{\mathcal{LP}}$ for the corresponding matrices and the cut-elimination theorem for these systems are well known results (see e.g. [4, 3]). That the latter theorem can be strengthened for them as described in the formulation of the present theorem can be shown in a way similar to that used below in case of G_r^3. This is the case on which the rest of this proof concentrates.

For simplicity, in what follows we drop the decorations on $\models$.

We start by showing that if $S \vdash_{\mathcal{G}_r^3} \Sigma$, then, for any legal valuation v in the Nmatrix $\mathcal{M}_r^3$ such that $v \models S$, we have $v \models \Sigma$. Obviously, for this it suffices to show the above for all $S \subseteq Seq$ and $\Sigma \in Seq$ such that S is the set of premises of some rule r in $\mathcal{C}_r^3$ and Σ is the consequence of that rule. This can be done by a simple check, which we illustrate below on the example of two non-standard rules in $\mathcal{G}_r^3$.

Rule $(\vee \Rightarrow)_A$**:** We have $S = \{\Gamma \Rightarrow \Delta, \neg\varphi;\ \Gamma, \psi \Rightarrow \Delta\}, \Sigma = \Gamma, \varphi \vee \psi \Rightarrow \Delta$. Assume now v is a legal valuation in the Nmatrix $\mathcal{M}_r^3$ such that $v \models S$. Then we have the following two cases:

Case 1: $v \not\models \Gamma$ or $v \models \Delta$. As $\Sigma = \Gamma, \varphi \vee \psi \Rightarrow \Delta$, then obviously $v \models \Sigma$ too.

Case 2: $v \models \neg\varphi$ and $v \not\models \psi$. As the sets of designated and non-designated values in $\mathcal{M}_r^3$ are respectively $\{1\}$ and $\{\mathrm{I}, 0\}$, this implies $v(\neg\varphi) = 1$ and $v(\psi) \in \{\mathrm{I}, 0\}$. Hence from the truth tables of $\mathcal{M}_r^3$ we obtain $v(\varphi) = 0$ and $v(\varphi \vee \psi) \in \{\mathrm{I}, 0\}$. Thus $v \models \Sigma$.

Rule $(\neg\wedge \Rightarrow)_B$**:** We have $S = \{\Gamma \Rightarrow \Delta, \psi;\ \Gamma, \neg\varphi \Rightarrow \Delta\}, \Sigma = \Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta$. Assume again v is a legal valuation in $\mathcal{M}_r^3$ such that $v \models S$. If either $v \not\models \Gamma$ or $v \models \Delta$, then obviously $v \models \Sigma$. Thus it suffices to consider the case when $v \models \psi$ and $v \not\models \neg\varphi$. Then $v(\psi) = 1$ and $v(\neg\varphi) \in \{\mathrm{I}, 0\}$. By the truth tables of $\mathcal{M}_r^3$, the latter yields $v(\varphi) \in \{\mathrm{I}, 1\}$, whence $v(\varphi \wedge \psi) \in \{\mathrm{I}, 1\}$ and $v(\neg(\varphi \wedge \psi)) \in \{\mathrm{I}, 0\}$. Consequently, $v \models \Sigma$ also in this case.

To show the converse as well as the cut-elimination part of the theorem, define $\underline{\mathcal{G}}_r^3$ to be the calculus obtained from $\mathcal{G}_r^3$ by limiting the applications of the cut rule to formulas occurring in the premises of sequent derivations. In other words: If $S = \{\Gamma_i \Rightarrow \Delta_i \mid i \in I\}$ then $S \vdash_{\underline{\mathcal{G}}_r^3} \Sigma$ if there is a proof of Σ from S in $\mathcal{G}_r^3$ where all cuts are made on formulas in $F(S) = \bigcup_{i \in I}(\Gamma_i \cup \Delta_i)$ (in particular: $\vdash_{\underline{\mathcal{G}}_r^3} \Sigma$ iff Σ has a cut-free proof in $\mathcal{G}_r^3$). We shall prove that $\underline{\mathcal{G}}_r^3$ is (strongly) complete for $\vdash_{\mathcal{M}_r^3}$, i.e., for any set of sequents $S \subseteq Seq$ and any sequent $\Sigma \in Seq$, $S \vdash_{\mathcal{M}_r^3} \Sigma$ iff $S \vdash_{\underline{\mathcal{G}}_r^3} \Sigma$. Obviously, this will end the proof of the theorem.

For simplicity, we first provide the proof in detail for the case in which S is finite. Until the end of this proof, by "derivable" we mean "derivable in $\underline{\mathcal{G}}_r^3$".

We argue by contradiction. Suppose that for a finite set of sequents S and a sequent $\Sigma_0 = \Gamma \Rightarrow \Delta$ we have $S \vdash_{\mathcal{M}_r^3} \Sigma_0$, but Σ_0 is not derivable from S. We shall construct a counter-valuation v such that $v \models S$ but $v \not\models \Sigma_0$.

Since S is finite, so is $F(S)$. Assume it has l elements. Let $\varphi_1, \varphi_2, \ldots, \varphi_l$ be an enumeration of the formulae in $F(S)$. We shall now define a sequence of sequents $\Gamma_n \Rightarrow \Delta_n, n = 0, 1, \ldots, l$, such that, for $n = 0, 1, \ldots, l$:

(i) $\Gamma \subseteq \Gamma_n, \Delta \subseteq \Delta_n$

(ii) If $n \neq 0$ then $\varphi_n \in (\Gamma_n \cup \Delta_n)$.

(iii) $\Gamma_n \Rightarrow \Delta_n$ is not derivable from S.

The above sequences are defined inductively as follows:

- We put $\Gamma_0 = \Gamma, \Delta_0 = \Delta$. As by our assumption $\Gamma \Rightarrow \Delta$ is not derivable from S, (i)–(iii) above are satisfied for $n = 0$.

- Suppose $n \leq l-1$ and we have defined the sequents $\Gamma_i \Rightarrow \Delta_i$ satisfying conditions (i)–(iii) for $i \leq n$. Then the sequents $\Sigma_1 = \Gamma_n \Rightarrow \Delta_n, \varphi_{n+1}$ and $\Sigma_2 = \varphi_{n+1}, \Gamma_n \Rightarrow \Delta_n$ cannot be both derivable from S, since then $\Gamma_n \Rightarrow \Delta_n$ would be derivable from them by an allowed cut on the formula $\varphi_{n+1} \in S$. We take $\Gamma_{n+1} \Rightarrow \Delta_{n+1}$ to be Σ_1, if Σ_1 is not derivable, and Σ_2 otherwise. Then, obviously, from the inductive assumption it follows that the sequence $\Gamma_{n+1} \Rightarrow \Delta_{n+1}$ satisfies conditions (i)–(iii).

By induction, the whole sequence $\Gamma_n \Rightarrow \Delta_n, n = 0, 1, \ldots, l$, satisfies the desired conditions (i)–(iii). What is more, from the inductive construction we can see that

(iv) $\Gamma_n \subseteq \Gamma_{n+1}, \Delta_n \subseteq \Delta_{n+1}$ for $n = 1, 2, \ldots, l-1$

Let $\Gamma^* \Rightarrow \Delta^*$ be the extension of $\Gamma_l \Rightarrow \Delta_l$ to a saturated sequent, i.e. a sequent containing $\Gamma_l \Rightarrow \Delta_l$ and closed under the logical rules in $\underline{\mathcal{G}}_r^3$ applied backwards. By way of example, a sequent $\Gamma' \Rightarrow \Delta'$ is closed under rule $(\vee \Rightarrow)_A$ applied backwards iff $\varphi \vee \psi \subseteq \Gamma'$ implies either $\neg\varphi \in \Delta'$ or $\psi \in \Gamma'$. Then we can easily see that:

(I) $\Gamma \subseteq \Gamma^*, \Delta \subseteq \Delta^*$;

(II) $F(S) \subseteq \Gamma^* \cup \Delta^*$;

(III) $\Gamma^* \Rightarrow \Delta^*$ is saturated and it is not derivable from S.

Now we define the valuation v as follows:

- For any atomic p,

$$\text{(i)}\quad v(p) = \begin{cases} 1 & \text{if } p \in \Gamma^* \\ 0 & \text{if } \neg p \in \Gamma^* \\ \mathrm{I} & \text{otherwise} \end{cases}$$

- For any formulas φ, ψ:

$$\text{(ii)}\quad v(\neg\varphi) = \begin{cases} 1 & \text{if } v(\varphi) = 0 \\ 0 & \text{if } v(\varphi) = 1 \\ \mathrm{I} & \text{otherwise} \end{cases}$$

$$\text{(iii)}\quad v(\varphi \vee \psi) = \begin{cases} 1 & \text{iff } v(\varphi) = 1 \text{ or } v(\psi) = 1 \\ & \text{or } v(\varphi) = v(\psi) = \mathrm{I} \text{ and } \varphi \vee \psi \in \Gamma^* \\ 0 & \text{iff } v(\varphi) = 0 \text{ and } v(\psi) = 0 \\ \mathrm{I} & \text{otherwise} \end{cases}$$

$$\text{(iv)}\quad v(\varphi \wedge \psi) = \begin{cases} 1 & \text{iff } v(\varphi) = 1 \text{ and } v(\psi) = 1 \\ 0 & \text{iff } v(\varphi) = 0 \text{ or } v(\psi) = 0 \\ & \text{or } v(\varphi) = v(\psi) = \mathrm{I} \text{ and } \neg(\varphi \wedge \psi) \in \Gamma^* \\ \mathrm{I} & \text{otherwise} \end{cases}$$

Clearly, v is well-defined, since p and $\neg p$ cannot both belong to Γ^* (for otherwise $\Gamma^* \Rightarrow \Delta^*$ would be an axiom). Moreover, it can be easily checked, by considering the truth tables of the Nmatrix $\mathcal{M}_r^3$, that v defined as above is legal valuation for that Nmatrix. It remains to prove that v is indeed the desired counter-valuation, i.e., that:

(I) $v \models \Sigma$ for each $\Sigma \in S$; (II) $v \not\models (\Gamma \Rightarrow \Delta)$;

We start with (II). As $\Gamma \subseteq \Gamma^*, \Delta \subseteq \Delta^*$, it suffices to prove that $v \not\models (\Gamma^* \Rightarrow \Delta^*)$. Since $\mathcal{D} = \{1\}$ and $v(\varphi) \in \{0, 1, \mathrm{I}\}$ for any formula, this means we have to show that:

$$v(\gamma) = 1 \text{ for any } \gamma \in \Gamma^*, \qquad v(\delta) \in \{0, \mathrm{I}\} \text{ for any } \delta \in \Delta^*$$

To do this, we will prove the following for any formula $\varphi \in \mathcal{F}$:

$$(\mathbf{A}) \qquad v(\varphi) = \begin{cases} 1 & \text{if } \varphi \in \Gamma^* \\ 0 & \text{if } \neg\varphi \in \Gamma^* \end{cases} \qquad (\mathbf{B}) \qquad v(\varphi) \in \begin{cases} \{0, \mathrm{I}\} & \text{if } \varphi \in \Delta^* \\ \{1, \mathrm{I}\} & \text{if } \neg\varphi \in \Delta^* \end{cases}$$

We argue by induction on the complexity of φ, proving simultaneously (A) and (B).

$\varphi = p \in \mathcal{A}$

(A) We have $v(\varphi) = 1$ by (i) in the definition of v if $p \in \Gamma^*$, and $v(\varphi) = 0$ by (ii) in that definition if $\neg p \in \Gamma^*$, whence (A) holds for φ.

(B) If $\varphi = p \in \Delta^*$, then, since $\Gamma^* \Rightarrow \Delta^*$ is not derivable, we have $p \notin \Gamma^*$, whence $v(\varphi) \neq 1$ by (i) in the definition of v, which yields $v(\varphi) \in \{0, \mathrm{I}\}$.
In turn, if $\neg\varphi \in \Delta^*$, then $\neg p \notin \Gamma^*$, for $\Gamma^* \Rightarrow \Delta^*$ is not derivable. Thus again by (i) we have $v(p) \neq 0$, which yields $v(\varphi) \in \{1, \mathrm{I}\}$. Hence (B) holds for φ.

$\varphi = \neg\psi$

(A) If $\varphi \in \Gamma^*$, then $\neg\psi \in \Gamma^*$, whence by the inductive hypothesis for ψ we get $v(\psi) = 0$. Hence by (ii) in the definition of v we have $v(\varphi) = v(\neg\psi) = 1$.
Suppose now $\neg\varphi \in \Gamma^*$. Then $\neg\neg\psi \in \Gamma^*$, and, as $\Gamma^* \Rightarrow \Delta^*$ is saturated, by rule $(\neg\neg \Rightarrow)$ we have $\psi \in \Gamma^*$. Hence by the inductive hypothesis $v(\psi) = 1$, which in turn yields $v(\varphi) = v(\neg\psi) = 0$ by an application of (ii) in the definition of v. Thus (A) holds in this case too.

(B) If $\varphi \in \Delta^*$, then $\neg\psi \in \Delta^*$, whence by the inductive hypothesis for ψ we get $v(\psi) \in \{1, \mathrm{I}\}$. By the definition of v, the latter implies $v(\varphi) = v(\neg\psi) \in \{0, \mathrm{I}\}$.
If $\neg\varphi \in \Delta^*$, then $\neg\neg\psi \in \Delta^*$, and, as $\Gamma^* \Rightarrow \Delta^*$ is saturated, by rule $(\Rightarrow \neg\neg)$ we have $\psi \in \Delta^*$. Thus by the inductive hypothesis $v(\psi) \in \{0, \mathrm{I}\}$, whence $v(\varphi) = v(\neg\psi) \in \{1, \mathrm{I}\}$ by the definition of v. Hence (B) holds for φ.

$\varphi = \psi_1 \vee \psi_2$

(A) If $\varphi \in \Gamma^*$, then, since $\Gamma^* \Rightarrow \Delta^*$ is saturated, by rule $(\vee \Rightarrow)_A$ we have either (a1) $\neg\psi_1 \in \Delta^*$ or (a2) $\psi_2 \in \Gamma^*$, while by rule $(\vee \Rightarrow)_B$ we get either (b1) $\neg\psi_2 \in \Delta^*$ or (b2) $\psi_1 \in \Gamma^*$.
If either (a2) or (b2) hold, then by inductive hypothesis we have either $v(\psi_2) = 1$ for (a2), or $v(\psi_1) = 1$ for (b2). Hence in both cases $v(\varphi) = v(\psi_1 \vee \psi_2) = 1$ by the first line in (iii) of the definition of v.
If neither (a2) or (b2) hold, then (a1) and (b1) do, i.e. $\neg\psi_1 \in \Delta^*$ and $\neg\psi_2 \in \Delta^*$. Hence by the inductive hypothesis for (B) we have $v(\psi_1), v(\psi_2) \in \{1, \mathrm{I}\}$. If $v(\psi_i) = 1$ for some i, then again $v(\varphi) = v(\psi_1 \vee \psi_2) = 1$ by the first line in (iii) of the above definition. Finally, if $v(\psi_i) = \mathrm{I}$ for $i = 1, 2$, then, in view of $\psi_1 \vee \psi_2 \in \Gamma^*$, by the last clause of the above-mentioned definition line we get again $v(\varphi) = v(\psi_1 \vee \psi_2) = 1$.
In turn, if $\neg\varphi \in \Gamma^*$, then $\neg(\psi_1 \vee \psi_2) \in \Gamma^*$. As $\Gamma^* \Rightarrow \Delta^*$ is saturated, by rule $(\neg\vee \Rightarrow)$ this implies $\neg\psi_1, \neg\psi_2 \in \Gamma^*$. Hence by the inductive hypothesis $v(\psi_1) = v(\psi_2) = 0$, which implies $v(\varphi) = v(\psi_1 \vee \psi_2) = 0$ by the second line of (iii) in the definition of (v). This ends the proof of (A) for φ.

(B) If $\varphi \in \Delta^*$, then $\psi_1 \vee \psi_2 \in \Delta^*$. Thus, as Δ^* is saturated, by rule $(\Rightarrow \vee)$ we have $\psi_i \in \Delta^*$ for $i = 1, 2$. Hence by the inductive hypothesis $v(\psi_i) \in \{0, \mathrm{I}\}$ for $i = 1, 2$. Moreover, as $\psi_1 \vee \psi_2 \in \Delta^*$ and $\Gamma^* \vdash \Delta^*$ is not derivable, then $\psi_1 \vee \psi_2 \notin \Gamma^*$. Hence by the definition of v we have $v(\psi_1 \vee \psi_2) \in \{0, \mathrm{I}\}$, so (B) is satisfied.

Next, if $\neg\varphi \in \Delta^*$, then $\neg(\psi_1 \vee \psi_2) \in \Delta^*$. Since Δ^* is saturated, from rule $(\Rightarrow \neg\vee)$ we obtain $\neg\psi_i \in \Delta^*$ for some $i \in \{1, 2\}$. Hence by the inductive hypothesis $v(\psi_i) \in \{1, \mathrm{I}\}$ for some $i \in \{1, 2\}$, and so $v(\psi_1 \vee \psi_2) \in \{1, \mathrm{I}\}$ by the definition of v. Thus (B) holds for φ.

$\varphi = \psi_1 \wedge \psi_2$

The proof in this case is similar to that in the previous one, and is left for the reader.

It remains to prove (I), i.e., to show that $v \models \Sigma$ for each $\Sigma \in S$. So let $\Sigma \in S$. Then $\Sigma = \varphi_1, \ldots, \varphi_k \Rightarrow \psi_1, \ldots, \psi_l$ for some integers k, l and formulas $\varphi_i, \psi_j, i = 1, \ldots, k, j = 1, \ldots, l$. Clearly, we cannot have both $\{\varphi_1, \ldots, \varphi_k\} \subseteq \Gamma^*$ and $\{\psi_1, \ldots, \psi_l\} \subseteq \Delta^*$, for then $\Gamma^* \Rightarrow \Delta^*$ would be derivable from Σ, and hence from S, by weakening. Since $F(S) \subseteq \Gamma^* \cup \Delta^*$, this implies that either $\varphi_i \in \Delta^*$ for some i, or $\psi_j \in \Gamma^*$ for some j. Hence by (A) and (B), which we have already proved, we have either $v \not\models \varphi_i$ for some i, or $v \models \psi_j$ for some j, which implies that $v \models \Sigma$.

This ends the proof in case S is finite. The proof in case S is infinite is similar. Using Zorn Lemma (or an infinitary version of the construction used above for the finite case), we first extend Γ and Δ to a maximal pair of theories T_Γ and T_Δ such that $\Gamma \subseteq T_\Gamma$, $\Delta \subseteq T_\Delta$, and there is no sequent $\Gamma' \Rightarrow \Delta'$ such that $\Gamma' \subseteq T_\Gamma$, $\Delta' \subseteq T_\Delta$, and $\Gamma' \Rightarrow \Delta'$ is derivable from S. It can easily be shown that $F(S) \subseteq T_\Gamma \cup T_\Delta$, and that the infinite "sequent" $T_\Gamma \Rightarrow T_\Delta$ is closed under the logical rules in $\underline{\mathcal{G}}^3_r$ applied backwards. It easily follows that we can use T_Γ and T_Δ to define a counterexample v in exactly the same way as Γ^* and Δ^* have been used for this task in the finite case. □

Corollary 5.1. The cut rule is admissible in all the systems considered in Theorem 5.1.

6. Conclusions

A natural next step in our research will be to extend the language of EIP structures with suitably interpreted quantifiers to obtain a first-order logic of information processing structures. Another direction is to consider other strategies for collecting information from sources, or to use more than four truth values to characterize the information they provide about formulas.

Further, the problems considered in our work bear an obvious relationship to the works on on social choice, e.g. [13, 14], where a group of individuals aggregate their individual judgments on some interconnected propositions into the corresponding collective judgment. Hence another direction of future work would be to try to apply our approach to the problems of social choice.

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