Reasoning Inside The Box
Gentzen Calculi for Herbrand Logics

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MUGS
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Semantically speaking, logic is based on models. FO classical logic: FO structures. Finite model theory: finite FO structures. Herbrand logic: Herbrand FO structures.

Herbrand structures = FO structures with a fixed domain. The domain = the set of closed terms.

Example

<table>
<thead>
<tr>
<th>Language</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>⟨0, s⟩</td>
<td>{0, s(0), s(s(0)), s(s(s(0))), \ldots}</td>
</tr>
<tr>
<td>⟨0, 1, f⟩</td>
<td>{0, 1, f(0), f(1), f(f(1)), \ldots}</td>
</tr>
<tr>
<td>⟨c, d⟩</td>
<td>{c, d}</td>
</tr>
</tbody>
</table>
Herbrand Logic

- Herbrand logic = the logic that is induced by Herbrand structures
- Studied in [Genesereth & Kao’15]

Advantages:

- Natural sub-class of structures
- Simpler than classical FOL
- $\mathbb{N}$ is axiomatizable
- TC is axiomatizable
- MM of SSLP is axiomatizable

Disadvantages:

- Not compact
- Not R.E.
- Inherently incomplete
- $\vdash$ is language-dependent
- No proof theory so far
Main Results

Semantics:
- Decomposition of "Herbrandness"
- General treatment of equality

Proof Theory:
- Infinitary proof systems
  - Soundness
  - Completeness
- Finitary approximations
  - Soundness
  - (Completeness is impossible)
Herbrand Logics

**Semantics**

- Semi-Herbrand Structures
- Herbrand Structures
- Semi-Herbrand Structures with =
- Herbrand Structures with =

**Proof Systems**

<table>
<thead>
<tr>
<th>Infinitary Systems</th>
<th>Completeness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finitary Systems</td>
<td>Effectiveness</td>
</tr>
</tbody>
</table>
Herbrand and semi-Herbrand Structures

First Order Structures

- $M = \langle D, I \rangle$
- $D$ is a set of elements (domain)
- $I$ is an interpretation function for terms and predicates

Semi-Herbrand Structures

For every $d \in D$ there is $t \in cterms(\mathcal{L})$ s.t. $I(t) = d$

Herbrand Structures

For every $d \in D$ there is a unique $t \in cterms(\mathcal{L})$ s.t. $I(t) = d$

Herbrand Structures (equivalent definition)

- $D = cterms(\mathcal{L})$
- $I(t) = t$ for every $t \in cterms(\mathcal{L})$
### Consequence Relations

Let $\mathcal{T}$ be a set of formulas and $A$ a formula

- $\mathcal{T} \vdash_{\text{cla}} A$: every structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash_{\text{sHer}} A$: every semi-Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash_{\text{Her}} A$: every Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$

Are these relations different from one another?

---

Proposition $\vdash_{\text{cla}} \subset \vdash_{\text{sHer}} = \vdash_{\text{Her}}$

Thus, (semi-)Herbrand logic is super-classical.

Example: $\{A\{t\}_x | t \in \text{cterms}(L)\} \not\vdash_{\text{cla}} \forall x A\{t\}_x$

$\vdash_{\text{Her}} \forall x A$
Consequence Relations

Let $\mathcal{T}$ be a set of formulas and $A$ a formula

- $\mathcal{T} \vdash_{\text{cla}} A$: every structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash_{s\text{Her}} A$: every semi-Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash_{\text{Her}} A$: every Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$

Are these relations different from one another?

**Proposition**

$\vdash_{\text{cla}} \not\subseteq \vdash_{s\text{Her}} = \vdash_{\text{Her}}$

Thus, (semi-)Herbrand logic is super-classical.

**Example**

- $\{ A \{ \frac{t}{x} \} \mid t \in \text{cterms}(\mathcal{L}) \} \vdash_{\text{cla}} \forall xA$
- $\{ A \{ \frac{t}{x} \} \mid t \in \text{cterms}(\mathcal{L}) \} \vdash_{\text{Her}} \forall xA$
Herbrand Logics

**Semantics**

- Semi-Herbrand Structures
- Herbrand Structures

**Proof Systems**

- Infinitary
- Finitary

- Completeness
- Effectiveness

Semi-Herbrand Structures with $=$

Herbrand Structures with $=$
Suppose “=” is a predicate symbol of $\mathcal{L}$.

**Normal Structures**

A structure $M = \langle D, I \rangle$ is normal if $I(=)$ is $\{\langle a, a \rangle \mid a \in D\}$ ("===")

**Consequence Relations**

- $\mathcal{T} \vdash_{cla} A$: every structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash_{sHer} A$: every semi-Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash_{Her} A$: every Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$
Equality

Suppose “=” is a predicate symbol of $\mathcal{L}$.

### Normal Structures

A structure $M = \langle D, I \rangle$ is **normal** if $I(=)$ is $\{\langle a, a \rangle \mid a \in D\}$ ("===")

### Consequence Relations

- $\mathcal{T} \vdash \equiv_{\text{cla}} A$: every **normal** structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash \equiv_{\text{sHer}} A$: every **normal** semi-Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash \equiv_{\text{Her}} A$: every **normal** Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$
Suppose “=” is a predicate symbol of $\mathcal{L}$.

### Normal Structures

A structure $M = \langle D, I \rangle$ is **normal** if $I(\bar{=})$ is $\{\langle a, a \rangle \mid a \in D\}$ ("$==$")

### Consequence Relations

- $\mathcal{T} \vdash_{cla} A$: every normal structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash_{sHer} A$: every normal semi-Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$
- $\mathcal{T} \vdash_{Her} A$: every normal Herbrand structure that satisfies $\mathcal{T}$ satisfies $A$

### Without Equality

$\vdash_{cla} \not\vdash_{sHer} \vdash_{Her}$

### With Equality

$\vdash_{cla} \not\vdash_{sHer} \vdash_{Her}$

### Example

$\vdash_{Her} c \neq d$, while $\not\vdash_{sHer} c \neq d$. 

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Gentzen Calculi for Herbrand Logics
Herbrand Logics

**Semantics**

Semi-Herbrand Structures

\[ \uparrow \]

Herbrand Structures

\[ \downarrow \]

Semi-Herbrand Structures with \( = \)

\[ \downarrow \]

Herbrand Structures with \( = \)

**Proof Systems**

Infinitary

Completeness

Finitary Systems

Effectiveness

Finitary Systems
Axiomatization of Equality

Equality Formulas

- \textit{Equiv} consists of the following formulas:
  - \( x = x \)
  - \( x = y \supset y = x \)
  - \( (x = y \land y = z) \supset x = z \)

- \( \text{Con}(\mathcal{L}) \) consists of the following formulas:
  - \( (x_1 = y_1 \land \ldots \land x_n = y_n) \supset (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)) \)
  - \( (x_1 = y_1 \land \ldots \land x_n = y_n) \supset (P(x_1, \ldots, x_n) \supset P(y_1, \ldots, y_n)) \)

Fact

\( \vdash_{=\text{cla}} \approx \vdash_{\text{cla}} \)

Question

Does the same hold for \textit{semi-Herbrand logic}? \textit{Herbrand logic}?

Yes. No.
### Axiomatization of Equality

#### Equality Formulas
- **Equiv** consists of the following formulas:
  - \( x = x \)
  - \( x = y \supset y = x \)
  - \( (x = y \land y = z) \supset x = z \)
- **Con(Ł)** consists of the following formulas:
  - \( (x_1 = y_1 \land \ldots \land x_n = y_n) \supset (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)) \)
  - \( (x_1 = y_1 \land \ldots \land x_n = y_n) \supset (P(x_1, \ldots, x_n) \supset P(y_1, \ldots, y_n)) \)

#### Fact
\[
\mathcal{T} \vdash_{cla} A \quad \text{iff} \quad \mathcal{T}, \text{Equiv}, \text{Con(Ł)} \vdash_{cla} A
\]

#### Question
Does the same hold for semi-Herbrand logic? Herbrand logic?

Yes. No.
Equality in Herbrand Logics

**semi-Herbrand Logic**

\[ \mathcal{T} \vdash_{sHer} A \iff \mathcal{T}, \text{Equiv}, \text{Con}(\mathcal{L}) \vdash_{sHer} A \]

**Herbrand Logic**

\[ \mathcal{T} \vdash_{cla} A \iff \mathcal{T}, \text{Equiv}, \text{Con}(\mathcal{L}) \vdash_{cla} A \]

For Herbrand structures, **inequalities** are also needed (surprise surprise?)
Equality in Herbrand Logics

**semi-Herbrand Logic**

\[ \mathcal{T} \vdash_{s\text{Her}} A \iff \mathcal{T}, \text{Equiv}, \text{Con}(\mathcal{L}) \vdash_{s\text{Her}} A \]

**Herbrand Logic**

\[ \mathcal{T} \vdash_{\text{cla}} A \iff \mathcal{T}, \text{Equiv}, \text{Con}(\mathcal{L}) \vdash_{\text{cla}} A \]

**Inequality Formulas**

\( \text{inEq}(\mathcal{L}) \) consists of the following formulas:

- \( f(x_1, \ldots, y_n) \neq g(y_1, \ldots, y_n) \) for every distinct \( f \) and \( g \)
- \( x_i \neq y_i \supset f(\ldots, x_i, \ldots) \neq f(\ldots, y_i, \ldots) \)

**Herbrand Logic**

\[ \mathcal{T} \vdash_{\text{Her}} A \iff \mathcal{T}, \text{Equiv}, \text{Con}(\mathcal{L}), \text{inEq}(\mathcal{L}) \vdash_{\text{Her}} A \]

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Gentzen Calculi for Herbrand Logics
Equality in Herbrand Logics

**semi-Herbrand Logic**

\[ \mathcal{T} \vdash_{sHer} \ A \quad \text{iff} \quad \mathcal{T}, \ \text{Equiv}, \ \text{Con}(\mathcal{L}) \vdash_{sHer} \ A \]

**Herbrand Logic**

\[ \mathcal{T} \vdash_{cla} \ A \quad \text{iff} \quad \mathcal{T}, \ \text{Equiv}, \ \text{Con}(\mathcal{L}) \vdash_{cla} \ A \]

**Inequality Formulas**

\( \text{inEq}(\mathcal{L}) \) consists of the following formulas:

- \( f(x_1, \ldots, y_n) \neq g(y_1, \ldots, y_n) \) for every distinct \( f \) and \( g \)
- \( x_i \neq y_i \supset f(\ldots, x_i, \ldots) \neq f(\ldots, y_i, \ldots) \)

**Herbrand Logic**

\[ \mathcal{T} \vdash_{Her} \ A \quad \text{iff} \quad \mathcal{T}, \ x = x, \ \text{inEq}(\mathcal{L}) \vdash_{Her} \ A \]
Example: Natural Numbers

Inequality Formulas

\( inEq(\mathcal{L}) \) consists of the following formulas:

- \( f(x_1, \ldots, y_n) \neq g(y_1, \ldots, y_n) \) for every distinct \( f \) and \( g \)
- \( x_i \neq y_i \supset f(\ldots, x_i, \ldots) \neq f(\ldots, y_i, \ldots) \)

Herbrand Logic

\( \mathcal{T} \models_{Her} A \) iff \( \mathcal{T}, x = x, inEq(\mathcal{L}) \models_{Her} A \)

Example (Relational PA without Induction)

- \( \forall x. equal(x, x) \)
- \( \forall x. (\neg equal(0, s(x)) \land \neg equal(s(x), 0)) \)
- \( \forall x. \forall y. (\neg equal(x, y) \Rightarrow \neg equal(s(x), s(y))) \)
Example: Natural Numbers

Inequality Formulas

\( inEq(\mathcal{L}) \) consists of the following formulas:
- \( f(x_1, \ldots, y_n) \neq g(y_1, \ldots, y_n) \) for every distinct \( f \) and \( g \)
- \( x_i \neq y_i \supset f(\ldots, x_i, \ldots) \neq f(\ldots, y_i, \ldots) \)

Herbrand Logic

\( T \models_{\text{Her}} A \iff T, x = x, inEq(\mathcal{L}) \models_{\text{Her}} A \)

Example (Relational PA without Induction)

\[
\forall x.\text{equal}(x,x) \\
\forall x. (\neg \text{equal}(0,s(x)) \land \neg \text{equal}(s(x),0)) \\
\forall x. \forall y. (\neg \text{equal}(x,y) \Rightarrow \neg \text{equal}(s(x),s(y)))
\]

The induction scheme is **valid** in Herbrand structures!
Herbrand Logics

**Semantics**

- Semi-Herbrand Structures
- Herbrand Structures
- Semi-Herbrand Structures with =
- Herbrand Structures with =

**Proof Systems**

- Infinitary
- Completeness
- Effectiveness
- Finitary Systems
- Finitary Systems
Herbrand Logics

**Semantics**

Thinking inside the box

No nameless elements

**Proof Theory**

Reasoning inside the box

No free variables

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Gentzen Calculi for Herbrand Logics
What Are Sequents?

- Sequents have the form $\Gamma \Rightarrow \Delta$, where $\Gamma$ and $\Delta$ are finite sets of formulas.
- Intuition:

$$\begin{align*}
A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m & \iff A_1 \land \cdots \land A_n \rightarrow B_1 \lor \cdots \lor B_m
\end{align*}$$

- Special instance 1: $\Delta$ has one element: $\Gamma \Rightarrow A$
- Special instance 2: $\Gamma$ is empty: $\Rightarrow A$

Example

- $A, B \Rightarrow A \land B$
- $A \Rightarrow A \lor B$
- $\Rightarrow A \lor \neg A$
- $A, \neg A \Rightarrow$
- $\Rightarrow A, \neg A$
- $A \Rightarrow A, B, C$
Sequent Calculus for Classical Logic \textbf{LK}

\(\Gamma\) and \(\Delta\) are finite sets of formulas

Proof trees are finite

\[(id)\]
\[\frac{\Gamma, A \Rightarrow A, \Delta}{\Gamma, A \Rightarrow A, \Delta}\]

\[(\text{cut})\]
\[\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\]

\[(W \Rightarrow)\]
\[\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\]

\[(\Rightarrow W)\]
\[\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\]

\[\neg \Rightarrow \]
\[\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}\]

\[\neg \Rightarrow \]
\[\frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \neg A, \Delta}\]

\[\wedge \Rightarrow \]
\[\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\]

\[\Rightarrow \wedge \]
\[\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\]

\[\vee \Rightarrow \]
\[\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\]

\[\Rightarrow \vee \]
\[\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\]

\[\supset \Rightarrow \]
\[\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}\]

\[\Rightarrow \supset \]
\[\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}\]

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Gentzen Calculi for Herbrand Logics
Sequent Calculus for Classical Logic \( \text{LK} \)

\( \Gamma \) and \( \Delta \) are finite sets of formulas

Proof trees are finite

\[
\begin{align*}
(\forall \Rightarrow) & \quad \frac{\Gamma, A \{ \frac{t}{x} \} \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta} \\
(\Rightarrow \forall) & \quad \frac{\Gamma \Rightarrow A \{ \frac{y}{x} \}, \Delta}{\Gamma \Rightarrow \forall x A, \Delta \quad \text{y is fresh}} \\
(\exists \Rightarrow) & \quad \frac{\Gamma, A \{ \frac{y}{x} \} \Rightarrow \Delta \quad \text{y is fresh}}{\Gamma \Rightarrow \exists x A, \Delta} \\
(\Rightarrow \exists) & \quad \frac{\Gamma \Rightarrow A \{ \frac{t}{x} \}, \Delta}{\Gamma \Rightarrow \exists x A, \Delta}
\end{align*}
\]
Sequent Calculus for Herbrand Logic $G_{\text{Her}}$

$\Gamma$ and $\Delta$ are finite sets of closed formulas

Proof trees are of finite height

\[
\Gamma, A \{ \frac{t}{x} \} \Rightarrow \Delta \\
\Gamma, \forall x A \Rightarrow \Delta
\]
t is closed

\[
\Gamma \Rightarrow A \{ \frac{y}{x} \}, \Delta \\
\Gamma \Rightarrow \forall x A, \Delta
\]
y is fresh

\[
\Gamma, A \{ \frac{y}{x} \} \Rightarrow \Delta \\
\Gamma, \forall x A \Rightarrow \Delta
\]
y is fresh

\[
\Gamma \Rightarrow A \{ \frac{t}{x} \}, \Delta \\
\Gamma \Rightarrow \exists x A, \Delta
\]
t is closed

\[
\Gamma \Rightarrow A \{ \frac{t}{x} \}, \Delta \\
\Gamma \Rightarrow \forall x A, \Delta
\]
\[
\Gamma \Rightarrow \exists x A \Rightarrow \Delta
\]
\[
\Gamma, \forall x A \Rightarrow \Delta
\]
\[
\Gamma, \exists x A \Rightarrow \Delta
\]

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Gentzen Calculi for Herbrand Logics
Intuition Behind $G_{\text{Her}}$

\[
\frac{\{\Gamma \Rightarrow A \{\frac{t}{x}\}, \Delta \mid t \in cterms(\mathcal{L})\}}{\Gamma \Rightarrow \forall x A, \Delta}
\]

We have seen that:

- \{A \{\frac{t}{x}\} \mid t \in cterms(\mathcal{L})\} \vdash_{\text{cla}} \forall x A
- \{A \{\frac{t}{x}\} \mid t \in cterms(\mathcal{L})\} \vdash_{\text{Her}} \forall x A

This is known as the \textit{\(\omega\)-rule}.

It suffices to add this rule to characterize Herbrand semantics.
We have seen that:

\[ \{ \Rightarrow A \{ \frac{t}{x} \} \mid t \in cterms(\mathcal{L}) \} \Rightarrow \forall x A \]

This is known as the \( \omega \)-rule.

It suffices to add this rule to characterize Herbrand semantics.
Intuition Behind $G_{\text{Her}}$

$$\{ \Rightarrow A \{ \frac{t}{x} \} \mid t \in cterms(\mathcal{L}) \} \Rightarrow \forall x A$$

**ω-rule**

- We have seen that:
  - $\{ A \{ \frac{t}{x} \} \mid t \in cterms(\mathcal{L}) \} \vdash_{\text{cl}a} \forall x A$
  - $\{ A \{ \frac{t}{x} \} \mid t \in cterms(\mathcal{L}) \} \vdash_{\text{Her}} \forall x A$

- This is known as the $\omega$-rule
- It suffices to add this rule to characterize Herbrand semantics

**Theorem (soundness and completeness)**

$\mathcal{T} \vdash_{\text{Her}} A$ iff the sequent $\mathcal{T} \Rightarrow A$ is derivable in $G_{\text{Her}}$.

**Bonus**

The system is also complete for $\vdash_{s\text{Her}}$. 

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Gentzen Calculi for Herbrand Logics
**Herbrand Logics**

### Semantics

- Semi-Herbrand Structures
- Herbrand Structures
- Semi-Herbrand Structures with $=\quad$
- Herbrand Structures with $=$

### Proof Systems

- Infinitary $G_{\text{Her}}$
- Finitary
Rules for Equality: Systems $G_{sHer_\equiv}$ and $G_{Her_\equiv}$

- $G_{sHer_\equiv} = G_{Her} + (\Rightarrow=) + (paramodulation)$
- $G_{Her_\equiv} = G_{Her} + (\Rightarrow=) + (\Rightarrow\Rightarrow)$

(paramodulation) \[ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, s = t \Rightarrow, A', \Delta} \quad s, t \in cterms(\mathcal{L}) \]

(A’ is obtained from A by replacing s by t)

(\Rightarrow=) \[ \frac{\Gamma \Rightarrow t = t, \Delta}{\Gamma \Rightarrow} \quad t \in cterms(\mathcal{L}) \]

(\Rightarrow\Rightarrow) \[ \frac{\Gamma, s = t \Rightarrow \Delta}{\Gamma, s \neq t \not\in cterms(\mathcal{L})} \]

Theorem (soundness and completeness)

- $\mathcal{T} \vdash_{sHer} A$ iff the sequent $\mathcal{T} \Rightarrow A$ is derivable in $G_{sHer_\equiv}$.
- $\mathcal{T} \vdash_{Her} A$ iff the sequent $\mathcal{T} \Rightarrow A$ is derivable in $G_{Her_\equiv}$. 
Herbrand Logics

**Semantics**

Semi-Herbrand Structures

Herbrand Structures

Semi-Herbrand Structures with $=$

Herbrand Structures with $=$

**Proof Systems**

Infinitary

$G_{\text{Her}}$

Finitary

$G_{\text{sHer}=}$

$G_{\text{Her}=}$
Herbrand Logics

**Semantics**

- Thinking inside the box
- No nameless elements

**Proof Theory**

- Reasoning inside the box
- No free variables

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Gentzen Calculi for Herbrand Logics
Herbrand Logics

**Semantics**

*Thinking inside the box*

*No nameless elements*

**Proof Theory**

*Peeking outside the box*

*free variables*
Γ and Δ are finite sets of closed formulas.

Proof trees are of finite height.

Sequent Calculus for Herbrand Logic $G_{\text{Her}}$

$\forall \Rightarrow$

- \[
\Gamma, A \{ \frac{t}{x} \}, \Delta \Rightarrow \Delta \quad \text{t is closed}
\]
- \[
\Gamma, \forall x A \Rightarrow \Delta
\]

$\forall \Rightarrow_H$

- \[
\Gamma \Rightarrow A \{ \frac{y}{x} \}, \Delta \quad y \text{ is fresh}
\]
- \[
\Gamma \Rightarrow \forall x A, \Delta
\]

$\exists \Rightarrow$

- \[
\Gamma, A \{ \frac{y}{x} \} \Rightarrow \Delta \quad y \text{ is fresh}
\]
- \[
\Gamma, \exists x A \Rightarrow \Delta
\]

$\exists \Rightarrow_H$

- \[
\Gamma \Rightarrow A \{ \frac{t}{x} \}, \Delta \quad t \in \text{cterms}(\mathcal{L})
\]
- \[
\Gamma \Rightarrow \forall x A, \Delta
\]

$\Rightarrow \forall$

- \[
\Gamma \Rightarrow A \{ \frac{t}{x} \}, \Delta \quad t \text{ is closed}
\]
- \[
\Gamma \Rightarrow \exists x A, \Delta
\]
Sequent Calculus for Herbrand Logic $G^{\text{IND}}_{\text{Her}}$

\[ \Gamma \text{ and } \Delta \text{ are finite sets of formulas} \]

\[ \text{Proof trees are finite} \]

\[ \begin{align*}
(\forall \Rightarrow) & \quad \Gamma, A \{ \frac{t}{x} \} \Rightarrow \Delta \\
& \quad \Gamma, \forall x A \Rightarrow \Delta \\
\quad & \quad \Gamma \Rightarrow A \{ \frac{y}{x} \}, \Delta \quad \text{y is fresh} \\
& \quad \Gamma \Rightarrow \forall x A, \Delta \\
(\exists \Rightarrow) & \quad \Gamma, A \{ \frac{y}{x} \} \Rightarrow \Delta \\
& \quad \Gamma, \exists x A \Rightarrow \Delta \\
\quad & \quad \Gamma \Rightarrow A \{ \frac{t}{x} \}, \Delta \\
& \quad \Gamma \Rightarrow \exists x A, \Delta \quad (\Rightarrow \forall)_{\text{IND}} \text{ induction rule} \\
\quad & \quad (\Rightarrow \exists)_{\text{IND}} \text{ induction rule} \\
\end{align*} \]
Effective Systems: \( \forall \)

\[
(\Rightarrow \forall)_H \quad \frac{\{ \Rightarrow A \{ \frac{t}{x} \} \mid t \in cterms(\mathcal{L}) \}}{\Rightarrow \forall xA}
\]

\[\Rightarrow \]

\[
(\Rightarrow \forall)_{IND} \quad \frac{\{ A \{ \frac{x}{x} \} \Rightarrow A \{ \frac{f(x)}{x} \} \mid f \in func(\mathcal{L}) \}}{\Rightarrow \forall xA}
\]

Example

\( func(\mathcal{L}) = \{0, s\} \)

\[
\Rightarrow A \{ \frac{0}{x} \} \quad A \{ \frac{x}{x} \} \Rightarrow A \{ \frac{s(x)}{x} \}
\]

\[\Rightarrow \forall xA\]
Effective Systems: \( \forall \)

\[
(\Rightarrow \forall)_H \quad \frac{\Rightarrow \forall A \{ \frac{t}{x} \} \mid t \in cterms(\mathcal{L})}{\Rightarrow \forall xA}
\]

\[
(\Rightarrow \forall)_{IND} \quad \frac{\{ A \{ \frac{x_1}{x} \}, \ldots, A \{ \frac{x_n}{x} \} \Rightarrow A \{ \frac{f(x_1, \ldots, x_n)}{x} \} \mid f \in \text{func}(\mathcal{L}) \}}{\Rightarrow \forall xA}
\]

**Example**

\( func(\mathcal{L}) = \{0, s\} \)

\[
\Rightarrow A \{ \frac{0}{x} \} \quad A \{ \frac{x}{x} \} \Rightarrow A \{ \frac{s(x)}{x} \}
\]

\[
\Rightarrow \forall xA
\]
\[(\Rightarrow=) \quad \frac{\Gamma \Rightarrow t = t, \Delta}{t \in \text{cterms}(\mathcal{L})} \quad \xrightarrow{\sim} \quad (\Rightarrow=)_{\text{IND}} \quad \frac{\Gamma \Rightarrow x = x, \Delta}{\Gamma \Rightarrow x = x, \Delta} \]

\[(\text{paramodulation}) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, s = t \Rightarrow, A', \Delta} \quad s, t \in \text{cterms}(\mathcal{L})\]
Effective Systems: equality (2)

\[(\Rightarrow) \quad \Gamma, s = t \Rightarrow \Delta \quad s \neq t \in cterms(\mathcal{L}) \]

\[\sim \sim \Rightarrow\]

\[(\Rightarrow)_1 \quad \Gamma, f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m) \Rightarrow \Delta \quad f \neq g \in func(\mathcal{L})\]

\[(\Rightarrow)_2 \quad \Gamma, x_i = y_i \Rightarrow \Delta \quad \Gamma, f(\ldots, x_i, \ldots) = f(\ldots, y_i, \ldots) \Rightarrow \Delta\]
(⇒⇒) \[ \frac{s = t}{s \neq t \in cterms(\mathcal{L})} \]

(⇒⇒) \[\frac{f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)}{f \neq g \in func(\mathcal{L})} \]

(⇒⇒) \[\frac{x_i = y_i}{f(\ldots, x_i, \ldots) = f(\ldots, y_i, \ldots)} \]
Effective Systems: equality (2)

\[(\Rightarrow) \quad \frac{s = t}{s \neq t \in cterms(\mathcal{L})}\]

\[\sim\sim\]

\[(\Rightarrow)_1 \quad \frac{f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)}{f \neq g \in func(\mathcal{L})}\]

\[(\Rightarrow)_2 \quad \frac{x_i = y_i}{f(\ldots, x_i, \ldots) = f(\ldots, y_i, \ldots)}\]

Example

\[func(\mathcal{L}) = \{0, s\}\]

\[(\Rightarrow)_1 \quad \frac{0 = s(x)}{}\]
Effective Systems: equality (2)

\[
(=\Rightarrow) \quad \frac{s = t}{s \neq t \in cterms(\mathcal{L})}
\]

\[
\sim\sim\sim
\]

\[
(=\Rightarrow)_1 \quad \frac{f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)}{f \neq g \in func(\mathcal{L})}
\]

\[
(=\Rightarrow)_2 \quad \frac{x_i = y_i}{f(\ldots, x_i, \ldots) = f(\ldots, y_i, \ldots)}
\]

**Example**

\[func(\mathcal{L}) = \{0, s\}\]

\[
(=\Rightarrow)_2 \quad \frac{x = y}{s(x) = s(y)}
\]
Soundness Theorem

- \( G_{\text{Her}} \)
- \( G_{\text{shHer}} = G_{\text{Her}} + (\Rightarrow=)_{\text{IND}} + (\text{paramodulation}) \)
- \( G_{\text{Her}} = G_{\text{shHer}} + (\Rightarrow_1 + (\Rightarrow_2) \)

**Theorem (soundness)**

- \( \mathcal{T} \vdash_{\text{Her}} A \) whenever the sequent \( \mathcal{T} \Rightarrow A \) is derivable in \( G_{\text{Her}} \).
- \( \mathcal{T} \vdash_{\text{shHer}} A \) whenever the sequent \( \mathcal{T} \Rightarrow A \) is derivable in \( G_{\text{shHer}} \).
- \( \mathcal{T} \vdash_{\text{Her}} A \) whenever the sequent \( \mathcal{T} \Rightarrow A \) is derivable in \( G_{\text{Her}} \).

**Properties**

- \( G_{\text{Her}}, G_{\text{shHer}} \) and \( G_{\text{Her}} \) are finite whenever \( \text{func}(\mathcal{L}) \) is
- Strictly stronger than \( \text{LK} \)
- Full \( (\Rightarrow) \) is derivable in \( G_{\text{Her}} \) using \( (\Rightarrow_1) \) and \( (\Rightarrow_2) \)
Herbrand Logics

**Semantics**

- Semi-Herbrand Structures
- Herbrand Structures
- Semi-Herbrand Structures with $=$
- Herbrand Structures with $=$

**Proof Systems**

- Infinitary
  - $G_{\text{Her}}$
  - $G_{\text{IND}_{\text{Her}}}$
- Finitary
  - $G_{\text{Her}_=}$
  - $G_{\text{IND}_{\text{Her}_=}}$
  - $G_{\text{sHer}_=}$
  - $G_{\text{IND}_{\text{sHer}_=}}$
Intuitionistic Herbrand Logic

- Herbrand logic carries **definitional** content
- Intuitionistic logic carries **computational** content
- Can the two be combined?

- An idea: Kripke models with Herbrand structures in each world
- Problem:
  - Domains are allowed to expand in intuitionistic logic
  - Domains are fixed in Herbrand logic
- Possible Resolutions:
  - Settle with CD (compromising intuitionism)
  - Semi-Herbrand structures (compromising Herbrandness)
  - A middle ground: Each world has a different language
Transitive Closure (TC)

- TC of a binary relation $R$ is the minimal transitive relation that contains $R$
- TC Logic = FOL + TC Operator
- TC Logic is stronger than FOL and weaker than SOL

Example

<table>
<thead>
<tr>
<th>Relation</th>
<th>TC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Successor</td>
<td>$&lt;$</td>
</tr>
<tr>
<td>Edge</td>
<td>Path</td>
</tr>
<tr>
<td>Parent</td>
<td>Predecessor</td>
</tr>
<tr>
<td>Tomorrow</td>
<td>Future</td>
</tr>
</tbody>
</table>

Perhaps $TC \approx Herbrand$?
Herbrand Logic and The Transitive Closure

Herbrand Logic can express TC

Definition of helper relation $qq$:

$$\forall x.\forall z.(qq(x,z,0) \iff p(x,z))$$
$$\forall x.\forall z.(qq(x,z,s(n)) \iff qq(x,z,n) \lor \exists y.(qq(x,y,n) \land qq(y,z,n)))$$

Definition of $q$ in terms of $qq$:

$$\forall x.\forall z.(q(x,z) \iff \exists n.qq(x,z,n))$$

But what about TC Operator?

TC can express “Herbrandness”

$$\forall w. \bigvee_{c \in \text{const}(\mathcal{L})} \left( \left( \bigvee_{f \in \text{func}(\mathcal{L})} y = f(x) \right)^* (c, w) \right)$$

But what about general functions?

Yoni Zohar  Gentzen Calculi for Herbrand Logics
Conclusions

We have seen:

- A modular definition of Herbrand semantics
- Infinitary proof systems
- Finitary approximations

Future work:

- Applications
- Proof theoretical properties
- Semantics for the effective systems
- Transitive Closure
Conclusions

We have seen:

- A **modular** definition of Herbrand semantics
- Infinitary proof systems
- Finitary approximations

Future work:

- Applications
- Proof theoretical properties
- Semantics for the effective systems
- Transitive Closure

Thank you!
Proof Systems

- Infinitary proof systems
  - Consistency Proofs [Schutte’51, Pohlers’09]
  - Basis for practical subsystems [Feferman’62, Baker’93]
- Finitary approximations
  - Sound proof search
  - Algorithm for above applications