Unitary Space-Time Coding Using Group Related Structures

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by

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Abstract

Data communication is an inseparable part of our life today, and a continuing effort has been done to develop fast and reliable transmission schemes. A regular communication scheme involves one transmitter and one receiver. Utilization of multiple antennas in wireless communication is one of the techniques that increase the data rate while maintaining low error probabilities. This technique is known as signal diversity. In order to exploit the advantages of using several transmitters, we design efficient codes which achieve these rates while maintaining low encoding and decoding complexities. For most applications, a major obstacle to achieve high throughput lies in the inabilities of the transmitter to have instantaneous and updated information on the fading of the broadcast channel. Therefore, the transmitter has to employ a channel code that guarantees good performance within the majority of possible channel realizations. Such a channel code is inherently multi-spatial-dimensional and thus it is called a space-time code.

Space-time coding and modulation can use multiple antennas to improve the performance of the communication link, especially in situations where the signal to noise ratio (SNR) is high. Each codeword is represented by a matrix. The columns are input signals to each antenna for consecutive time intervals. In recent studies, group structures and their matrix representations proved to be excellent candidate for space time coding. In this thesis, we develop some group related structures to design high-rate constellations. Specifically, we explore the projective groups and combinations of quaternion groups to generate new and efficient set of codes. The criterion, which we use to estimate the performance of a space time code, is a measurement called the diversity product. It represents the minimal distance between the codewords of the space-time code. Therefore, it gives the error probability to get the wrong coded word. We show that our space-time codes outperform many other known space-time codes.
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1 Introduction

Using multiple antennas in wireless communication is an emerging technology for increasing the data rate while maintaining a low error probability. In order to exploit the advantages of using several transmitters, one has to design advanced transmission codes that can exploit the transmission antennas to achieve high rates and still maintain low encoding and decoding complexity.

The ability to achieve good performance from cellular communications today is highly dependent on developing these codes, since unlike the Gaussian channel, the wireless channel suffers from attenuation due to destructive addition of multipaths in the propagation media and due to interference from other users.

In multiple antenna communications, the data is encoded by a channel code and the encoded data is split into $n$ streams that are simultaneously transmitted using $n$ transmit antennas. The received signal at each receive antenna is a linear superposition of the $n$ transmitted signals added with some noises.

Recent studies have explored the ultimate limit of multiple antenna system from the information-theoretic point of view [10]. Assume a multiple antenna that has $n$ transmitting and $m$ receiving antennas. It is shown that, if the narrowband slow fading channel can be modelled as an $n \times m$ matrix with i.i.d. complex Gaussian random entries, the average channel capacity of such system is approximately $\min(n, m)$-times higher than that of a single antenna system for the same overall transmitting power.
In most applications, a major obstacle to utilizing this high throughput is that the transmitter cannot have the instantaneous information about the fading channel. This information is highly important in order to decode correctly transmitted codeword, while overcoming the channel interferences. Then, the transmitter must employ a channel code that can guarantee a good performance with the majority of possible channel realizations. Such a channel code is inherently multi-spaced over the same time and thus is called a space-time code [2]. In space time block coding, each codeword is a matrix which is transmitted over a time block. Each element in the matrix is the input signal to some transmitter, over a time interval in the block. The receivers try to reconstruct the matrix that was transmitted, and decode the correct codeword.

As defined in [1], the design problem for unitary space-time coding is the following: let $M$ be the number of transmitter antennas and $R$ the desired transmission rate (in bits per channel use). Construct a set $\mathcal{V}$ of $L = 2^{RM}$ unitary $M \times M$ matrices such that for any two
distinct elements $A$ and $B$ in $\mathcal{V}$, the quantity $|\det(A - B)|$ is as large as possible.

Since the effort to find an optimal solution to this problem in this generality is unapproachable and impossibly hard, we need to focus on some structure on the constellation set $\mathcal{V}$. For this discussion we will assume that $\mathcal{V}$ is a set of unitary matrices that forms a group with respect to matrix multiplication. The reason that we consider unitary matrices is to avoid the need to normalize. Each row represents the energies that will be transmitted in each antennas, in a specific time frame. In this case, the sum of square roots of these energies remains the same as the total energy which is being used in that time frame.

Using a group structure offers certain advantages. First, there is more potential for good performance in the following sense: if $\mathcal{V}$ is not a group, $|\det(A - B)|$ generally may take on $L(L - 1)/2$ distinct values for $A \neq B \in \mathcal{V}$. The minimal value (equivalent to the minimal distance of the constellation) may, therefore, be quite small. But if $\mathcal{V}$ is a group, the determinant takes on at most $L - 1$ distinct values given by $|\det(I - A)|$ for $I \neq A \in \mathcal{V}$, possibly yielding a larger minimum distance. This can be observed from the fact that for each $V_\ell \neq V_{\ell'} \in \mathcal{V}$

$$|\det(V_\ell - V_{\ell'})| = |\det(V_\ell(I - V_\ell^{-1}V_{\ell'}))| = |\det(I - V_\ell V_{\ell'}^*)| = |\det(I - V)|.$$  \hfill (1.1)

The second advantage is a practical one. As we will see in the following chapters, differential space-time modulation multiplies matrices in $\mathcal{V}$ to form the transmitted signal matrix. If $\mathcal{V}$ is a group, every transmitted signal matrix is always an element of $\mathcal{V}$. Therefore, explicit matrix multiplication is replaced by the simpler group table lookup.

In this thesis we come up with new constellation designs, which perform well in terms of diversity and decoding complexity. The remainder of this thesis is organized as follows: Section 3 presents the diversity criterion and the techniques for of finding good constellations. In section 2, we elaborate on the potential of space-time coding, and using DPSK in order to
overcome the problem of not knowing the behavior of the channel at the receiver. In section 4, we show how the behavior of group structures can help to construct fast and efficient space time codes, and we describe some groups that were investigated in previous works. Section 5 elaborates on non groups codes that achieve good performance as well. Sections 6 and 7 present some new structures that are studied in this work, and explain how to construct them. The results and performance analysis of these structures with comparison to other known codes is presented in section 8.
2 Multiple Antenna Space Time Modulation

Assume we have a communication link with $M$ transmitter antennas and $N$ receiver antennas operating in a Rayleigh flat-fading environment. The $n$-th receiver antenna responds to the symbol sent on the $m$-th transmitter antenna through a statistically independent multiplicative complex-Gaussian fading coefficient $h_{mn}$. The received signal at the $n$-th antenna is corrupted at time $t$ by an additive complex-Gaussian noise $w_{tn}$ that is statistically independent among the receiver antennas and also independent from one symbol to the next. $t$ represents discrete time frames $t = 0, 1, 2, \ldots$.

It is convenient to group the symbols transmitted over the $M$ antennas in blocks of $M$ channel uses. We will use $\tau = 0, 1, \ldots$ to index these blocks; within the $\tau$th block, $t = \tau M, \ldots, \tau M + M - 1$, the transmitted signal is written as an $M \times M$ matrix $S_\tau$ whose $m$th column contains the symbols transmitted from the $m$-th antenna as a function of time; equivalently, the rows contain the symbols transmitted from the $M$ antennas at any given time. The matrices are normalized so that the expected square Euclidean norm of each row is equal to one. Hence, the total transmitted power does not depend on the number of antennas. The fading coefficients $h_{mn}$ (see Eq. 2.1) are assumed to be constant over the usage of this channel $M$ consecutive times.

Similarly, the received signals are organized in $M \times N$ matrices $X_\tau$. Since we assume that the fading coefficients are constant within the block of $M$ symbols, the action of the channel is given by the matrix equation

$$X_\tau = \sqrt{\rho}S_\tau H_\tau + W_\tau, \quad \tau = 0, 1, \ldots$$

(2.1)

where $S_\tau$ is the transmitted matrix, $H_\tau = \{h_{mn}\}$ and $W_\tau = \{w_{tn}\}$ are $M \times N$ matrices of independent $\mathcal{CN}(0,1)$ distributed random variables. Because of the power normalization, $\rho$ is the expected signal-to-noise ratio (SNR) at each receiver antenna.
We first discuss signal encoding and decoding when the receiver knows the channel $H_r$. We assume that the data that will be transmitted is the sequence $z_0, z_1, \ldots$ of codewords, where $z_\tau \in \{0, \ldots, L-1\}$. Then, each codeword dictates which matrix $S_\tau = V_{z_\tau}$ is transmitted. Each transmitted matrix occupies $M$ time samples of the channel, implying that transmitting at a rate of $R$ bits per channel use, requires a constellation $\mathcal{V} = \{V_1, \ldots, V_L\}$ of $L = 2^{RM}$ unitary signal matrices. The receiver knows $H_r$ and computes the maximum-likelihood (ML) estimate of the transmitted data as

$$\hat{z}_\tau = \arg \min_{\ell=0,\ldots,L-1} \| X_\tau - V_\ell H_r \|.$$

(2.2)

The quality of a constellation $\mathcal{V}$ is determined by the error probability of mistaking one symbol of $\mathcal{V}$ for another.

When the receiver does not know the channel, one can communicate using multiple-antenna differential modulation [4]. Multiple-antenna differential modulation is formally similar to standard single-antenna differential phase-shift keying. In standard DPSK, the transmitted symbol has unit-modulus and is the product of the previously transmitted symbol and the current data symbol. The data symbol typically is one of $L$ equally-spaced points on the complex unit circle. As a generalization, $M$-antenna differential unitary space-time modulation differentially encodes $M \times M$ unitary matrix-valued signals. We transmit an $M \times M$ unitary matrix that is the product of the previously transmitted matrix and a unitary data matrix taken from the constellation. In other words,

$$S_\tau = V_{z_\tau} S_{\tau-1}, \quad \tau = 1, 2, \ldots$$

(2.3)

where $S_0 = I_M$. We immediately see why it is useful in practice to have for $\mathcal{V}$ a group structure under matrix multiplication. If $\mathcal{V}$ is a group then from Eq. 2.3 all the transmitted matrices $S_\tau$ also belong to $\mathcal{V}$. Therefore, the transmitter sends matrices $S_\tau$ from a finite set and does not need to explicitly multiply $S_\tau = V_{z_\tau} S_{\tau-1}$, but rather can use a group lookup table.
If the fading coefficients are approximately constant over 2M time samples ($H_\tau \approx H_{\tau-1}$), the received matrices turn out to obey

$$X_\tau = V_{\tau}X_{\tau-1} + \sqrt{2}W'_\tau$$

(2.4)

where $W'_\tau$ is an $M \times N$ matrix of additive independent $\mathcal{CN}(0,1)$ noise, uncorrelated with the signal. As shown in [1, 5], the ML decoder has the form:

$$\hat{z}_\tau = \arg \min_{\ell=0, \ldots, L-1} \|X_\tau - V_\ell X_{\tau-1}\|.$$  

(2.5)
3 The diversity product criterion

Let $M$ be the number of transmitting antennas and $R$ is the desired transmission rate. We want to find a set of $L = 2^{RM}$ matrices, where each matrix represents a codeword, such that the probability of transmitting codeword $V_\ell$ and decoding it as $V_{\ell'}$, is as small as possible. In [1, 2] it is shown that for high SNR channels, the error bound for mistaking $V_\ell$ for $V_{\ell'}$, averaged over all possible channels, depend primarily on the products of the singular values of the matrix $V_\ell - V_{\ell'}$, and therefore depends on $\det(V_\ell - V_{\ell'})$. It is then convenient to measure the quality of a constellation $\mathcal{V}$ by what is called the diversity product, which is defined as:

$$\zeta_{\mathcal{V}} = \frac{1}{2} \min_{0 \leq \ell < \ell' < L} |\det(V_\ell - V_{\ell'})|^{\frac{1}{M}}. \quad (3.1)$$

The scaling factor $\frac{1}{2}$ guarantees that $0 \leq \zeta_{\mathcal{V}} \leq 1$. The exponent $\frac{1}{M}$ is the geometric normalization of the distance between the matrices with dimension $M$. A constellation with $\zeta_{\mathcal{V}} > 0$ is said to have full diversity. When $\zeta_{\mathcal{V}} > 0$ and the SNR is high, we note that no distinct transmitted signals can give the same received signal, for any channel behavior.

We therefore want to find structures which provide good constellations in terms of high diversity product. Different methods have been proposed to generate these constellations. Some of them include optimization methods [8, 9], while others introduced some characteristics for the set $\mathcal{V}$. We present in this work a new construction that is based on related group structures, which provides good constellations according to the above criterions.
4 Using group structures for space time coding

As explained in section 1, using groups for designing unitary space-time constellations, has high potential of providing constellations with good diversity product. We will review some of those groups which were analyzed in prior works.

**cyclic groups:** cyclic groups are used for differential modulation in [5], and also referred as “diagonal designs”. The elements of these groups, are diagonal $L$th roots of unity.

$$V_\ell = V_s^\ell \quad \text{and} \quad V_s = \text{diag}[e^{i2\pi u_1/L}, ..., e^{i2\pi u_m/L}]$$

The $u_m$ are chosen to maximize the diversity product $\zeta_V$, which is given by

$$\zeta_V = \frac{1}{2} \min_{0 \leq \ell < \ell' < L} |\det(V_\ell - V_{\ell'})|^\frac{1}{M}$$

$$= \frac{1}{2} \min_{\ell = 1, ..., L-1} |\det(I - V_\ell)|^\frac{1}{M}$$

$$= \frac{1}{2} \min_{\ell = 1, ..., L-1} |\det(\text{diag}[1 - e^{i2\pi u_1/L}, ..., 1 - e^{i2\pi u_m/L}])|^\frac{1}{M}$$

$$= \frac{1}{2} \min_{\ell = 1, ..., L-1} \left| \prod_{i=1}^M \frac{1}{1 - e^{i2\pi u_i/L}} \right|^{\frac{1}{M}}$$

$$= \frac{1}{2} \min_{\ell = 1, ..., L-1} \left| \prod_{i=1}^M e^{i\pi u_i/L}(e^{-i\pi u_i/L} - e^{i\pi u_i/L}) \right|^{\frac{1}{M}}$$

$$= \min_{\ell = 1, ..., L-1} \left| \prod_{i=1}^M \frac{e^{-i\pi u_i/L} - e^{i\pi u_i/L}}{2} \right|^{\frac{1}{M}}$$

$$= \min_{\ell = 1, ..., L-1} \left| \prod_{i=1}^M \sin \frac{\pi u_i\ell}{L} \right|^{\frac{2}{M}}.$$ 

In this constellation, the transmitter antennas are activated one at a time, sequentially, and in the same order. Notice that these groups are *Abelian* (commutative), and that they have *full diversity*. Diagonal designs perform well in low rates, but other works such as in [1] tried to find designs with matrices that are not “sparse” and will achieve good diversity at high rates.
**Quaternion Groups:** The quaternion groups are also called “dicyclic groups” in [4], and have the form

\[ Q_p = \langle \alpha, \beta | \alpha^{2^p} = 1, \beta^2 = \alpha^{2^{p-1}}, \beta \alpha = \alpha^{-1} \beta \rangle, \quad p \geq 1 \]

where \langle \cdot \rangle refers to the groups generated from the elements that are in the brackets. The group order is \( L = 2^{p+1} \). The matrix representation for the \( 2 \times 2 \) quaternion group appears in [4], so the rate of the constellations will be \( R = (p+1)/2 \) and the group is generated from the two unitary matrices

\[ \begin{pmatrix} e^{2\pi i/2^p} & 0 \\ 0 & e^{-2\pi i/2^p} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \] .

These groups will be used in our work for the development of some new constellations, which are not necessarily groups, and outperform in diversity results.

**Fixed Point Free Groups:** A classification of all possible finite groups of matrices which have full diversity (\( \zeta > 0 \)) is given in [1]. Those type of groups are named fix-point-free groups, and explored by Burnside, Zassenhaus, Amitsur and others. Cyclic groups and quaternion groups are specific examples of such groups. As seen in their work, there are six types of fix-point free groups: \( G_{m,r}, D_{m,r,t}, E_{m,r}, F_{m,r,t}, J_{m,r} \) and \( K_{m,r,t} \). \( G_{m,r} \), for example, has the form of

\[ G_{m,r} = \langle \alpha, \beta | \alpha^m = 1, \beta^n = \alpha^t, \beta \alpha = \alpha^r \beta \rangle \quad (4.1) \]

where \( n \) is the order of \( r \) modulo \( m \) (i.e. \( n \) is the smallest positive integer such that \( r^n \equiv 1 \mod m \)), \( t = m/\gcd(r-1, m) \), and \( \gcd(n, t) = 1 \). The matrix representation of \( G_{m,r} \) is \( A^sB^k, s = 0, \ldots, m-1, k = 0, \ldots, n-1 \) where

\[ A = \begin{pmatrix} \xi & 0 & 0 & \ldots & 0 \\ 0 & \xi^r & 0 & \ldots & 0 \\ 0 & 0 & \xi^{r^2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \xi^{rn-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ \xi^t & 0 & 0 & \ldots & 0 \end{pmatrix} \quad (4.2) \]
and $\xi = e^{2\pi i/m}$.

It can be proven that a finite group is fixed-point-free if and only if it is isomorphic to one of the above six groups [1]. Many of the fixed-point-free group representations have good diversity product and therefore provide high rate constellations.
5 Extensions for group-related Structures

Although we saw that group constellations have better chance to have full diversity, sometimes the group constraint does not really help, and we may want to look at matrix sets that are not necessarily groups. Moreover, there is a strong motivation to develop group-related structures, that still have low number of distinct pairwise distances.

Orthogonal Designs: orthogonal designs were introduced in the early stages of space-time codes development (see [3, 11]). A complex orthogonal design of size $N$ is an $N \times N$ unitary matrix whose rows are permutations of real numbers $\pm x_1, \pm x_2, ..., \pm x_n$, their conjugates $\pm x_1^*, \pm x_2^*, ..., \pm x_n^*$, or multiples of these indeterminates by $\pm \sqrt{-1}$. The matrix representation of orthogonal designs for two transmit antennas is:

$$OD(x, y) = \frac{1}{\sqrt{2}} \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix}.$$  \hspace{1cm} (5.1)

where $x$ and $y$ are subject to a power constraint. In order to have unitary matrices we can use the constraint that $|x|^2 = |y|^2 = 1$. The unitary constellations are then obtained by letting $x, y$ be the $Q$th roots of unity, so

$$\mathcal{V} = \{OD(x, y) \mid x, y \in \{1, e^{2\pi i/Q}, ..., e^{2\pi i(Q-1)/Q}\} \}.$$  \hspace{1cm} (5.2)

These constellations do not generally form a group, but preform well as space-time block codes [3].

Nongroup Generalizations of $G_{m,r}$: As shown in section 4, $G_{m,r}$ has a matrix representation of dimension $n$, where $n$ is a function of $m$ and $r$. We can try and relax the constraint on the elements of the matrices that form $G_{m,r}$ and look at the general case of the set $S_{m,s}$, consisting of the matrices $A^\ell B^k$ where $\ell = 0, ..., m - 1$, $k = 0, ..., s - 1$. 

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where \( \alpha, \beta \) are the primitive \( m \)th and \( s \)th roots of unity, and \( u_1, \ldots, u_n \) are integers. Deeper analysis of these constellations can be found in [1].

**Products of Group Representations:** Consider two fixed-point free groups \( S_A \) and \( S_B \) and their unitary \( M \times M \) matrix representations \( \{ A_1, \ldots, A_{L_A} \} \) and \( \{ B_1, \ldots, B_{L_B} \} \), respectively. Now we look at the set of pairwise product

\[
S_{A,B} = \{ A_j B_k \mid j = 1, \ldots, L_A, \ k = 1, \ldots, L_B \}.
\] (5.4)

\( S_{A,B} \) has at most \( L_A L_B \) distinct elements, and the constellation rate is at most \( R = R_A + R_B \). The diversity product is then:

\[
\zeta_s = \frac{1}{2} \min_{(j,k) \neq (j',k')} |\det(A_j B_k - A_{j'} B_{k'})|^\frac{1}{M}
\]

\[
= \frac{1}{2} \min_{(j,k) \neq (j',k')} \left|\det(A_{j'}^{-1}) \det(A_j B_k - A_{j'} B_{k'}) \det(B_{k'}^{-1})\right|^\frac{1}{M}
\]

\[
= \frac{1}{2} \min_{(j,k) \neq (j',k')} \left|\det(A_{j'}^{-1} A_j - B_{k'} B_{k'}^{-1})\right|^\frac{1}{M}
\]

\[
= \frac{1}{2} \min_{(\ell,\ell') \neq (0,0)} |\det(A_{\ell} - B_{\ell})|^\frac{1}{M}.
\]

Notice on how we used the fact that the determinant of the unitary matrix equals 1, and that \( S_A \) and \( S_B \) are groups. Finally, we see that even though \( S_{A,B} \) is not a group, it has at most \( L - 1 \), rather then \( L(L - 1) / 2 \), distinct pairwise distances. Therefore, it has a good chance of having full diversity. Deeper analysis of group products can be found in [1].
Sets generated by Simulated Annealing Algorithms: Simulated annealing is a method which imitate the process of melted metal getting cooled off. In the annealing process of the melted metal, first the metal is heated to melt, then the temperature is slowly lowered. The metal will get to a minimized energy state if the temperature is lowered enough. We can use this idea as a general method of optimization: given an optimization problem, we choose an initial solution in some way, and then in each step we consider the neighborhood of this solution. We will accept a new solution in this neighborhood according to some predefined criterion. On each step we also narrow the neighborhood of the solution, since the energy needed to leap over to a distant solution decreases. Since finding a set of matrices with maximal diversity is an optimization problem, it might be a good idea to try this algorithm. For the initial solution we can take a group that has high diversity product. This idea and some experimental results appear in [9].

Sets generated by Genetic Algorithms: genetic algorithm is another optimization algorithm which emulates the species evolutionary model. The algorithm does not assume much on the given optimization problem. We first choose a set of candidates in some way. Then, we consider the offsprings of the candidates (the mating processing is also defined with respect to the specific problem). We might also add in some new random candidates. Now we need to take the candidates with the best “survival index” which is another criterion of the specific problem. After losing the “bad” elements, we iterate this step with the new “improved” species. The idea is that the elements with the best survival score, will keep staying in the group while the “weaker” ones will be discarded. This algorithm optimized the diversity product of some initial group of matrices (see [9]).
6 Projective groups and their performances

In order to achieve high performance space-time coding we design a code that is based on unitary matrices that provide us with good diversity product values. One of the structure that we explore is the *projective groups*. Projective group is a collection of elements $V$ that satisfies the property that if $A, B \in V$, then $AB = \alpha C$ where $C \in V$ and $\alpha$ is a scalar. In other words, the product of every two elements in $V$, is also an element of $V$, multiplied by a scalar. Projective groups have a less strict requirement to satisfy for a collection of signal matrices, but can still maintain the decoding complexity criterion. This is because for differential space time modulation, the matrix multiplication can be replaced by a group lookup table. For example, $S_{a_1}S_{a_2}S_{a_3}S_{a_4} = \alpha_1\alpha_2\alpha_3S_{a_5}$.

We design a set of matrices that form a projective group by the following construction: let $\alpha, \beta$ be some non rational numbers, and let $\zeta = e^{2\pi i/3}$. The elements of the projective group $P_{\alpha,\beta,3}$ are given by $A^iB^j$ where $j=0,1,2$, $i = 0,1,2$ and

$$
B = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\beta & 0 & 0
\end{pmatrix}, \quad
A = \begin{pmatrix}
\sqrt[3]{\alpha} & 0 & 0 \\
0 & \zeta\sqrt[3]{\alpha} & 0 \\
0 & 0 & \zeta^2\sqrt[3]{\alpha}
\end{pmatrix}.
$$

We used MATLAB to verify that this is a projective group, by computing the multiplication table for $P_{\alpha,\beta,3}$. $P_{\alpha,\beta,3}$ has only 9 elements, so the transmission rate is $R = \log_2(9)/3 = 1.06$, which is relatively small. We explore, through the usage of an optimization algorithm, different $\alpha$ and $\beta$ in order to identify the best achievable diversity product.
The relation between $\alpha, \beta$ and the diversity product $\zeta$ seems to be chaotic. Some of the diversity products for different $\alpha$ and $\beta$ are given in table 1.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$L$</th>
<th>$M$</th>
<th>$\zeta$</th>
<th>Group structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.06</td>
<td>9</td>
<td>3</td>
<td>0.3971</td>
<td>$P_{\alpha,\beta,3}$ with $\alpha = e^{2\pi i \sqrt{5}}, \beta = e^{2\pi i \sqrt{11}}$</td>
</tr>
<tr>
<td>1.06</td>
<td>9</td>
<td>3</td>
<td>0.4273</td>
<td>$P_{\alpha,\beta,3}$ with $\alpha = e^{2\pi i \sqrt{6}}, \beta = e^{2\pi i \sqrt{32}}$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>0.3741</td>
<td>$P_{\alpha,\beta,2}$ with $\alpha = e^{2\pi i \sqrt{5}}, \beta = e^{2\pi i \sqrt{7}}$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>0.7053</td>
<td>$P_{\alpha,\beta,2}$ with $\alpha = e^{2\pi i \sqrt{42}}, \beta = e^{2\pi i \sqrt{90}}$</td>
</tr>
</tbody>
</table>

Table 1: The relations between the transmission rate $R$, size of the group $L$, number of transmitting antennas $M$, diversity product $\zeta$ and the group structure.

Projective groups seems to be a promise structure that can have high diversity product and still have low complexity for DPSK constellations. The finite groups that can occur as
groups of inner automorphisms of finite dimensional division algebras have been classified (see [12]) but realizing them seems difficult and have not been carried out so far.
7 Extensions of the quaternion groups

When we examine the diversity product of the matrices in the constellations that form a group with respect to matrix multiplication, we realize that the performance of the quaternion group outperforms other structures. This fact led us to the assumption that we can extend the quaternion group to have more elements while preserving a high diversity product. We may lose the group structure in this construction. Furthermore, if we think of the diversity product as the minimum distance between the elements of the constellation, we can design a structure where the elements are “well-spaced”, in the sense that the distance between every two elements is sufficiently big.

We begin with $2 \times 2$ matrix representation of the quaternion group $Q_2$. It has 8 elements. If the transmission rate is $R = \log_2 8/2$ then diversity product is $\zeta_{Q_2} = 0.7071$. The group is constructed from the set of matrices $\pm Q_i, \pm Q_j, \pm Q_k, \pm Q_l$ where:

$$Q_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q_k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Q_l = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

We now define the super-quaternion of $Q_2$ with $n$ layers (explained later), for matrices of degree 2, as a set of all linear combinations of the matrices $Q_i$ that satisfy:

$$S_{Q_2,n,2} = \left\{ x_1Q_i + x_2Q_j + x_3Q_k + x_4Q_l \frac{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}}{x_1^2 + x_2^2 + x_3^2 + x_4^2} \mid 0 < x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq n, \quad x_i \text{ are integers} \right\}. \quad (7.1)$$

The matrices in $S_{Q_2,n,2}$ are unitary. Indeed, if $q$ is a linear combination of matrices in $Q_2$, $q = x_1Q_i + x_2Q_j + x_3Q_k + x_4Q_l$, and $\bar{q}$ is the conjecture matrix, $\bar{q} = x_1Q_i - x_2Q_j - x_3Q_k - x_4Q_l$ then $q\bar{q} = \|q\|^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2)Q_i = kI$. Therefore, all the elements in $S_{Q_2,n,2}$ are normalized by $\frac{1}{\sqrt{k}}$ and they become unitary matrices.

Another way to represent the structure of the super-quaternion is through a collection of Layers. Assume that the quaternion group $Q_p$ has a matrix representation of $m \times m$ matrices. Let $S_{Q_p,n,m}$ be the collection of $m \times m$ matrices, which are all linear combinations
of the matrices in $Q_p$, where the coefficients of the matrices are all integers and the sum of square roots of them is at most $n$. Then, $S_{Q_p,n,m}$ is written as a sequence of the layers $\bigcup_{i=1}^n L_i$, where

$$L_i = \left\{ \sum_{s=1}^{2^p} x_s Q_s \middle| \sum_{s=1}^{2^p} x_s^2 = i, x_s \text{ are integers} \right\}. \quad (7.2)$$

When $p = 2$ and $m = 2$ then $S_{Q_2,n,2}$ is a special case of this family. In this case, the set is the union of the layers $L_1, ..., L_n$. $L_i$ is the set of all linear combinations of matrices in $Q_2$, where the coefficients $x_i$ are integers that satisfy

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = i. \quad (7.3)$$

In this case $L_1$ is $Q_2$ in Eq. 7.2 and it has eight elements. The calculation of the number of elements in each layer (which is the number of solutions to Eq. 7.3), shows that $|L_2| = 24$, $|L_3| = 32$, $|L_4| = 24$, etc.

By examining the layers $L_i$ of the super quaternion, we observe that in some cases the same matrix element can exist in more than one layer. If $(x_1, ..., x_n)$ is in layer $L_i$, then $(\alpha x_1, \alpha x_2, ..., \alpha x_n)$ must be in layer $L_{\alpha^2 i}$, and since the elements are normalized, the matrices are equal. Therefore, for $\alpha > 1$ we have $L_i \subseteq L_{\alpha^2 i}$. For example, the element $Q_a \in L_1$ (the solution $(1, 0, 0, 0)$), is equal to element $2Q_a/\sqrt{4} \in L_4$ (the solution $(2, 0, 0, 0)$). To eliminate the duplicate elements, we have to reduce these solutions in order to calculate correctly $S_{Q_2,n,2}$ and its diversity product.

While the super-quaternion structure outperforms in term of diversity product, they do not form a group, therefore the decoding involves matrix multiplication. We explore the structure of these sets in order to have a unique fast decoding method such as lookup table instead of matrix multiplication. Since every element in $S_{Q_2,n,2}$ is a linear combination of the quaternion matrices, it can be expressed by the scalar coefficients of each $Q_i, q = (x_1, x_2, x_3, x_4) \in S_{Q_2,n,2}$. Therefore, multiplying two elements in $S_{Q_2,n,2}$ is the same as multiplying two vectors of size four, where the basis of these vector is $Q_2$. The result is
another vector from the same basis, and this result can be calculated with multiplication
table of $Q_2$ (see table 2).

<table>
<thead>
<tr>
<th>$\otimes$</th>
<th>$Q_i$</th>
<th>$Q_j$</th>
<th>$Q_k$</th>
<th>$Q_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_i$</td>
<td>$Q_i$</td>
<td>$Q_i$</td>
<td>$Q_i$</td>
<td>$Q_i$</td>
</tr>
<tr>
<td>$Q_j$</td>
<td>$Q_i$</td>
<td>$-Q_i$</td>
<td>$Q_l$</td>
<td>$-Q_k$</td>
</tr>
<tr>
<td>$Q_k$</td>
<td>$Q_i$</td>
<td>$-Q_i$</td>
<td>$-Q_i$</td>
<td>$Q_j$</td>
</tr>
<tr>
<td>$Q_l$</td>
<td>$Q_i$</td>
<td>$Q_k$</td>
<td>$-Q_j$</td>
<td>$-Q_i$</td>
</tr>
</tbody>
</table>

Table 2: Multiplication table for $Q_2$

Using the multiplication table (table 2) gives the products of any two elements in the set
$S_{Q_2,n,2}$ without the need to multiply matrices. Instead we add and multiply the components.

If $p = (x_1, x_2, x_3, x_4)$, $q = (y_1, y_2, y_3, y_4)$ where $p, q \in S_{Q_2,n,2}$ then

$$pq = (x_1, x_2, x_3, x_4) \otimes (y_1, y_2, y_3, y_4) = (z_1, z_2, z_3, z_4)$$  \hspace{1cm} (7.4)

where

$$z_1 = x_1y_1 + x_2y_1 + x_3y_1 + x_4y_1 + x_1y_2 + x_1y_3$$

$$+ \ x_1y_4 - x_2y_2 - x_3y_3 - x_4y_4$$

$$z_2 = x_3y_4 - x_4y_3$$

$$z_3 = x_4y_2 - x_2y_4$$

$$z_4 = x_2y_3 - x_3y_2.$$

Since all the $z_i$ are integers, then to calculate the matrix $pq$ we take an integer number
of each $Q_i$ and add them together.
8 Results

We find all the elements of each layer that creates $S_{Q_2,n,2}$ and calculate its diversity product. Table 3 summarizes the results obtained for these groups. The diversity of the Quaternion groups $Q_2, Q_4$ and $Q_5$ are included for comparison.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$L$</th>
<th>$M$</th>
<th>$\zeta$</th>
<th>Group structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>8</td>
<td>2</td>
<td>0.7071</td>
<td>Quaternion group $Q_2$</td>
</tr>
<tr>
<td>2.5</td>
<td>32</td>
<td>2</td>
<td>0.1951</td>
<td>Quaternion group $Q_4$</td>
</tr>
<tr>
<td>3</td>
<td>64</td>
<td>2</td>
<td>0.0951</td>
<td>Quaternion group $Q_5$</td>
</tr>
<tr>
<td>2.5</td>
<td>32</td>
<td>2</td>
<td>0.3827</td>
<td>$S_{Q_2,2,2} = Q_2 + L_2$</td>
</tr>
<tr>
<td>3</td>
<td>64</td>
<td>2</td>
<td>0.3029</td>
<td>$S_{Q_2,3,2} = Q_2 + L_2 + L_3$</td>
</tr>
<tr>
<td>3.161</td>
<td>80</td>
<td>2</td>
<td>0.2588</td>
<td>$S_{Q_2,4,2} = Q_2 + L_2 + L_3 + L_4$</td>
</tr>
<tr>
<td>3.5</td>
<td>128</td>
<td>2</td>
<td>0.1602</td>
<td>$S_{Q_2,5,2}$</td>
</tr>
<tr>
<td>3.9037</td>
<td>224</td>
<td>2</td>
<td>0.1602</td>
<td>$S_{Q_2,6,2}$</td>
</tr>
</tbody>
</table>

Table 3: The super quaternion group and their diversity products $\zeta$, transmission rate $R$, size of the constellation $L$ and $M$ the number of antenna.

In addition, we studied structures that contain a subset of the Layers $L_1, \ldots L_n$ of the super quaternion group. Their diversity product results are:

<table>
<thead>
<tr>
<th>$R$</th>
<th>$L$</th>
<th>$M$</th>
<th>$\zeta$</th>
<th>Group structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>32</td>
<td>2</td>
<td>0.4082</td>
<td>$L_3$</td>
</tr>
<tr>
<td>2.29</td>
<td>24</td>
<td>2</td>
<td>0.5</td>
<td>$L_2$</td>
</tr>
<tr>
<td>2.79</td>
<td>48</td>
<td>2</td>
<td>0.3827</td>
<td>$Q_2 + L_2 + L_4$</td>
</tr>
</tbody>
</table>

Table 4: The diversity product for subset of layers

Some of the sets in table 4 have very good diversity in comparison to the quaternion
groups. They are also better than the results in sections 4 and 5.

Figure 3: Diversity product as a function of bit Rate when different space time codes are used

Each point in the graph 3 represents the diversity product against the bit rate for different group structure constructions. All the matrix representations are $2 \times 2$. We can see that our construction outperforms the diversity product performance of the other groups and constellations presented in graph 3.
9 Conclusions

Multiple antennas can enhance the data rate for wireless communication system without increasing the error probability. In most applications today, it is difficult to estimate the channel behavior and its feature ahead of time. We are required to design communication techniques that do not rely on channel estimates at the transmitter or receiver antennas. Space time coding satisfy these requirements. When this methodology (space time coding) is combined with the insights of group theories, it can extend the capacity of the channel. In cellular systems, for example, this can provide broadband data access to highly mobile users.

The diversity product, which was presented in this work, helps us find constellation with superior performances. Some of the structures which were studied in this work, can be simulated as real channel codes, and might be even tuned to outperform known space time codes used today. In our work we encounter some new groups which need to be analyzed for theoretic reasons. We would like to mention for example, that $S_{Q_2,\infty,2}$,

$$S_{Q_2,\infty,2} = \left\{ \frac{x_1 Q_i + x_2 Q_j + x_3 Q_k + x_4 Q_l}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}} \mid x_i \text{ are integers} \right\}, \quad (9.1)$$

the super-quaternion group which contains all the normalized integral quaternion, is an interesting group by itself (the multiplication of every two elements in $S_{Q_2,\infty,2}$ is also in $S_{Q_2,\infty,2}$), and must be explored in the future.
References


