Extensions of Analytic Pure Sequent Calculi with Modal Operators

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(joint work with Ori Lahav)

GeTFun 4.0

Motivation

C_1 [Avron, Konikowska, Zamansky '12]

Positive rules of **LK** + the following rules:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma, \neg(A \wedge \neg A) \Rightarrow \Delta} \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, B, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad \frac{\Gamma, A, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta}$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta} \qquad \frac{\Gamma, A, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta}$$

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$$(\mathrm{K}) \quad \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$$

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$$(S5): \quad (\mathrm{B4})\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow A, \Box\Delta_1, \Box\Delta_2}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box A, \Box\Delta_1, \Delta_2} \quad + \quad (\mathrm{T})\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}$$

Analytic Pure Sequent Calculi

Analyticity

- A calculus is *analytic* if $\vdash \Gamma \Rightarrow \Delta$ implies that there is a derivation of $\Gamma \Rightarrow \Delta$ using only subformulas of $\Gamma \cup \Delta$.
- May be based on "liberal" definitions of subformulas (e.g. usual subformulas and their negations).
- If a pure calculus is analytic then it is decidable.

Pure Sequent Calculi Calculi

- Sequents: objects of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets.
- *Pure sequent calculi*: propositional sequent calculi that include all usual structural rules, and a finite set of pure logical rules.
- *Pure logical rules*: allow any context [Avron '91].

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \quad \text{but not} \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Analytic Pure Sequent Calculi

- Prominent proof theoretic framework.
- Suitable for many logics.
- For example:
  - Classical Logic
  - Three-valued logics
  - Four-valued logics
  - Paraconsistent logics (e.g. C_1)
  - Primal infon logic
  - Dolev-Yao Intruder model

Our Question

Given an arbitrary analytic pure calculus, will it stay analytic after adding:

Modal Rules

$$(K)\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \quad (4)\frac{\Box\Gamma_1, \Gamma_2 \Rightarrow A}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box A} \quad (45)\frac{\Box\Gamma_1, \Gamma_2 \Rightarrow A, \Box\Delta}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box A, \Box\Delta}$$

$$(B)\frac{\Gamma \Rightarrow A, \Box\Delta}{\Box\Gamma \Rightarrow \Box A, \Delta} \quad (B4)\frac{\Box\Gamma_1, \Gamma_2 \Rightarrow A, \Box\Delta_1, \Box\Delta_2}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box A, \Box\Delta_1, \Delta_2} \quad (alt1)\frac{\Gamma \Rightarrow A, \Delta}{\Box\Gamma \Rightarrow \Box A, \Box\Delta}$$

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T Rule

$$(T)\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \qquad S4 = K + 4 + T \qquad S5 = K + B4 + T$$

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$$(T)\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \qquad S4 = \mathrm{K} + 4 + \mathrm{T} \qquad S5 = \mathrm{K} + \mathrm{B4} + \mathrm{T}$$

D Rules

$$(D_K)\frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow} \quad (D_4)\frac{\Box\Gamma_1, \Gamma_2 \Rightarrow}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow} \quad (D_{45})\frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\Delta}$$

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General Notations

In what follows:

- $\mathcal{L}$ is an arbitrary propositional language, not containing $\Box$.
- $\mathcal{L}_{\Box} = \mathcal{L} \cup \{\Box\}$.
- For every pure calculus G for $\mathcal{L}$ and every modal rule X from above, G_X denotes the addition of X to G.

Analyticity Survives

Main Theorem

Let G be a pure calculus. If G is analytic, then so is G_X.

- Holds for the generalized notions of analyticity (e.g. C_1)
- Valid for multi-modal logics (e.g. Primal infon logic)

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Example: Classical modal logics

This theorem provides a new and short proof for the analyticity of the classical modal logics above, by deriving it from the analyticity of **LK**.

Analyticity Survives

Main Theorem

Let G be a pure calculus. If G is analytic, then so is G_X.

Example: Primal Infon Logic with quotations

- "*said*" operators are indispensable for applications.
- Each principle q has an operator "q *said*".

$(\wedge \Rightarrow)$ $\dfrac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$ $(\Rightarrow \wedge)$ $\dfrac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$

$(\vee \Rightarrow)$ *none* $(\Rightarrow \vee)$ $\dfrac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$

$(\supset \Rightarrow)$ $\dfrac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$ $(\Rightarrow \supset)$ $\dfrac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$

$$\frac{\Gamma \Rightarrow \Delta}{q \ said \ \Gamma \Rightarrow q \ said \ \Delta}$$ for every principal q

Analyticity Survives

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Example: C_1 with necessity

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$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$$

Proof

Main Theorem

Let G be a pure calculus. If G is analytic, then so is G_X.

- The proof of the theorem does not constructively take a proof of a sequent and transforms it to an analytic proof.
- Instead, it goes through semantics.
- This detour provides further insights.

Semantics for Pure Calculi

The Semantic Framework

- Pure calculi correspond to *two-valued valuations* [Béziau '01].
- Each pure rule is read as a semantic condition.
- By joining the semantic conditions of all rules in a calculus G, we obtain the set of *G-legal* valuations.

Soundness and Completeness

The sequent $\Gamma \Rightarrow \Delta$ is provable in G iff every G-legal valuation is a model of $\Gamma \Rightarrow \Delta$.

Example (Sequent Calculus for $\mathbf{C_1}$)

Corresponding semantic conditions for $\dfrac{A \Rightarrow}{\Rightarrow \neg A}$ $\dfrac{A \Rightarrow}{\neg\neg A \Rightarrow}$

1. If $v(A) = \mathrm{F}$ then $v(\neg A) = \mathrm{T}$
2. If $v(A) = \mathrm{F}$ then $v(\neg\neg A) = \mathrm{F}$

This semantics is non-deterministic.

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The sequent $\Gamma \Rightarrow \Delta$ is provable in G using only formulas of $\mathcal{F}$ iff every G-legal valuation whose domain is $\mathcal{F}$ is a model of $\Gamma \Rightarrow \Delta$.

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Semantic Meaning of Analyticity

Soundness and Completeness

The sequent $\Gamma \Rightarrow \Delta$ is provable in G using only formulas of $\mathcal{F}$ iff every G-legal valuation whose domain is $\mathcal{F}$ is a model of $\Gamma \Rightarrow \Delta$.

Corollary

G is analytic iff every G-legal partial valuation whose domain is closed under subformulas can be extended to a full G-legal valuation.

Example

Consider the rules $\frac{\Rightarrow A}{\neg A \Rightarrow}$ and $\frac{\Rightarrow A}{\Rightarrow \neg A}$.

The partial valuation given by $\mathbf{v}(\mathbf{p}) = \mathrm{T}$ cannot be extended.

Semantics for Pure Calculi with Modal Operators

A Kripke model is a triple $\langle W, R, \mathcal{V}\rangle$:

- W is a set of states (possible worlds).
- R is a relation over W.
- $\mathcal{V}$ assigns a valuation $\mathcal{V}_w : Frm_{\mathcal{L}} \to \{\mathrm{F}, \mathrm{T}\}$ to every $w \in W$, s.t.:
$$\mathcal{V}_w(\Box A) = \mathrm{T} \text{ iff } \mathcal{V}_{w'}(A) = \mathrm{T} \text{ for every } wRw'$$

A Kripke model is

- "G-legal" if $\mathcal{V}_w$ is G-legal for every $w \in W$.
- "4" if R is transitive, "5" if R is euclidian, "B" if R is symmetric, "*alt*1" if R is a partial function.
- "T" if R is reflexive and "D" if R is serial.

Extension of Completeness to Modal Logics

Soundness and Completeness

Let G be a pure sequent calculus and X one of the combinations of modal rules discussed. The sequent $\Gamma \Rightarrow \Delta$ is provable in G_X iff every G-legal X-Kripke model satisfies $\Gamma \Rightarrow \Delta$.

Corollary

G_X is analytic iff every partial G-legal X Kripke model whose domain is closed under subformulas can be extended to a full G-legal X Kripke model.

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(Semantic) Analyticity

G is analytic

↕

Every partial G-legal valuation can be extended

Every partial G-legal X Kripke model can be extended

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G_X is analytic

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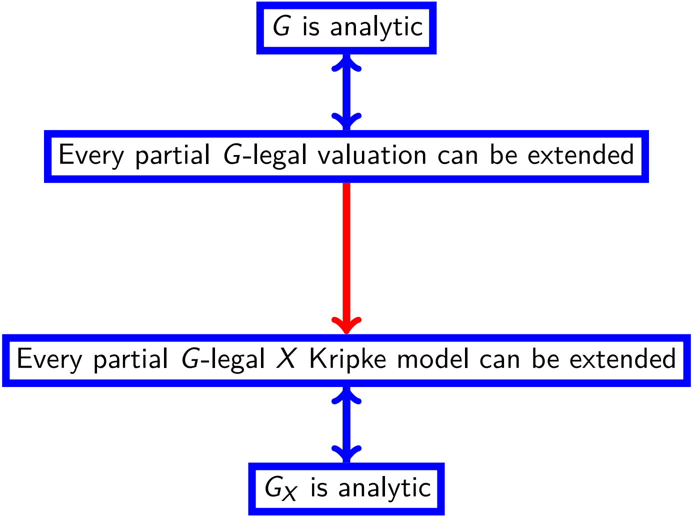

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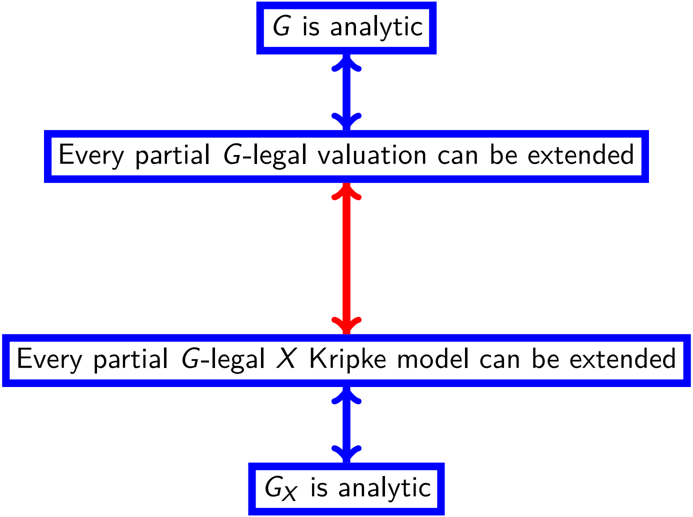

Extending Kripke Models

- Suppose every partial G-legal valuation can be extended.
- Let $\mathcal{W} = \langle W, R, \mathcal{V} \rangle$ be a partial G-legal Kripke model.
- We extend $\mathcal{W}$ incrementally:

$$p, \ldots, \varphi, \Box\varphi, \ldots, \varphi_1, \ldots, \varphi_n, \sharp(\varphi_1, \ldots, \varphi_n) \ldots$$

- For atoms: add p to all worlds and assign T to it.
- For every φ in the domain, add $\Box\varphi$ to all worlds, and assign it in each world the only value it can get.
- For an n-ary connective $\sharp$ and $\varphi_1, \ldots, \varphi_n$ in the domain, add $\sharp(\varphi_1, \ldots, \varphi_n)$ to all worlds.
- With what value? Use analyticity of G.
- But G is only for $\mathcal{L}$, not for $\mathcal{L}_\Box$.

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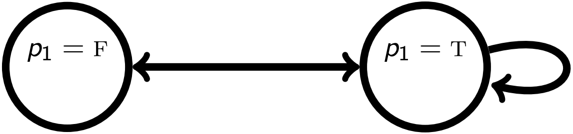
Getting rid of the boxes

For every $w \in W$:

- Take some bijection $\alpha : At \cup \{\Box\varphi \mid \varphi \in \mathcal{L}\} \to At$.
- Extend it to $\alpha : \mathcal{L}_{\Box} \to \mathcal{L}$.
- Define a partial valuation in $\mathcal{L}$ according to α and $\mathcal{V}_w$.
- Extend it (you can - G is analytic)!
- Assign the value given for $\alpha(\sharp(\varphi_1, \ldots, \varphi_n))$ to $\sharp(\varphi_1, \ldots, \varphi_n)$.

Example



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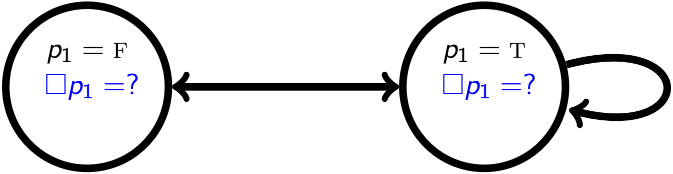

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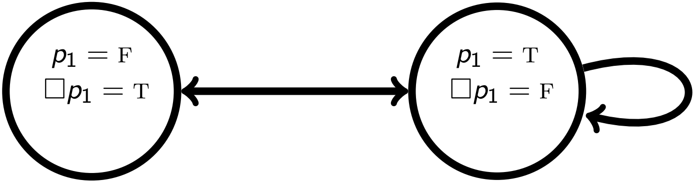

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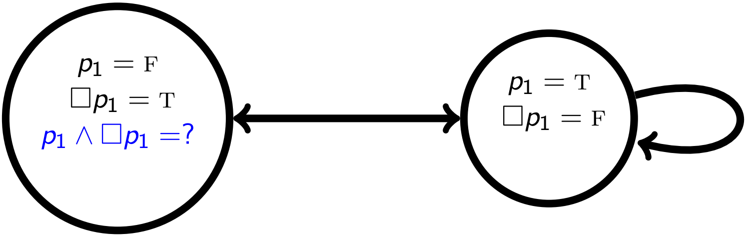
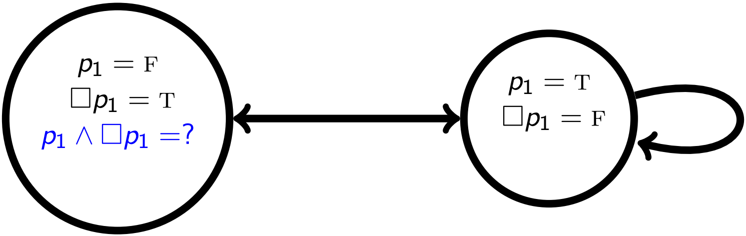


$p_1 = \mathrm{F}$
$\Box p_1 = \mathrm{T}$
$p_1 \wedge \Box p_1 =?$
$p_1 = \mathrm{T}$
$\Box p_1 = \mathrm{F}$

Example



$$\vdots$$

$$p_1 \mapsto p_{17}$$

$$\vdots$$

$$\Box p_1 \mapsto p_{25}$$

$$\vdots$$

$$\{p_{17} = \mathrm{F}, p_{25} = \mathrm{T}, p_{17} \wedge p_{25} = ?\}$$

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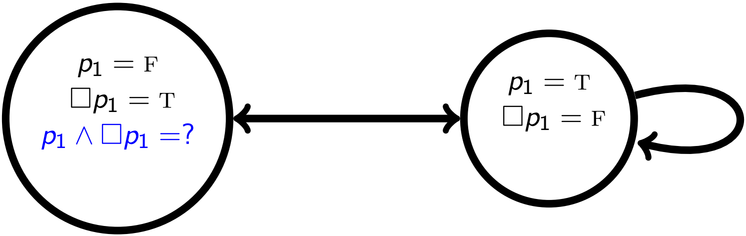

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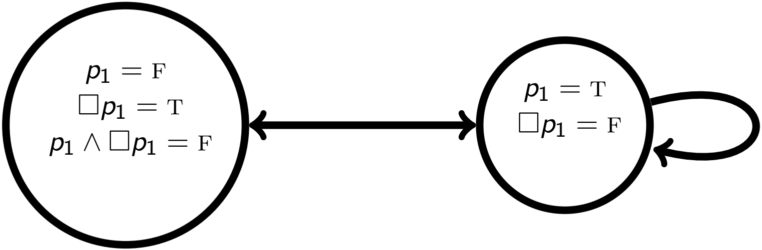

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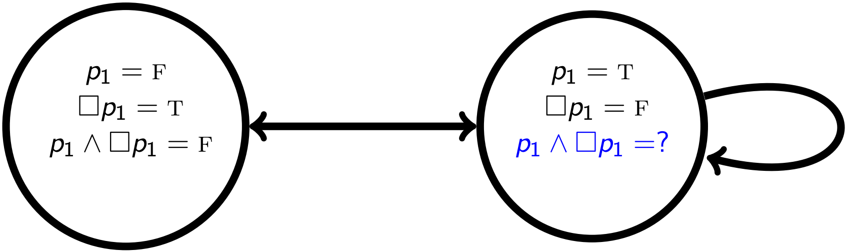

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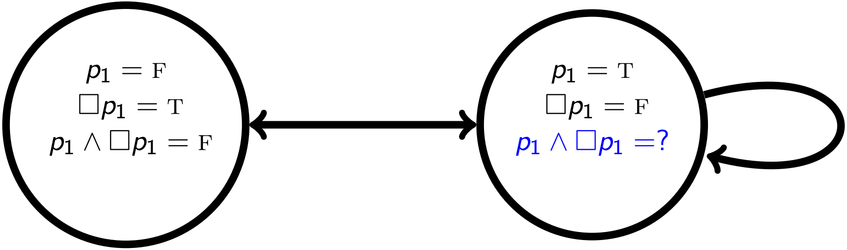

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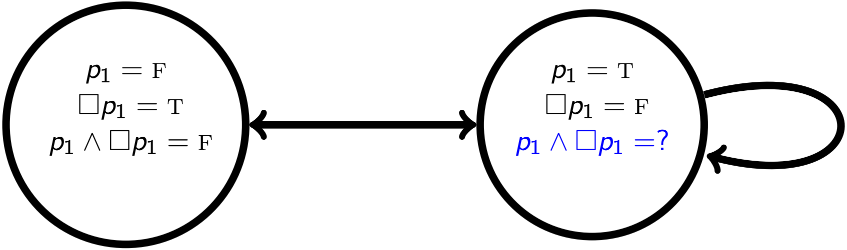
$\vdots$

$$\{p_{17} = \mathrm{T}, p_{25} = \mathrm{F}, p_{17} \wedge p_{25} = ?\}$$

Example



$$\vdots$$

$$p_1 \mapsto p_{17}$$

$$\vdots$$

$$\Box p_1 \mapsto p_{25}$$

$$\vdots$$

$$\{p_{17} = \mathrm{T}, p_{25} = \mathrm{F}, p_{17} \wedge p_{25} = \mathrm{F}\}$$

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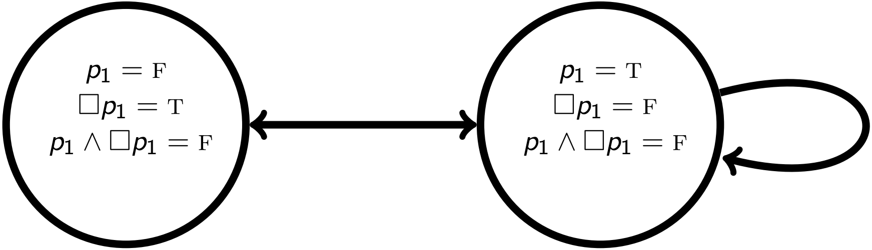

$$\vdots$$

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$$\vdots$$

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Extension Method

The main point:
use the ability to extend valuations to extend Kripke models.

- For generalized analyticity, things get more complicated:
- If the subformula relation is not anti-symmetric, we cannot have an enumeration $\varphi_1, \varphi_2, \ldots,$ such that:

$$\varphi_i \text{ is a subformula of } \varphi_j \quad \longrightarrow \quad i \leq j$$

- Still, a similar method works.

Conclusion

We have seen:

- A theorem: analytic pure calculi + modal operators are analytic.
- The proof is semantic: extending bivaluations $\longrightarrow$ extending frames.
- Valid for the usual modalities, and for general analyticity.

Future work:

- Cut-elimination
- First order
- Reveal the essential properties of the modal rules that made this work.

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- Cut-elimination
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- Reveal the essential properties of the modal rules that made this work.

Thank You, and Bon Appétit!