

# Some results on $(a : b)$ -choosability

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## Abstract

A solution to a problem of Erdős, Rubin and Taylor is obtained by showing that if a graph  $G$  is  $(a : b)$ -choosable, and  $c/d > a/b$ , then  $G$  is not necessarily  $(c : d)$ -choosable. The simplest case of another problem, stated by the same authors, is settled, proving that every 2-choosable graph is also  $(4 : 2)$ -choosable. Applying probabilistic methods, an upper bound for the  $k^{\text{th}}$  choice number of a graph is given. We also prove that a directed graph with maximum outdegree  $d$  and no odd directed cycle is  $(k(d + 1) : k)$ -choosable for every  $k \geq 1$ . Other results presented in this article are related to the strong choice number of graphs (a generalization of the strong chromatic number). We conclude with complexity analysis of some decision problems related to graph choosability.

# 1 Introduction

All graphs considered are finite, undirected and simple. A graph  $G = (V, E)$  is  $(a : b)$ -choosable if for every family of sets  $\{S(v) : v \in V\}$ , where  $|S(v)| = a$  for all  $v \in V$ , there exist subsets  $C(v) \subseteq S(v)$ , where  $|C(v)| = b$  for every  $v \in V$ , and  $C(u) \cap C(v) = \emptyset$  whenever  $u, v \in V$  are adjacent. The  $k^{\text{th}}$  choice number of  $G$ , denoted by  $ch_k(G)$ , is the smallest integer  $n$  for which  $G$  is  $(n : k)$ -choosable. A graph  $G = (V, E)$  is  $k$ -choosable if it is  $(k : 1)$ -choosable. The choice number of  $G$ , denoted by  $ch(G)$ , is equal to  $ch_1(G)$ .

The concept of  $(a : b)$ -choosability was defined and studied by Erdős, Rubin and Taylor in [10]. In the present paper we present some new results related to that topic. Part of our work generalizes previous results from [1], [2], [4] and [10]. We list our results in this introduction section. The detailed proofs are given separately in later sections.

The following theorem examines the behavior of  $ch_k(G)$  when  $k$  is large.

**Theorem 1.1** *Let  $G$  be a graph. For every  $\epsilon > 0$  there exists an integer  $k_0$  such that for every  $k \geq k_0$ ,  $ch_k(G) \leq k(\chi(G) + \epsilon)$ .*

The following question is stated in [10]:

If  $G$  is  $(a : b)$ -choosable, and  $\frac{c}{d} > \frac{a}{b}$ , does it imply that  $G$  is  $(c : d)$ -choosable?

The following is a negative answer to this question:

**Corollary 1.2** *If  $l > m \geq 3$ , then there exists a graph  $G$  which is  $(a : b)$ -choosable but not  $(c : d)$ -choosable, where  $\frac{c}{d} = l$  and  $\frac{a}{b} = m$ .*

Let  $K_{m*r}$  denote the complete  $r$ -partite graph with  $m$  vertices in each vertex class, and let  $K_{m_1, \dots, m_r}$  denote the complete  $r$ -partite graph with  $m_i$  vertices in the  $i^{\text{th}}$  vertex class. It is shown in [2] that there exist two positive constants  $c_1$  and  $c_2$  such that  $c_1 r \log m \leq ch(K_{m*r}) \leq c_2 r \log m$ , for every  $m \geq 2$  and  $r \geq 2$ . The following theorem generalizes the upper bound.

**Theorem 1.3** *If  $r \geq 1$  and  $m_i \geq 2$  for every  $1 \leq i \leq r$ , then*

$$ch_k(K_{m_1, \dots, m_r}) \leq 948r \left( k + \log \frac{m_1 + \dots + m_r}{r} \right).$$

Logarithms are in the natural base  $e$ . Following are two applications of the above.

**Corollary 1.4** *For every graph  $G$  and  $k \geq 1$*

$$ch_k(G) \leq 948\chi(G)(k + \log(\frac{|V|}{\chi(G)} + 1)).$$

The second corollary generalizes a result from [2] regarding the choice numbers of random graphs. We refer to the standard model  $G_{n,p}$  (see, e.g., [7]), a graph on  $n$  vertices, every pair of which is expected to be the endvertices of an edge, randomly and independently, with probability  $p$ .

**Corollary 1.5** *For every two constants  $k \geq 1$  and  $0 < p < 1$ , the probability that  $ch_k(G_{n,p}) \leq 475 \log(1/(1-p))n^{\frac{\log \log n}{\log n}}$  tends to 1 as  $n$  tends to infinity.*

A theorem stated in [4] reveals the connection between the choice number of a graph and its orientations. We present here a generalization of this theorem for a specific case:

**Theorem 1.6** *Let  $D = (V, E)$  be a digraph and  $k$  a positive integer. For each  $v \in V$ , let  $S(v)$  be a set of size  $k(d_D^+(v) + 1)$ , where  $d_D^+(v)$  is the outdegree of  $v$ . If  $D$  contains no odd directed (simple) cycle, then there exist subsets  $C(v) \subseteq S(v)$ , where  $|C(v)| = k$  for all  $v \in V$ , and  $C(u) \cap C(v) = \emptyset$  for every two adjacent vertices  $u, v \in V$ . The subsets  $C(v)$  can be found in polynomial time with respect to  $|V|$  and  $k$ .*

**Corollary 1.7** *Let  $G$  be an undirected graph. If  $G$  has an orientation  $D$  which contains no odd directed (simple) cycle and maximum outdegree  $d$ , then  $G$  is  $(k(d+1) : k)$ -choosable for every  $k \geq 1$ .*

**Corollary 1.8** *An even cycle is  $(2k : k)$ -choosable for every  $k \geq 1$ .*

The last corollary enables us to generalize a variant of Brooks' Theorem which appears in [10].

**Corollary 1.9** *If a connected graph  $G$  is not  $K_n$ , and not an odd cycle, then  $ch_k(G) \leq k\Delta(G)$  for every  $k \geq 1$ , where  $\Delta(G)$  is the maximum degree of  $G$ .*

For a graph  $G = (V, E)$ , define  $M(G) = \max(|E(H)|/|V(H)|)$ , where  $H = (V(H), E(H))$  ranges over all subgraphs of  $G$ . The following two corollaries are generalizations of results which appear in [4].

**Corollary 1.10** *Every bipartite graph  $G$  is  $(k(\lceil M(G) \rceil + 1) : k)$ -choosable for every  $k \geq 1$ .*

**Corollary 1.11** *Every bipartite planar graph  $G$  is  $(3k : k)$ -choosable for every  $k \geq 1$ .*

Following are some more applications:

**Corollary 1.12** *If every induced subgraph of a graph  $G$  has a vertex of degree at most  $d$ , then  $G$  is  $(k(d+1) : k)$ -choosable for every  $k \geq 1$ .*

**Corollary 1.13** *If  $G$  is a triangulated (chordal) graph, then  $ch_k(G) = k\chi(G) = k\omega(G)$  for every  $k \geq 1$ , where  $\omega(G)$  is the clique number of  $G$ .*

The list-chromatic conjecture asserts that for every graph  $G$ ,  $ch(L(G)) = \chi(L(G))$ , where  $L(G)$  denotes the line graph of  $G$ . The list-chromatic conjecture is easy to verify for trees, graphs of degree at most 2, and  $K_{2,m}$ . It has also been proven true for snarks [13],  $K_{3,3}$  [8],  $K_{4,4}$ ,  $K_{6,6}$  [4], and 2-connected regular planar graphs [9]. Galvin now proved the list-chromatic conjecture for all bipartite multigraphs. The following corollary shows that the assertion of the list-chromatic conjecture is true for a graph whose 2-connected components are at most triangles:

**Corollary 1.14** *If a graph  $G$  contains no simple circuit of size 4 or more then  $ch(L(G)) = \chi(L(G))$ .*

The *core* of a graph  $G$  is the graph obtained from  $G$  by successively deleting vertices of degree 1 until there are no such vertices left. The graph  $\Theta_{a,b,c}$  consists of three paths of lengths  $a, b$ , and  $c$ , which share a pair of endvertices and they are otherwise vertex disjoint. The following theorem from [10] gives a complete characterization of 2-choosable graphs:

**Theorem 1.15** *A connected graph  $G$  is 2-choosable if, and only if, the core of  $G$  belongs to  $\{K_1, C_{2m+2}, \Theta_{2,2,2m} : m \geq 1\}$ .*

The following is asked in [10] :

If  $G$  is  $(a : b)$ -choosable, does it follow that  $G$  is  $(am : bm)$ -choosable?

We give an affirmative answer to the first instance of that question:

**Theorem 1.16** *If a graph  $G$  is 2-choosable, then  $G$  is also  $(4 : 2)$ -choosable.*

Our proof makes use of theorem 1.15 and it is based on a surprisingly complicated case analysis.

In the other direction we obtain:

**Theorem 1.17** *Suppose that  $k$  and  $m$  are positive integers and that  $k$  is odd. If a graph  $G$  is  $(2mk : mk)$ -choosable, then  $G$  is also  $2m$ -choosable.*

A graph  $G = (V, E)$  is  $f$ -choosable for a function  $f : V \mapsto N$  if for every family of sets  $\{S(v) : v \in V\}$ , where  $|S(v)| = f(v)$  for all  $v \in V$ , there is a proper vertex-coloring of  $G$  assigning to each vertex  $v \in V$  a color from  $S(v)$ . It is shown in [10] that the following problem is  $\Pi_2^p$ -complete: ( for terminology see [12] )

**BIPARTITE GRAPH  $(2, 3)$ -CHOOSABILITY (BG  $(2, 3)$ -CH)**

INSTANCE: A bipartite graph  $G = (V, E)$  and a function  $f : V \mapsto \{2, 3\}$ .

QUESTION: Is  $G$   $f$ -choosable?

We consider the following decision problem:

**BIPARTITE GRAPH  $k$ -CHOOSABILITY (BG  $k$ -CH)**

INSTANCE: A bipartite graph  $G = (V, E)$ .

QUESTION: Is  $G$   $k$ -choosable?

It follows from theorem 1.15 that this problem is solvable in polynomial time for  $k = 2$ .

**Theorem 1.18 BIPARTITE GRAPH  $k$ -CHOOSABILITY** *is  $\Pi_2^p$ -complete for every constant  $k \geq 3$ .*

A graph  $G = (V, E)$  is *strongly  $k$ -colorable* if every graph obtained from  $G$  by appending a union of vertex disjoint cliques of size at most  $k$  ( on the vertex set  $V$  ) is  $k$ -colorable. An analogous definition of *strongly  $k$ -choosable* is made by replacing colorability with choosability. The *strong chromatic number* of a graph  $G$ , denoted by  $s\chi(G)$ , is the minimum  $k$  such that  $G$  is strongly  $k$ -colorable. Define  $s\chi(d) = \max(s\chi(G))$ , where  $G$  ranges over all graphs with maximum degree at most  $d$ . (The definition of strong colorability given in [1] is slightly different. It is claimed there that if  $G$  is strongly  $k$ -colorable, then it is also strongly  $(k + 1)$ -colorable. However, it is not known how to prove this if the original definition given in [1] is used).

**Theorem 1.19** *If  $G$  is strongly  $k$ -colorable, then it is strongly  $(k + 1)$ -colorable as well.*

We give a weaker version of this theorem for choosability.

**Theorem 1.20** *If  $G$  is strongly  $k$ -choosable, then it is also strongly  $km$ -choosable for any integer  $m$ .*

**Theorem 1.21** *Let  $G = (V, E)$  be a graph, and suppose that  $km$  divides  $|V|$ . If the choice number of any graph obtained from  $G$  by appending a union of vertex disjoint  $k$ -cliques (on the vertex set  $V$ ) is  $k$ , then the choice number of any graph obtained from  $G$  by appending a union of vertex disjoint  $km$ -cliques is  $km$ .*

**Corollary 1.22** *Let  $n$  and  $k$  be positive integers, and let  $G$  be a  $(3k + 1)$ -regular graph on  $3kn$  vertices. Assume that  $G$  has a decomposition into a Hamiltonian circuit and  $n$  pairwise vertex disjoint  $3k$ -cliques. Then  $ch(G) = 3k$ .*

It is proved in [1] that there is a constant  $c$  such that for every  $d$ ,  $3\lfloor d/2 \rfloor < s\chi(d) \leq cd$ . The following theorem improves the lower bound.

**Theorem 1.23** *For every  $d \geq 1$ ,  $s\chi(d) \geq 2d$ .*

## 2 A solution to a problem of Erdős, Rubin and Taylor

In this section we prove an upper bound for the  $k^{\text{th}}$  choice number of a graph when  $k$  is large and apply this bound to settle a problem raised in [10].

**Proof of Theorem 1.1** Let  $G = (V, E)$  be a graph and  $\epsilon > 0$ . Let  $r$  stand for the chromatic number of  $G$  and let  $\{V_1, \dots, V_r\}$  be a partition of  $V$  into stable sets. Assign a set  $S(v)$  of  $\lfloor k(\chi(G) + \epsilon) \rfloor$  distinct colors to every  $v \in V$ . Let  $S = \cup_{v \in V} S(v)$  be the set of all colors. Define  $R = \{1, 2, \dots, r\}$  and let  $f : S \mapsto R$  be a random function, obtained by randomly selecting, the value of  $f(c)$ , independently for each color  $c \in S$ , according to a uniform distribution on  $R$ . The colors  $c$  for which  $f(c) = i$  will be used to color the vertices in  $V_i$ . To complete the proof, it thus suffices to show that the probability of the following event is positive: For every  $i$ ,  $1 \leq i \leq r$ , and for every vertex  $v \in V_i$  there are at least  $k$  colors  $c \in S(v)$  for which  $f(c) = i$ .

For a fixed vertex  $v$ , included in a set  $V_i$ , define  $X = |S(v) \cap f^{-1}(i)|$ . The probability that there are less than  $k$  colors  $c \in S(v)$  for which  $f(c) = i$  is equal to  $Pr(X < k)$ . Since  $X$  is a random

variable with distribution  $B(\lfloor k(r + \epsilon) \rfloor, 1/r)$ , Chebyshev's inequality (see, e.g., [3]) implies

$$Pr(X < k) \leq Pr\left(|X - \frac{\lfloor k(r + \epsilon) \rfloor}{r}| \geq \frac{\lfloor k\epsilon \rfloor}{r}\right) \leq \frac{\lfloor k(r + \epsilon) \rfloor \frac{1}{r} (1 - \frac{1}{r})}{(\frac{\lfloor k\epsilon \rfloor}{r})^2} = O\left(\frac{1}{k}\right).$$

It follows that there is an integer  $k_0$  such that for every  $k \geq k_0$ ,  $P(X < k) < 1/|V|$ . There are  $|V|$  vertices from which  $v$  is selected (and  $i$  is determined) and hence, the probability that for some  $i$  and some  $v \in V_i$  there are less than  $k$  colors  $c \in S(v)$  for which  $f(c) = i$  is smaller than 1.  $\square$

**Proof of Corollary 1.2** Suppose that  $l > m \geq 3$ , and let  $G$  be a graph such that  $ch(G) = l + 1$  and  $\chi(G) = m - 1$  ( it is proven in [15] that for every  $l \geq m \geq 2$  there is a graph  $G$ , where  $ch(G) = l$  and  $\chi(G) = m$ . Take for example the disjoint union of  $K_m$  and  $K_{n,n}$  for an appropriate value of  $n$  ). By theorem 1.1, for  $\epsilon = 1$  there exist an integer  $k$  such that  $G$  is  $(k(\chi(G) + 1) : k)$ -choosable. Hence  $G$  is  $(km : k)$ -choosable but not  $(l : 1)$ -choosable, as claimed.  $\square$

Note that it is not true that for every graph  $G$  there exists an integer  $k_0$  such that  $ch_k(G) \leq k\chi(G)$  for every  $k \geq k_0$ . For example the chromatic number of  $G = K_{3,3}$  is 2, but that graph is not 2-choosable and therefore, by theorem 1.17, it is not  $(2k : k)$ -choosable for any odd  $k$ . Thus  $ch_k(G) > k\chi(G)$  for every odd  $k$ .

### 3 An upper bound for the $k$ th choice number

In this section we establish an upper bound for  $ch_k(K_{m_1, \dots, m_r})$ , and use it to prove two consequences. The following lemma appears in [3].

**Lemma 3.1** *If  $X$  is a random variable with distribution  $B(n, p)$ ,  $0 < p \leq 1$ , and  $k < pn$  then*

$$Pr(X < k) < e^{-\frac{(np-k)^2}{2pn}}.$$

In the rest of this section we denote  $t = \frac{m_1 + \dots + m_r}{r}$ ,  $t_1 = \frac{m_1 + \dots + m_{r/2}}{r/2}$ , and  $t_2 = \frac{m_{r/2+1} + \dots + m_r}{r/2}$ . Notice that  $t = (t_1 + t_2)/2$ , and therefore  $\log t_1 t_2 \leq 2 \log t$ .

**Lemma 3.2** *If  $1 \leq r \leq t$ ,  $k \geq 1$ , and  $m_i \geq 2$  for every  $i$ ,  $1 \leq i \leq r$ , then  $ch_k(K_{m_1, \dots, m_r}) \leq 4r(k + \log t)$ .*

**Proof** Let  $V_1, V_2, \dots, V_r$  be the stable sets of  $K = K_{m_1, \dots, m_r}$ , where  $|V_i| = m_i$  for all  $i$ , and let  $V = V_1 \cup \dots \cup V_r$  be the set of all vertices of  $K$ . For each  $v \in V$ , let  $S(v)$  be a set of  $\lfloor 4r(k + \log t) \rfloor$

distinct colors. Define  $R = \{1, 2, \dots, r\}$  and let  $f : S \mapsto R$  be a random function, obtained by choosing, for each color  $c \in S$ , randomly and independently, the value of  $f(c)$  according to a uniform distribution on  $R$ . The colors  $c$  for which  $f(c) = i$  will be the ones to be used for coloring the vertices in  $V_i$ . To complete the proof it thus suffices to show that with positive probability for every  $i$ ,  $1 \leq i \leq r$ , and every vertex  $v \in V_i$  there are at least  $k$  colors  $c \in S(v)$  so that  $f(c) = i$ .

Fix an  $i$  and a vertex  $v \in V_i$ , and define  $X = |S(v) \cap f^{-1}(i)|$ . The probability that there are less than  $k$  colors  $c \in S(v)$  so that  $f(c) = i$  is equal to  $\Pr(X < k)$ . Since  $X$  is a random variable with distribution  $B(\lfloor 4r(k + \log t) \rfloor, 1/r)$ , by lemma 3.1

$$\Pr(X < k) < e^{-\frac{(E(X)-k)^2}{2E(X)}} \leq e^{-\frac{(4(k+\log t)-1-k)^2}{8(k+\log t)}} < e^{-\frac{16(k+\log t)^2-8(k+1)(k+\log t)}{8(k+\log t)}} \leq e^{-2 \log t} = \frac{1}{t^2} \leq \frac{1}{rt},$$

where the last inequality follows the fact that  $r \leq t$ . There are  $rt$  possible ways to choose  $i$ ,  $1 \leq i \leq r$  and  $v \in V_i$ , and hence, the probability that for some  $i$  and some  $v \in V_i$  there are less than  $k$  colors  $c \in S(v)$  so that  $f(c) = i$  is smaller than 1, completing the proof.  $\square$

**Lemma 3.3** *Suppose that  $r$  is even,  $r > t$ ,  $k \geq 1$ ,  $d \geq 244$ , and  $m_i \geq 2$  for every  $i$ ,  $1 \leq i \leq r$ . If  $ch_k(K_{m_1, \dots, m_{r/2}}) \leq d(1 - \frac{1}{5r^{1/3}})\frac{r}{2}(k + \log t_1)$  and  $ch_k(K_{m_{r/2+1}, \dots, m_r}) \leq d(1 - \frac{1}{5r^{1/3}})\frac{r}{2}(k + \log t_2)$ , then  $ch_k(K_{m_1, \dots, m_r}) \leq dr(k + \log t)$ .*

**Proof** Let  $V_1, V_2, \dots, V_r$  be the stable sets of  $K = K_{m_1, \dots, m_r}$ , where  $|V_i| = m_i$  for all  $i$ , and let  $V = V_1 \cup \dots \cup V_r$  be the vertex set of  $K$ . For each  $v \in V$ , let  $S(v)$  be a set of  $\lfloor dr(k + \log t) \rfloor$  distinct colors. Define  $R = \{1, 2, \dots, r\}$ , and let  $S = \cup_{v \in V} S(v)$  be the set of all colors. Define  $R_1 = \{1, 2, \dots, r/2\}$  and  $R_2 = \{r/2 + 1, \dots, r\}$ . Let  $f : S \mapsto \{1, 2\}$  be a random function obtained by choosing, for each  $c \in S$  randomly and independently,  $f(c) \in \{1, 2\}$  where for all  $j \in \{1, 2\}$

$$\Pr(f(c) = j) = \frac{k + \log t_j}{2k + \log t_1 t_2}.$$

The colors  $c$  for which  $f(c) = 1$  will be used for coloring the vertices in  $\cup_{i \in R_1} V_i$ , whereas the colors  $c$  for which  $f(c) = 2$  will be used for coloring the vertices in  $\cup_{i \in R_2} V_i$ .

For every vertex  $v \in V$ , define  $C(v) = S(v) \cap f^{-1}(1)$  if  $v$  belongs to  $\cup_{i \in R_1} V_i$ , and  $C(v) = S(v) \cap f^{-1}(2)$  if  $v$  belongs to  $\cup_{i \in R_2} V_i$ . Because of the assumptions of the lemma, it remains to show that with positive probability,

$$|C(v)| \geq d(1 - \frac{1}{5r^{1/3}})\frac{r}{2}(k + \log t_j) \tag{1}$$

for all  $j \in \{1, 2\}$  and  $v \in \cup_{i \in R_j} V_i$ .

Fix a  $j \in \{1, 2\}$  and a vertex  $v \in \cup_{i \in R_j} V_i$ , and define  $X = |C(v)|$ . The expectation of  $X$  is

$$\lfloor dr(k + \log t) \rfloor \frac{k + \log t_j}{2k + \log t_1 t_2} \geq (dr(k + \log t) - 1) \frac{k + \log t_j}{2k + 2 \log t} \geq d \frac{r}{2} (k + \log t_j) - 1 = T.$$

It follows from lemma 3.1 and the inequality  $E(X) \geq T$  that

$$Pr(X < T - T^{2/3}) < e^{-\frac{(E(X) - T + T^{2/3})^2}{2E(X)}} \leq e^{-\frac{1}{2}T^{1/3}} \leq e^{-\frac{1}{2}(d \frac{r}{2})^{1/3}}.$$

Since  $|\cup_{i \in R_j} V_i| \leq rt < r^2$ , the probability that  $|C(v)| < T - T^{2/3}$  holds for some  $v \in \cup_{i \in R_j} V_i$  is at most

$$r^2 \cdot e^{-\frac{1}{2}(d \frac{r}{2})^{1/3}} < 1/2,$$

where the last inequality follows the fact that  $d \geq 244$ . One can easily verify that

$$T - T^{2/3} = T(1 - \frac{1}{T^{1/3}}) \geq d \frac{r}{2} (k + \log t_j) (1 - \frac{1}{5r^{1/3}}),$$

and therefore, with positive probability (1) holds for all  $j \in \{1, 2\}$  and  $v \in \cup_{i \in R_j} V_i$ .  $\square$

**Proof of Theorem 1.3** Define for every  $r$  which is a power of 2

$$f(r) = \prod_{j=0}^{\log_2 r} (1 - \frac{1}{5 \cdot 2^{j/3}}) / \prod_{j=0}^2 (1 - \frac{1}{5 \cdot 2^{j/3}}).$$

We claim that for every  $r$  which is a power of 2

$$ch_k(K_{m_1, \dots, m_r}) \leq \frac{244r(k + \log t)}{f(r)}. \quad (2)$$

The proof is by induction on  $r$ .

**Case 1:**  $r \leq t$ .

The result follows from lemma 3.2 since

$$\frac{244}{f(r)} \geq 244 \prod_{j=1}^2 (1 - \frac{1}{5 \cdot 2^{j/3}}) > 4.$$

**Case 2:**  $r > t$ .

Notice that  $t \geq 2$ , and therefore  $r \geq 4$ . By the induction hypothesis

$$ch_k(K_{m_1, \dots, m_{r/2}}) \leq \frac{244(1 - \frac{1}{5r^{1/3}}) \frac{r}{2} (k + \log t_1)}{f(r)}$$

and

$$ch_k(K_{m_{r/2+1}, \dots, m_r}) \leq \frac{244(1 - \frac{1}{5r^{1/3}})^{\frac{r}{2}}(k + \log t_2)}{f(r)}.$$

Since  $r \geq 4$ , we have  $244/f(r) \geq 244$  and it follows from lemma 3.3 that (2) holds, as claimed.

It is easy to check that

$$\prod_{j=3}^{\log_2 r} (1 - \frac{1}{5 \cdot 2^{j/3}}) \geq 1 - \sum_{j=3}^{\log_2 r} \frac{1}{5 \cdot 2^{j/3}} \geq 1 - \frac{1}{10(1 - 2^{-1/3})},$$

and therefore  $244/f(r) \leq 474$ . It follows from (2) that for every  $r$  which is a power of 2

$$ch_k(K_{m_1, \dots, m_r}) \leq 474r(k + \log t). \quad (3)$$

Returning to the general case, assume that  $r \geq 1$ . Choose an integer  $r'$  which is a power of 2 and  $r \leq r' < 2r$ . By applying (3), we get

$$\begin{aligned} ch_k(K_{m_1, \dots, m_r}) &\leq ch_k(K_{m_1, \dots, m_r, \underbrace{2, \dots, 2}_{r'-r}}) \\ &\leq 474r'(k + \log \frac{m_1 + \dots + m_r + 2(r'-r)}{r'}) \leq 948r(k + \log \frac{m_1 + \dots + m_r}{r}), \end{aligned}$$

completing the proof.  $\square$

Define  $K = K_{\underbrace{m, s, \dots, s}_r}$ , where  $m \geq 2$  and  $s \geq 2$ . Every induced subgraph of  $K$  has a vertex of degree at most  $rs$ , and therefore by corollary 1.12  $ch_k(K) \leq k(rs + 1)$  for all  $k \geq 1$ . Note that this upper bound for  $ch_k(K)$  does not depend of  $m$ , which means that a good lower bound for  $ch_k(K_{m_1, \dots, m_r})$  has a more complicated form than the upper bound given in theorem 1.3.

**Proof of Corollary 1.4** Let  $G = (V, E)$  be a graph and  $k \geq 1$ . Define  $r = \chi(G)$ , and let  $V = V_1 \cup \dots \cup V_r$  be a partition of the vertices, such that each  $V_i$  is a stable set. Define  $m_i = |V_i|$  for all  $i$ ,  $1 \leq i \leq r$ . By theorem 1.1

$$ch_k(G) \leq ch_k(K_{m_1+1, \dots, m_r+1}) \leq 948r(k + \log \frac{m_1 + \dots + m_r + r}{r}) = 948\chi(G)(k + \log(\frac{|V|}{\chi(G)} + 1)),$$

as claimed.  $\square$

**Proof of Corollary 1.5** As proven by Bollobás in [6], for a fixed probability  $p$ ,  $0 < p < 1$ , almost surely (i.e., with probability that tends to 1 as  $n$  tends to infinity), the random graph  $G_{n,p}$  has chromatic number

$$(\frac{1}{2} + o(1)) \log(1/(1-p)) \frac{n}{\log n}.$$

By corollary 1.4, for every  $\epsilon > 0$  almost surely

$$ch_k(G_{n,p}) \leq 948\left(\frac{1}{2} + \epsilon\right) \log(1/(1-p)) \frac{n}{\log n} \left(k + \log\left(\frac{3 \log n}{\log(1/(1-p))} + 1\right)\right).$$

The result follows since  $k$  and  $p$  are constants.  $\square$

Note that in the proof of the last corollary we have not used any knowledge concerning independent sets of  $G_{n,p}$ , as was done in [2] for the proof of the special case.

## 4 Choice numbers and orientations

Let  $D = (V, E)$  be a digraph. We denote the set of out-neighbors of  $v$  in  $D$  by  $N_D^+(v)$ . A set of vertices  $K \subseteq V$  is called a *kernel* of  $D$  if  $K$  is an independent set and  $N_D^+(v) \cap K \neq \emptyset$  for every vertex  $v \notin K$ . Richardson's theorem (see, e.g., [5]) states that any digraph with no odd directed cycle has a kernel.

**Proof of Theorem 1.6** Let  $D = (V, E)$  be a digraph which contains no odd directed (simple) cycle and  $k \geq 1$ . For each  $v \in V$ , let  $S(v)$  be a set of size  $k(d_D^+(v) + 1)$ . We claim that the following algorithm finds subsets  $C(v) \subseteq S(v)$ , where  $|C(v)| = k$  for all  $v \in V$ , and  $C(u) \cap C(v) = \emptyset$  for every two adjacent vertices  $u, v \in V$ .

1.  $S \leftarrow \cup_{v \in V} S(v)$ ,  $W \leftarrow V$  and for every  $v \in V$ ,  $C(v) \leftarrow \emptyset$ .
2. Choose a color  $c \in S \cap \cup_{v \in W} S(v)$  and put  $S \leftarrow S - \{c\}$ .
3. Let  $K$  be a kernel of the induced subgraph of  $D$  on the vertex set  $\{v \in W : c \in S(v)\}$ .
4.  $C(v) \leftarrow C(v) \cup \{c\}$  for all  $v \in K$ .
5.  $W \leftarrow W - \{v \in K : |C(v)| = k\}$ .
6. If  $W = \emptyset$ , stop. If not, go to step 2.

During the algorithm,  $W$  is equal to  $\{v \in V : |C(v)| < k\}$ , and  $S$  is the set of remaining colors. We first prove that in step 2,  $S \cap \cup_{v \in W} S(v) \neq \emptyset$ . When the algorithm reaches step 2, it is obvious that  $W \neq \emptyset$ . Suppose that  $w \in W$  in this step, and therefore  $|C(w)| < k$ . It follows easily from

the definition of a kernel that every color from  $S(w)$ , which has been previously chosen in step 2, belongs either to  $C(w)$  or to  $\cup_{v \in N_D^+(w)} C(v)$ . Since

$$|C(w)| + \left| \bigcup_{v \in N_D^+(w)} C(v) \right| < k + k \cdot d_D^+(w) = |S(w)|,$$

not all the colors of  $S(w)$  have been used. This means that  $S \cap S(w) \neq \emptyset$ , as needed. It follows easily that the algorithm always terminates.

Upon termination of the algorithm,  $|C(v)| = k$  for all  $v \in V$ . In step 4 the same color is assigned to the vertices of a kernel which is an independent set, and therefore  $C(u) \cap C(v) = \emptyset$  for every two adjacent vertices  $u, v \in V$ . This proves the correctness of the algorithm.

In step 4, the operation  $C(v) \leftarrow C(v) \cup \{c\}$  is performed for at least one vertex. Upon termination  $|\cup_{v \in V} C(v)| \leq k|V|$ , which means that the algorithm performs at most  $k|V|$  iterations. There is a polynomial time algorithm for finding a kernel in a digraph with no odd directed cycle. Thus, the algorithm is of polynomial time complexity in  $|V|$  and  $k$ , completing the proof.  $\square$

**Proof of Corollary 1.7** This is an immediate consequence of theorem 1.6, since  $k(d_D^+(v) + 1) \leq k(d + 1)$  for every  $v \in V$ .  $\square$

**Proof of Corollary 1.8** The result follows from 1.7 by taking the cyclic orientation of the even cycle.  $\square$

The proof of corollary 1.9 is similar to the proof of the special case which appears in [10]. A graph  $G = (V, E)$  is *k-degree-choosable* if for every family of sets  $\{S(v) : v \in V\}$ , where  $|S(v)| = kd(v)$  for all  $v \in V$ , there are subsets  $C(v) \subseteq S(v)$ , where  $|C(v)| = k$  for all  $v \in V$ , and  $C(u) \cap C(v) = \emptyset$  for every two adjacent vertices  $u, v \in V$ .

**Lemma 4.1** *If a graph  $G = (V, E)$  is connected, and  $G$  has a connected induced subgraph  $H = (V', E')$  which is k-degree-choosable, then  $G$  is k-degree-choosable.*

**Proof** For each  $v \in V$ , let  $S(v)$  be a set of size  $kd(v)$ . The proof is by induction on  $|V|$ . In case  $|V| = |V'|$  there is nothing to prove. Assuming that  $|V| > |V'|$ , let  $v$  be a vertex of  $G$  which is at maximal distance from  $H$ . This guarantees that  $G - v$  is connected. Choose any subset  $C(v) \subseteq S(v)$  such that  $|C(v)| = k$ , and remove the colors of  $C(v)$  from all the vertices adjacent to  $v$ . The choice can be completed by applying the induction hypothesis on  $G - v$ .  $\square$

**Lemma 4.2** *If  $c \geq 2$ , then  $\Theta_{a,b,c}$  is k-degree-choosable for every  $k \geq 1$ .*

**Proof** Suppose that  $\Theta_{a,b,c}$  has vertex set  $V = \{u, v, x_1, \dots, x_{a-1}, y_1, \dots, y_{b-1}, z_1, \dots, z_{c-1}\}$  and contains the three paths  $u-x_1-\dots-x_{a-1}-v$ ,  $u-y_1-\dots-y_{b-1}-v$ , and  $u-z_1-\dots-z_{c-1}-v$ . For each  $w \in V$ , let  $S(w)$  be a set of size  $kd(w)$ . For the vertex  $u$  we choose a subset  $C(u) \subseteq S(u) - S(z_1)$  of size  $k$ . For each vertex according to the sequence  $x_1, \dots, x_{a-1}, y_1, \dots, y_{b-1}, v, z_{c-1}, \dots, z_1$  we choose a subset of  $k$  colors that were not chosen in adjacent earlier vertices.  $\square$

For the proof of corollary 1.9, we shall need the following lemma which appears in [10].

**Lemma 4.3** *If there is no vertex which disconnects  $G$ , then  $G$  is an odd cycle, or  $G = K_n$ , or  $G$  contains, as a vertex induced subgraph, an even cycle without chord or with only one chord.*

**Proof of Corollary 1.9** Suppose that a connected graph  $G$  is not  $K_n$ , and not an odd cycle. If  $G$  is not a regular graph, then every induced subgraph of  $G$  has a vertex of degree at most  $\Delta(G) - 1$ , and by corollary 1.12  $ch_k(G) \leq k\Delta(G)$  for all  $k \geq 1$ . If  $G$  is a regular graph, then there is a part of  $G$  not disconnected by a vertex, which is neither an odd cycle nor a complete graph. It follows from lemma 4.3 that  $G$  contains, as a vertex induced subgraph, an even cycle or a particular kind of  $\Theta_{a,b,c}$  graph. We know from corollary 1.8 and lemma 4.2 that both an even cycle and  $\Theta_{a,b,c}$  are  $k$ -degree-choosable for every  $k \geq 1$ . The result follows from lemma 4.1.  $\square$

**Proof of Corollary 1.10** It is proved in [4] that a graph  $G = (V, E)$  has an orientation  $D$  in which every outdegree is at most  $d$  if and only if  $M(G) \leq d$ . Therefore, there is an orientation  $D$  of  $G$  in which the maximum outdegree is at most  $\lceil M(G) \rceil$ . Since  $D$  contains no odd directed cycles, the result follows from corollary 1.7.  $\square$

**Proof of Corollary 1.11**  $M(G) \leq 2$ , since any planar bipartite (simple) graph on  $r$  vertices contains at most  $2r - 2$  edges. The result follows from corollary 1.10.  $\square$

**Proof of Corollary 1.12** We claim that if every induced subgraph of a graph  $G = (V, E)$  has a vertex of degree at most  $d$ , then  $G$  has an acyclic orientation in which the maximum outdegree is  $d$ . The proof is by induction on  $|V|$ . If  $|V| = 1$ , the result is trivial. If  $|V| > 1$ , let  $v$  be a vertex of  $G$  with degree at most  $d$ . By the induction hypothesis,  $G - v$  has an acyclic orientation in which the maximum outdegree is  $d$ . We complete this orientation of  $G - v$  by orienting every edge incident to  $v$  from  $v$  to its appropriate neighbor and obtain the desired orientation of  $G$ , as claimed. The result follows from corollary 1.7.  $\square$

An undirected graph  $G$  is called *triangulated* if  $G$  does not contain an induced subgraph iso-

morphic to  $C_n$  for  $n \geq 4$ . Being triangulated is a hereditary property inherited by all the induced subgraphs of  $G$ . A vertex  $v$  of  $G$  is called *simplicial* if its adjacency set  $Adj(v)$  induces a complete subgraph of  $G$ . It is proved in [14] that every triangulated graph has a simplicial vertex.

**Proof of Corollary 1.13** Suppose that  $G$  is a triangulated graph, and let  $H$  be an induced subgraph of  $G$ . Since  $H$  is triangulated, it has a simplicial vertex  $v$ . The set of vertices  $\{v\} \cup Adj_H(v)$  induces a complete subgraph of  $H$ , and therefore  $v$  has degree at most  $\omega(G) - 1$  in  $H$ . It follows from corollary 1.12 that  $ch_k(G) \leq k\omega(G)$  for every  $k \geq 1$ . For every graph  $G$  and  $k \geq 1$ ,  $ch_k(G) \geq k\omega(G)$  and hence  $ch_k(G) = k\omega(G)$  for every  $k \geq 1$ . Since  $G$  is triangulated, it is also perfect, which means that  $\chi(G) = \omega(G)$ , as needed.  $\square$

**Proof of Corollary 1.14** It is easy to see that  $L(G)$  is triangulated if and only if  $G$  contains no  $C_n$  for every  $n \geq 4$ . The result follows from corollary 1.13.  $\square$

The validity of the list-chromatic conjecture for graphs of class 2 with maximum degree 3 (and in particular for snarks) follows easily from corollary 1.9. Suppose that  $G$  is a graph of class 2 with  $\Delta(G) = 3$ . Let  $C$  be a connected component of  $L(G)$ . If  $C$  is not a complete graph, and not an odd cycle, then  $ch(C) \leq \Delta(C) \leq \Delta(L(G)) \leq 4$ . If  $C$  is a complete graph or an odd cycle, then it is easy to see that  $\Delta(C) \leq 2$ , and therefore by corollary 1.12  $ch(C) \leq \Delta(C) + 1 \leq 3$ . It follows that  $ch(L(G)) \leq 4$ . Since  $G$  is a graph of class 2,  $ch(L(G)) \geq \chi(L(G)) = \Delta(G) + 1 = 4$ , and hence,  $ch(L(G)) = \chi(L(G)) = 4$ .

## 5 Properties of $(2k : k)$ -choosable graphs

Let  $A$  and  $B$  be sets of size 4. We denote  $p(A, B) = \{(C, D) : C \subseteq A, D \subseteq B, |C| = |D| = 2\}$ . Suppose that  $S \subseteq p(A_1, B_1)$  and that  $T \subseteq p(A_2, B_2)$ . We say that  $S$  and  $T$  are isomorphic if there exist two bijections  $f : A_1 \mapsto A_2$  and  $g : B_1 \mapsto B_2$  so that  $(C, D) \in S$  iff  $(f(C), g(D)) \in T$  for every  $C \subseteq A_1$  and  $D \subseteq B_1$ , where  $|C| = |D| = 2$ .

Let  $A$  and  $B$  be sets of size 4, and suppose that  $S \subseteq p(A, B)$ . Suppose that  $H_1, \dots, H_6$  are all the subsets of  $A$  of size 2. For each  $i$ ,  $1 \leq i \leq 6$ , we denote  $c(H_i) = \{G : (H_i, G) \in S\}$  and  $d_i = |c(H_i)|$ . The sequence containing the numbers  $d_1, \dots, d_6$  in monotone decreasing order is called the degree sequence of  $S$ . We say that  $S$  is special if it has the following properties:

1. Its degree sequence is  $(6, 5, 5, 3, 3, 1)$ .

2. If  $H$  and  $G$  are the two subsets of  $A$  for which  $|c(H)| = |c(G)| = 3$ , then  $|H \cap G| = 1$ . Denote  $H = \{1, 2\}$ ,  $G = \{1, 3\}$ , and  $A = \{1, 2, 3, 4\}$ .
3.  $c(H) = c(G)$ .
4.  $c(H)$  has either the form  $\{\{a, b\}, \{a, c\}, \{a, d\}\}$  or the form  $\{\{a, b\}, \{a, c\}, \{b, c\}\}$ , where  $a, b, c, d$  are any distinct colors.
5. Either  $|c(\{2, 3\})| = 1$  and  $|c(\{1, 4\})| = 6$ , or  $|c(\{2, 3\})| = 6$  and  $|c(\{1, 4\})| = 1$ .

We say that  $S$  has property  $P_1$  iff  $c(H)$  has the form  $\{\{a, b\}, \{a, c\}, \{a, d\}\}$  and that it has property  $P_2$  iff  $|c(\{2, 3\})| = 1$ .

Suppose that  $K_{2,2}$  has vertex set  $V = X \cup Y$ , where  $X = \{x_1, x_2\}$ ,  $Y = \{y_1, y_2\}$ , and it has exactly the edges  $\{x_i, y_j\}$ . For each  $v \in V$ , let  $S(v)$  be a set of size 4. By  $C(v)$  we denote a subset of  $S(v)$  of size 2. We say that  $C(x_1)$  and  $C(x_2)$  are compatible if there exist two subsets  $C(y_1)$  and  $C(y_2)$ , so that  $C(u) \cap C(v) = \emptyset$  for every two adjacent vertices  $u, v \in V$ . A subset  $C(x_1) \subseteq S(x_1)$  is called bad if  $C(x_1)$  is not compatible with any  $C(x_2)$ . An analogous definition is made for  $C(x_2)$ . We say that a family of sets  $\{S(v) : v \in V\}$  is defected if there exist two bad subsets  $C(x_1)$  and  $C(x_2)$ . We denote by  $incomp(x_1, x_2)$  the set of incompatible pairs  $(C(x_1), C(x_2))$ .

**Lemma 5.1** *If the family of sets  $\{S(v) : v \in V\}$  is defected and  $C(x_1)$  is bad, then both  $S(y_1)$  and  $S(y_2)$  intersect  $C(x_1)$  and at least one of them contains  $C(x_1)$ .*

**Proof** Suppose that neither  $S(y_1)$  nor  $S(y_2)$  contain  $C(x_1)$ . Remove the colors of  $C(x_1)$  from  $S(y_1)$  and  $S(y_2)$ . Now both  $S(y_1)$  and  $S(y_2)$  have size at least 3. We can assume the worst case, in which both  $S(y_1)$  and  $S(y_2)$  are subsets of  $S(x_2)$ , and therefore  $|S(y_1) \cap S(y_2)| \geq 2$ . Let  $C$  be a subset of  $S(y_1) \cap S(y_2)$  of size 2. Choose a subset  $C(x_2) \subseteq S(x_2) - C$ . We have that  $C(x_1)$  and  $C(x_2)$  are compatible in contrast to the fact that  $C(x_1)$  is bad. This proves that at least one of  $S(y_1)$  and  $S(y_2)$  contains  $C(x_1)$ .

Suppose that  $S(y_1) \cap C(x_1) = \emptyset$ . Choose a subset  $C(y_2) \subseteq S(y_2) - C(x_1)$  and a subset  $C(x_2) \subseteq S(x_2) - C(y_2)$ . We have that  $C(x_1)$  and  $C(x_2)$  are compatible in contrast to the fact that  $C(x_1)$  is bad. This proves that both  $S(y_1)$  and  $S(y_2)$  intersect  $C(x_1)$ .  $\square$

**Lemma 5.2** *If the family of sets  $\{S(v) : v \in V\}$  is defected, then both  $S(x_1)$  and  $S(x_2)$  contain exactly one bad subset. Furthermore, at least one of the following is valid:*

1. The set  $\text{incomp}(x_1, x_2)$  is special and has properties  $P_1$  and  $P_2$ .

2.  $\text{incomp}(x_1, x_2)$  has degree sequence  $(6, 5, 5, 3, 2, 2)$ .

3.  $|\text{incomp}(x_1, x_2)| = 21$ .

**Proof** The set  $S(x_1)$  contains a bad subset, which we denote by  $C(x_1) = \{1, 2\}$ . Without loss of generality, we can assume by lemma 5.1 that  $C(x_1) \subseteq S(y_1)$  and that  $S(y_2)$  intersects  $C(x_1)$ . Denote  $S(y_1) = \{1, 2, 3, 4\}$ . Since  $C(x_1)$  is bad, we must have that  $|(S(y_1) \cap S(y_2)) - C(x_1)| < 2$ .

**Case 1:**  $C(x_1) \subseteq S(y_2)$  and  $|S(y_1) \cap S(y_2)| = 3$ .

Denote  $S(y_2) = \{1, 2, 3, 5\}$ . Since  $C(x_1)$  is bad, surely  $\{3, 4, 5\} \subseteq S(x_2)$ . The set  $S(x_2)$  contains a bad subset, which we denote by  $C(x_2)$ , and therefore  $\{1, 2\} \cap S(x_2) \neq \emptyset$ . We can assume, without loss of generality, that  $S(x_2) = \{1, 3, 4, 5\}$ . Since  $C(x_2)$  is bad and  $|S(y_1) \cap S(y_2)| = 3$ , we must have that  $C(x_2) \subseteq S(y_1) \cap S(y_2)$ . Hence,  $C(x_2) = \{1, 3\}$  and  $S(x_1) = \{1, 2, 4, 5\}$ . We have that

$$S(x_1) = \{1, 2, 4, 5\}, S(x_2) = \{1, 3, 4, 5\}, S(y_1) = \{1, 2, 3, 4\}, S(y_2) = \{1, 2, 3, 5\}.$$

The set  $\text{incomp}(x_1, x_2)$  is special and has properties  $P_1$  and  $P_2$ .

**Case 2:**  $C(x_1) \subseteq S(y_2)$  and  $|S(y_1) \cap S(y_2)| = 2$ .

Denote  $S(y_2) = \{1, 2, 5, 6\}$ . Since  $C(x_1)$  is bad, surely  $|S(x_2) \cap \{3, 4, 5, 6\}| \geq 3$ . Suppose without loss of generality that  $\{3, 4, 5\} \subseteq S(x_2)$ . The set  $S(x_2)$  contains a bad subset, which we denote by  $C(x_2)$ , and therefore  $\{1, 2\} \cap S(x_2) \neq \emptyset$ . We can assume, without loss of generality, that  $S(x_2) = \{1, 3, 4, 5\}$ . Since  $C(x_2)$  is bad and  $|S(y_1) \cap S(y_2)| = 2$ , we must have that  $C(x_2) \cap \{1, 2\} \neq \emptyset$ , and therefore  $1 \in C(x_2)$ . We can assume, without loss of generality, that  $C(x_2) = \{1, 3\}$ . Since  $C(x_2)$  is bad, we must have that  $4 \in S(x_1)$  and  $S(x_1) \cap \{5, 6\} \neq \emptyset$ . Suppose without loss of generality that  $S(x_1) = \{1, 2, 4, 5\}$ . This is a contradiction to the fact that  $C(x_2)$  is bad.

**Case 3:**  $|C(x_1) \cap S(y_2)| = 1$  and  $|S(y_1) \cap S(y_2)| = 2$ .

We can assume, without loss of generality, that  $1 \in S(y_2)$  and  $3 \in S(y_2)$ . Denote  $S(y_2) = \{1, 3, 5, 6\}$ . Since  $C(x_1)$  is bad, surely  $S(x_2) = \{3, 4, 5, 6\}$ . The set  $S(x_2)$  contains a bad subset, which we denote by  $C(x_2)$ . Since  $C(x_2)$  is bad and  $|S(y_1) \cap S(y_2)| = 2$ , we must have that  $C(x_2) \cap \{1, 3\} \neq \emptyset$ , and therefore  $3 \in C(x_2)$ . If  $C(x_2) = \{3, 4\}$ , then we must have that  $S(x_1) = \{1, 2, 5, 6\}$ , so

$$S(x_1) = \{1, 2, 5, 6\}, S(x_2) = \{3, 4, 5, 6\}, S(y_1) = \{1, 2, 3, 4\}, S(y_2) = \{1, 3, 5, 6\}.$$

The set  $incomp(x_1, x_2)$  has degree sequence  $(6, 5, 5, 3, 2, 2)$ . Otherwise, suppose without loss of generality that  $C(x_2) = \{3, 5\}$ . We must have that  $S(x_1) = \{1, 2, 4, 6\}$ , so

$$S(x_1) = \{1, 2, 4, 6\}, S(x_2) = \{3, 4, 5, 6\}, S(y_1) = \{1, 2, 3, 4\}, S(y_2) = \{1, 3, 5, 6\}.$$

In this case  $|incomp(x_1, x_2)| = 21$ .

**Case 4:**  $|C(x_1) \cap S(y_2)| = 1$  and  $|S(y_1) \cap S(y_2)| = 1$ .

We can assume, without loss of generality, that  $1 \in S(y_2)$ . Denote  $S(y_2) = \{1, 5, 6, 7\}$ . Since  $C(x_1)$  is bad, we must have that  $S(x_2) \cap \{3, 4\} \neq \emptyset$  and  $|S(x_2) \cap \{5, 6, 7\}| \geq 2$ . Suppose without loss of generality that  $\{3, 5, 6\} \subseteq S(x_2)$ . Since  $C(x_1)$  is bad, we must have that either  $S(x_2) = \{4, 3, 5, 6\}$  or  $S(x_2) = \{7, 3, 5, 6\}$ . It is easy to see that in both cases we have a contradiction to the fact that  $S(x_2)$  contains a bad subset.  $\square$

For every  $i$ ,  $1 \leq i \leq m$ , let  $A_i$  be a sequence of 4 distinct elements. The sequence  $A_1, \dots, A_m$  is called valid if whenever  $c \in A_i \cap A_{i+1}$ , then  $c$  appears in the same position in both  $A_i$  and  $A_{i+1}$ . A valid sequence  $A_1, \dots, A_m$  is called legal if whenever  $c \in A_{i+1} - A_i$ , then  $c \notin A_j$  for every  $j$ ,  $1 \leq j \leq i$ . By a subsequence of  $A_1, \dots, A_m$  we mean a sequence of the form  $A_i, A_{i+1}, \dots, A_j$ , where  $1 \leq i \leq j \leq m$ .

Let  $A_1, \dots, A_m$  be a valid sequence. The pair  $(A_i, A_{i+1})$  contains a change in the  $k^{th}$  position if the elements which appear in the  $k^{th}$  position of  $A_i$  and  $A_{i+1}$  are different. The sequence  $A_1, \dots, A_m$  contains a change in the  $k^{th}$  position if there exists a pair  $(A_i, A_{i+1})$  which contains a change in the  $k^{th}$  position.

Let  $A_1, \dots, A_m$  be a sequence. By  $C_i$  we denote a subset of  $A_i$  of size 2. We say that  $C_1$  and  $C_m$  are compatible if there exist subsets  $\{C_k : 1 < k < m\}$  so that  $C_p \cap C_{p+1} = \emptyset$  for every  $p$ ,  $1 \leq p < m$ . A subset  $C_1$  is called bad if  $C_1$  is not compatible with any  $C_m$ . A subset  $C_1$  is called good if  $C_1$  is compatible with every  $C_m$ . We denote by  $comp(C_1; A_1, \dots, A_m)$  the set which consists of all the subsets  $C_m$  which are compatible with  $C_1$ , and by  $comp(A_1, \dots, A_m)$  the set of all the compatible pairs  $(C_1, C_m)$ . By  $good(A_1, \dots, A_m)$  we denote the set which consists of all the good subsets that  $A_1$  contains.

**Lemma 5.3** *If the valid sequence  $D_1, \dots, D_r$  contains a change in at least 3 positions and there is no  $i$ ,  $1 < i < r - 1$ , for which  $D_i = D_{i+1}$ , then it contains a subsequence  $A_1, \dots, A_m$ , so that the*

sequence  $A_1$  contains at least one good subset. Furthermore, the sequence  $A_1, \dots, A_m$  has at least one of the following properties:

1.  $|good(A_1, \dots, A_m)| \geq 3$ .
2.  $|comp(A_1, \dots, A_m)| > 23$ .
3. The set  $comp(A_1, \dots, A_m)$  is special. If  $m$  is odd, then  $comp(A_1, \dots, A_m)$  has exactly one of the properties  $P_1$  and  $P_2$ . If  $m$  is even then  $comp(A_1, \dots, A_m)$  has either both or none of the properties  $P_1$  and  $P_2$ .

**Proof** We consider the following cases.

**Case 1:** For some  $i$ ,  $|D_i \cap D_{i+1}| \leq 1$ .

In this case  $|good(D_i, D_{i+1})| \geq 3$ .

**Case 2:** For every  $j$ ,  $|D_j \cap D_{j+1}| \geq 2$ , and for some  $i$ ,  $|D_i \cap D_{i+1}| = 2$ .

Assume without loss of generality that the pair  $(D_i, D_{i+1})$  contains a change in the first and second positions. At least one of the pairs  $(D_{i-1}, D_i)$  and  $(D_{i+1}, D_{i+2})$  contains a change in some position. Suppose that the pair  $(D_{i-1}, D_i)$  contains a change in some position. The proof in case the pair  $(D_{i+1}, D_{i+2})$  contains a change in some position is similar. If the pair  $(D_{i-1}, D_i)$  contains a change in at least one of the first and second positions, then surely  $|good(D_{i-1}, D_i, D_{i+1})| \geq 3$ . If the only position in which the pair  $(D_{i-1}, D_i)$  contains a change is either the third or the fourth position, then  $comp(D_{i-1}, D_i, D_{i+1})$  is special, has property  $P_2$ , and does not have property  $P_1$ . If the pair  $(D_{i-1}, D_i)$  contains a change in the third and fourth positions, then  $|comp(D_{i-1}, D_i, D_{i+1})| = 27$ .

**Case 3:** For every  $j$ ,  $|D_j \cap D_{j+1}| \geq 3$ .

Let  $B_1, \dots, B_k$  be a subsequence of  $D_1, \dots, D_r$  which contains a change in at least 3 positions, but no proper subsequence of  $B_1, \dots, B_k$  has this property. This implies that the three pairs  $(B_1, B_2)$ ,  $(B_2, B_3)$  and  $(B_{k-1}, B_k)$  contain a change in three different positions. We can assume, without loss of generality, that the three pairs contain a change in the first, second and third positions respectively. Suppose that  $3 \leq i \leq k - 2$ , and consider the pair  $(B_i, B_{i+1})$ . If this pair contains a change in the first position, then the sequence  $B_2, \dots, B_m$  contains a change in at least 3 positions. If this pair contains a change in the third or fourth position, then the sequence  $B_1, \dots, B_{i+1}$  contains a change in at least 3 positions. Hence, the pair  $(B_i, B_{i+1})$  contains a change in the second position.

If  $k = 4$  then the set  $\text{comp}(B_1, \dots, B_4)$  is special and does not have neither property  $P_1$  nor property  $P_2$ . If  $k > 4$  then  $|\text{good}(B_1, \dots, B_4)| = 3$ .  $\square$

**Lemma 5.4** *If the set  $\text{comp}(A_1, \dots, A_m)$  is special, then both the set  $\text{comp}(A_1, A_1, \dots, A_m)$  and the set  $\text{comp}(A_1, \dots, A_m, A_m)$  are special. The set  $\text{comp}(A_1, A_1, \dots, A_m)$  has property  $P_1$  iff the set  $\text{comp}(A_1, \dots, A_m)$  has property  $P_1$ . The set  $\text{comp}(A_1, A_1, \dots, A_m)$  has property  $P_2$  iff the set  $\text{comp}(A_1, \dots, A_m)$  does not have property  $P_2$ . The set  $\text{comp}(A_1, \dots, A_m, A_m)$  has property  $P_1$  iff the set  $\text{comp}(A_1, \dots, A_m)$  does not have property  $P_1$ . The set  $\text{comp}(A_1, \dots, A_m, A_m)$  has property  $P_2$  iff the set  $\text{comp}(A_1, \dots, A_m)$  has property  $P_2$ .*

**Lemma 5.5** *If  $A_1, A_2, A_2, A_3$  is a legal sequence, then*

$$\text{comp}(A_1, A_2, A_2, A_3) = \text{comp}(A_1, A_3).$$

**Proof** Let  $k_1, \dots, k_n$  be all the positions in which  $A_1, A_2, A_2, A_3$  does not contain a change. It is easy to verify that  $(C, D) \in \text{comp}(A_1, A_3)$  iff there is no  $i$  for which  $C$  contains the  $k_i^{\text{th}}$  element of  $A_1$  and  $D$  contains the  $k_i$  element of  $A_3$ . The same property holds also for  $\text{comp}(A_1, A_2, A_2, A_3)$ .  $\square$

**Lemma 5.6** *If  $A_i, \dots, A_j$  is a subsequence of  $A_1, \dots, A_m$ , then*

$$|\text{comp}(A_1, \dots, A_m)| \geq |\text{comp}(A_i, \dots, A_j)|.$$

**Proof** By induction on  $m$ . If  $m = j - i + 1$ , there is nothing to prove. Suppose that  $m > j - i + 1$ . Assume that  $i > 1$ . The proof in case  $j < m$  is similar. Hence,

$$|\text{comp}(A_1, \dots, A_m)| \geq |\text{comp}(A_2, A_2, A_3, \dots, A_m)| = |\text{comp}(A_2, \dots, A_m)| \geq |\text{comp}(A_i, \dots, A_j)|,$$

where the last inequality follows from the induction hypothesis.  $\square$

**Lemma 5.7** *If  $A_i, \dots, A_j$  is a subsequence of  $A_1, \dots, A_m$ , then*

$$|\text{good}(A_1, \dots, A_m)| \geq |\text{good}(A_i, \dots, A_j)|.$$

**Proof** Similar to the proof of lemma 5.6.  $\square$

**Lemma 5.8** *Suppose that  $i \geq 0$ , and denote by  $F$  the sequence  $A_{i+1}, \dots, A_m$  together with an additional  $A_{i+1}$  as the first element of the sequence in case  $i \equiv 1 \pmod{2}$ . If  $A_{i+1}, \dots, A_m$  is a subsequence of  $A_1, \dots, A_m$  and  $|\text{comp}(A_{i+1}, \dots, A_m)| = |\text{comp}(A_1, \dots, A_m)|$ , then  $\text{comp}(A_1, \dots, A_m)$  is isomorphic to  $\text{comp}(F)$ .*

**Proof** We can assume that  $A_1, \dots, A_m$  is a valid sequence. Suppose that  $i \equiv 1 \pmod{2}$ . The proof in case  $i \equiv 0 \pmod{2}$  is similar. Suppose that  $C_1 \subseteq A_1$ . Denote by  $T$  the subset of  $A_{i+1}$  that appears in the two positions in which  $C_1$  does not appear in  $A_1$ . Since  $A_1, \dots, A_{i+1}$  is a valid sequence, we have that  $C_1$  is compatible with  $T$ . Hence,

$$V = \text{comp}(C_1; A_1, \dots, A_m) \supseteq \text{comp}(T; A_{i+1}, \dots, A_m) = W.$$

Since  $|\text{comp}(A_{i+1}, \dots, A_m)| = |\text{comp}(A_1, \dots, A_m)|$ , we must have that  $V = W$ . It is easy to see now that  $\text{comp}(A_1, \dots, A_m)$  is isomorphic to  $\text{comp}(A_{i+1}, A_{i+1}, \dots, A_m)$ .  $\square$

**Lemma 5.9** *Suppose that  $i, j \geq 0$ . Denote by  $F$  the sequence  $A_{i+1}, \dots, A_{m-j}$  together with an additional  $A_{i+1}$  as the first element of the sequence in case  $i \equiv 1 \pmod{2}$  and an additional  $A_{m-j}$  as the last element of the sequence in case  $j \equiv 1 \pmod{2}$ . If  $A_{i+1}, \dots, A_{m-j}$  is a subsequence of  $A_1, \dots, A_m$  and  $|\text{comp}(A_{i+1}, \dots, A_{m-j})| = |\text{comp}(A_1, \dots, A_m)|$ , then  $\text{comp}(A_{i+1}, \dots, A_{m-j})$  is isomorphic to  $\text{comp}(F)$ .*

**Proof** Apply lemma 5.8 twice.  $\square$

**Lemma 5.10** *Suppose that  $r$  is odd and that  $r \geq 3$ . If the valid sequence  $D_1, \dots, D_r$  contains a change in at least 3 positions, then the sequence  $D_1$  contains at least one good subset. Furthermore, at least one of the following is valid:*

1.  $|\text{good}(D_1, \dots, D_r)| \geq 3$ .
2.  $|\text{comp}(D_1, \dots, D_r)| > 23$ .
3. The set  $\text{comp}(D_1, \dots, D_m)$  is special and has exactly one of the properties  $P_1$  and  $P_2$ .

**Proof** We can assume, without loss of generality, that  $D_1, \dots, D_r$  is legal. Due to lemma 5.5, we can assume that there is no  $i$ ,  $1 < i < r - 1$ , for which  $D_i = D_{i+1}$ . It follows from lemma 5.3 that the sequence  $D_1, \dots, D_r$  contains a subsequence  $A_1, \dots, A_m$ , so that the sequence  $A_1$  contains

at least one good subset. It follows from lemma 5.7 that  $|good(D_1, \dots, D_r)| \geq 1$ . According to lemma 5.3, we consider the following cases:

**Case 1:**  $|good(A_1, \dots, A_m)| \geq 3$ .

It follows from lemma 5.7 that  $|good(D_1, \dots, D_r)| \geq 3$ .

**Case 2:**  $|comp(A_1, \dots, A_m)| \geq 27$ .

It follows from lemma 5.6 that  $|comp(D_1, \dots, D_r)| \geq 27$ .

**Case 3:** The set  $comp(A_1, \dots, A_m)$  is special.

We know that if  $m$  is odd, then  $comp(A_1, \dots, A_m)$  has exactly one of the properties  $P_1$  and  $P_2$ . Furthermore, if  $m$  is even then  $comp(A_1, \dots, A_m)$  has either both or none of the properties  $P_1$  and  $P_2$ . If  $|comp(D_1, \dots, D_r)| > |comp(A_1, \dots, A_m)|$ , then  $|comp(D_1, \dots, D_r)| > 23$ . Suppose that  $|comp(D_1, \dots, D_r)| = |comp(A_1, \dots, A_m)|$ . It follows from lemma 5.9 that  $comp(D_1, \dots, D_r)$  is isomorphic to  $comp(F)$  for some sequence  $F$ . Since  $r$  is odd and using lemma 5.4, it is easy to see that  $comp(D_1, \dots, D_r)$  is special and has exactly one of the properties  $P_1$  and  $P_2$ .  $\square$

**Proof of Theorem 1.16** It is easy to see that a graph  $G$  is  $(4 : 2)$ -choosable iff its core is  $(4 : 2)$ -choosable. Due to theorem 1.15, we need to prove that for every  $m \geq 1$ ,  $\Theta_{2,2,2m}$  is  $(4 : 2)$ -choosable. Suppose that  $m$  is odd and that  $m \geq 3$ . Assume that  $\Theta_{2,2,m-1}$  has vertex set  $V = \{u, v, z_1, \dots, z_m\}$  and contains the three paths  $z_1 - z_2 - \dots - z_m$ ,  $z_1 - u - z_m$ , and  $z_1 - v - z_m$ . For each  $w \in V$ , let  $S(w)$  be a set of size 4. We denote  $A_i = S(z_i)$  for every  $i$ ,  $1 \leq i \leq m$ . We can assume that  $A_1, \dots, A_m$  is a valid sequence.

Suppose first that the sequence  $A_1, \dots, A_m$  contains a change in at most 2 positions. This means that there is a set  $C$  of size 2 so that  $C \subseteq A_i$  for every  $i$ ,  $1 \leq i \leq m$ . From  $A_i$  when  $i$  is odd, choose the subset  $C$ . Complete the choice by choosing a subset of  $S(w) - C$  for every other vertex  $w$ .

Suppose next that the sequence  $A_1, \dots, A_m$  contains a change in at least 3 positions. The graph induced by the set of vertices  $\{z_1, z_m, u, v\} = W$  is isomorphic to  $K_{2,2}$ . Denote  $x_1 = z_1$ ,  $x_2 = z_m$ ,  $y_1 = u$ , and  $y_2 = v$ . We use the same terminology as before.

**Case 1:**  $\{S(w) : w \in W\}$  is not defected.

Suppose without loss of generality that  $S(z_1)$  contains no bad subset. It follows from lemma 5.10 that  $|good(A_1, \dots, A_m)| \geq 1$ , and therefore a choice is possible.

**Case 2:**  $\{S(w) : w \in W\}$  is defected.

According to lemma 5.10, we consider the following cases:

**Case 2a:**  $|good(A_1, \dots, A_m)| \geq 3$ .

It follows from lemma 5.2 that  $S(z_1)$  contains exactly one bad subset, and therefore a choice is possible.

**Case 2b:**  $|comp(D_1, \dots, D_r)| > 23$ .

It follows from lemma 5.2 that  $|incomp(z_1, z_m)| \leq 23$ , and therefore a choice is possible.

**Case 2c:** The set  $comp(D_1, \dots, D_m)$  is special.

We know that  $comp(D_1, \dots, D_m)$  has exactly one of the properties  $P_1$  and  $P_2$ . It is easy to see from lemma 5.2 that the set  $incomp(z_1, z_m)$  does not contain the set  $comp(D_1, \dots, D_m)$ , and therefore a choice is possible.  $\square$

**Proof of Theorem 1.17** Suppose that  $G = (V, E)$  is  $(2mk : mk)$ -choosable for  $k$  odd. We prove that  $G$  is  $2m$ -choosable as well. For each  $v \in V$ , let  $S(v)$  be a set of size  $2m$ . With every color  $c$  we associate a set  $F(c)$  of size  $k$ , such that  $F(c) \cap F(d) = \emptyset$  if  $c \neq d$ . For every  $v \in V$ , we define  $T(v) = \cup_{c \in S(v)} F(c)$ . Since  $G$  is  $(2mk : mk)$ -choosable, there are subsets  $C(v) \subseteq T(v)$ , where  $|C(v)| = mk$  for all  $v \in V$ , and  $C(u) \cap C(v) = \emptyset$  for every two adjacent vertices  $u, v \in V$ .

Fix a vertex  $v \in V$ . Since  $k$  is odd, there is a color  $c \in S(v)$  for which  $|C(v) \cap F(c)| > k/2$ , so we define  $f(v) = c$ . In case  $u$  and  $v$  are adjacent vertices for which  $c \in S(u) \cap S(v)$ , it is not possible that both  $|C(u) \cap F(c)|$  and  $|C(v) \cap F(c)|$  are greater than  $k/2$ . This proves that  $f$  is a proper vertex-coloring of  $G$  assigning to each vertex  $v \in V$  a color in  $S(v)$ .  $\square$

## 6 The complexity of graph choosability

Let  $G = (V, E)$  be a graph. We denote by  $G'$  the graph obtained from  $G$  by adding a new vertex to  $G$ , and joining it to every vertex in  $V$ . Consider the following decision problem:

**GRAPH  $k$ -COLORABILITY**

INSTANCE: A graph  $G = (V, E)$ .

QUESTION: Is  $G$   $k$ -colorable?

The standard technique to show a polynomial transformation from GRAPH  $k$ -COLORABILITY to GRAPH  $(k + 1)$ -COLORABILITY is to use the fact that  $\chi(G') = \chi(G) + 1$  for every graph  $G$ .

However, it is not true that  $ch(G') = ch(G) + 1$  for every graph  $G$ . To see that, we first prove that  $K'_{2,4}$  is 3-choosable.

Suppose that  $K'_{2,4}$  has vertex set  $V = \{v, x_1, x_2, y_1, y_2, y_3, y_4\}$ , and contains exactly the edges  $\{x_i, y_j\}$ ,  $\{v, x_i\}$ , and  $\{v, y_j\}$ . For each  $w \in V$ , let  $S(w)$  be a set of size 3.

**Case 1:** All the sets are the same.

A choice can be made since  $K'_{2,4}$  is 3-colorable.

**Case 2:** There is a set  $S(x_i)$  which is not equal to  $S(v)$ .

Without loss of generality, suppose that  $S(v) \neq S(x_1)$ . For the vertex  $v$ , choose a color  $c \in S(v) - S(x_1)$ , and remove  $c$  from the sets of the other vertices. We can assume that every set  $S(y_j)$  is of size 2 now.

Suppose first that  $S(x_1)$  and  $S(x_2)$  are disjoint. The number of different sets consisting of one color from each of the  $S(x_i)$  is at least 6, and therefore we can choose colors  $c_i \in S(x_i)$ , such that  $\{c_1, c_2\}$  does not appear as a set of  $S(y_j)$ . We complete the choice by choosing for every vertex  $y_j$  a color from  $S(y_j) - \{c_1, c_2\}$ . Suppose next that  $d \in S(x_1) \cap S(x_2)$ . For every vertex  $x_i$  we choose  $d$ , and for every vertex  $y_j$  we choose a color from  $S(y_j) - \{d\}$ .

**Case 3:** There is a set  $S(y_j)$  which is not equal to  $S(v)$ .

Without loss of generality, suppose that  $S(v) \neq S(y_1)$ . For the vertex  $v$ , choose a color  $c \in S(v) - S(y_1)$ , and remove  $c$  from the sets of the other vertices. Suppose first that  $S(x_1)$  and  $S(x_2)$  are disjoint. The number of different sets consisting of one color from each of the  $S(x_i)$  is at least 4, and since  $|S(y_1)| = 3$  we can choose colors  $c_i \in S(x_i)$ , such that  $S(y_j) - \{c_1, c_2\} \neq \emptyset$  for every vertex  $y_j$ . We can complete the choice as in case 2. In case  $S(x_1)$  and  $S(x_2)$  are not disjoint, we proceed as in case 2.

This completes the proof that  $K'_{2,4}$  is 3-choosable. It follows from theorem 1.15 and corollary 1.12 that  $ch(K_{2,4}) = 3$ , and therefore  $ch(K'_{2,4}) = ch(K_{2,4}) = 3$ . The following lemma exhibits a construction which increases the choice number of a graph by exactly 1.

**Lemma 6.1** *Let  $G = (V, E)$  be a graph. If  $H$  is the disjoint union of  $|V| + 1$  copies of  $G$ , then  $ch(H') = ch(G) + 1$ .*

**Proof** Let  $H$  be the disjoint union of the graphs  $\{G_i : 1 \leq i \leq |V| + 1\}$ , where each  $G_i$  is a copy of  $G$ . Suppose that  $H'$  is obtained from  $H$  by joining the new vertex  $v$  to all the vertices of  $H$ .

We claim that if  $G$  is  $k$ -choosable, then  $H'$  is  $(k + 1)$ -choosable. For each  $w \in V(H')$ , let  $S(w)$  be a set of size  $k + 1$ . Choose a color  $c \in S(w)$ , and remove  $c$  from the sets of the other vertices. We can complete the choice since  $G$  is  $k$ -choosable.

We now prove that if  $H'$  is  $k$ -choosable, then  $G$  is  $(k - 1)$ -choosable. By above,  $ch(H') \leq ch(G) + 1 \leq |V| + 1$ . Hence, we can assume that  $k \leq |V| + 1$ . For each  $w \in V$ , let  $S(w)$  be a set of size  $k - 1$ . Without loss of generality  $S(w) \cap \{1, 2, \dots, |V| + 1\} = \emptyset$ . For every  $i$ ,  $1 \leq i \leq |V| + 1$ , on the vertices of the graph  $G_i$  we put the sets  $S(w)$  together with the additional color  $i$ . The vertex  $v$  is given the set  $\{1, 2, \dots, k\}$ . Let  $f$  be a proper vertex-coloring of  $H'$  assigning to each vertex a color from its set. Denote  $f(v) = i$ , then  $f$  restricted to  $G_i$  is a proper vertex-coloring of  $G$  assigning to each vertex  $w \in V$  a color in  $S(w)$ . By corollary 1.9,  $ch(G) + 1 \leq |V|$  if  $G$  is not a complete graph and the proof still goes through if  $|V| + 1$  is replaced by  $|V|$  in the statement of the lemma.  $\square$

**Lemma 6.2 BIPARTITE GRAPH 3-CHOOSABILITY** *is  $\Pi_2^p$ -complete.*

**Proof** It is easy to see that **BG 3-CH**  $\in \Pi_2^p$ . We transform **BG (2,3)-CH** to **BG 3-CH**. Let  $G = (V, E)$  and  $f : V \mapsto \{2, 3\}$  be an instance of **BG (2,3)-CH**. We shall construct a bipartite graph  $H''$  such that  $H''$  is 3-choosable if and only if  $G$  is  $f$ -choosable.

Let  $H$  be the disjoint union of the graphs  $\{G_{i,j} : 1 \leq i, j \leq 3\}$ , where each  $G_{i,j}$  is a copy of  $G$ . Let  $(X, Y)$  be a bipartition of the bipartite graph  $H$ . The graph  $H''$  is obtained from  $H$  by adding two new vertices  $u$  and  $v$ , joining  $u$  to every vertex  $w \in X$  for which  $f(w) = 2$ , and joining  $v$  to every vertex  $w \in Y$  for which  $f(w) = 2$ .

Since  $H$  is bipartite,  $H''$  is also a bipartite graph. It is easy to see that if  $G$  is  $f$ -choosable, then  $H''$  is 3-choosable. We now prove that if  $H''$  is 3-choosable, then  $G$  is  $f$ -choosable. For every  $w \in V$ , let  $S(w)$  be a set of size  $f(w)$ , such that  $S(w) \cap \{1, 2, 3\} = \emptyset$ . For every  $i$  and  $j$ ,  $1 \leq i, j \leq 3$ , on the vertices of the graph  $G_{i,j}$  we put the sets  $S(w)$  with the vertices for which  $f$  is equal to 2 receiving another color as follows: to the vertices which belong to  $X$  we add the color  $i$ , whereas to the vertices which belong to  $Y$  we add the color  $j$ . The vertices  $u$  and  $v$  are both given the set  $\{1, 2, 3\}$ . Let  $f$  be a proper vertex-coloring of  $H''$  assigning to each vertex a color from its set. Denote  $f(u) = i$  and  $f(v) = j$ , then  $f$  restricted to  $G_{i,j}$  is a proper vertex-coloring of  $G$  assigning to each vertex  $w \in V$  a color in  $S(w)$ .  $\square$

**Proof of Theorem 1.18** The proof is by induction on  $k$ . For  $k = 3$ , the result follows from lemma 6.2. Assuming that the result is true for  $k$ ,  $k \geq 3$ , we prove it is true for  $k + 1$ . It is easy to see that **BG**  $(k + 1)$ -**CH**  $\in \Pi_2^p$ . We transform **BG**  $k$ -**CH** to **BG**  $(k + 1)$ -**CH**. Let  $G = (V, E)$  be an instance of **BG**  $k$ -**CH**. We shall construct a bipartite graph  $W$  such that  $W$  is  $(k + 1)$ -choosable if and only if  $G$  is  $k$ -choosable.

Let  $H$  be the disjoint union of the graphs  $\{G_{i,j} : 1 \leq i, j \leq k + 1\}$ , where each  $G_{i,j}$  is a copy of  $G$ . Let  $(X, Y)$  be a bipartition of the bipartite graph  $H$ . The graph  $W$  is obtained from  $H$  by adding two new vertices  $u$  and  $v$ , joining  $u$  to every vertex of  $X$ , and joining  $v$  to every vertex of  $Y$ .

It is easy to see that if  $G$  is  $k$ -choosable, then  $W$  is  $(k + 1)$ -choosable. In a similar way to the proof of lemma 6.2, we can prove that if  $W$  is  $(k + 1)$ -choosable, then  $G$  is  $k$ -choosable.  $\square$

## 7 The strong choice number

Let  $G = (V, E)$  be a graph, and let  $V_1, \dots, V_r$  be pairwise disjoint subsets of  $V$ . We denote by  $[G, V_1, \dots, V_r]$  the graph obtained from  $G$  by appending the union of cliques induces by each  $V_i$ ,  $1 \leq i \leq r$ .

Suppose that  $G = (V, E)$  is a graph with maximum degree at most 1. We claim that  $G$  is strongly  $k$ -choosable for every  $k \geq 2$ . To see that, let  $V_1, \dots, V_r$  be pairwise disjoint subsets of  $V$ , each of size at most  $k$ . The graph  $[G, V_1, \dots, V_r]$  has maximum degree at most  $k$ , and therefore by corollary 1.9 it is  $k$ -choosable.

**Proof of Theorem 1.19** Let  $G = (V, E)$  be a strongly  $k$ -colorable graph. Let  $V_1, \dots, V_r$  be pairwise disjoint subsets of  $V$ , each of size at most  $k + 1$ . Without loss of generality, we can assume that  $V_1, \dots, V_m$  are subsets of size exactly  $k + 1$ , and  $V_{m+1}, \dots, V_r$  are subsets of size less than  $k + 1$ . Let  $H$  be the graph  $[G, V_1, \dots, V_r]$ . To complete the proof, it suffices to show that  $H$  is  $(k + 1)$ -colorable. For every  $i$ ,  $1 \leq i \leq m$ , we define  $W_i = V_i - \{c\}$  for an arbitrary element  $c \in V_i$ , whereas for every  $j$ ,  $m + 1 \leq j \leq r$ , we define  $W_j = V_j$ . Since  $[G, W_1, \dots, W_r]$  is  $k$ -colorable, there exists an independent set  $S$  of  $H$  which is composed of exactly one vertex from each  $V_i$ ,  $1 \leq i \leq m$ . For every  $i$ ,  $1 \leq i \leq m$ , we define  $W'_i = V_i - S$ , whereas for every  $j$ ,  $m + 1 \leq j \leq r$ , we define  $W'_j = V_j$ . Since  $[G, W'_1, \dots, W'_r]$  is  $k$ -colorable, we can obtain a proper  $(k + 1)$ -vertex coloring of  $H$

by using  $k$  colors for  $V - S$  and another color for  $S$ .  $\square$

**Lemma 7.1** *Suppose that  $k, l \geq 1$ . If  $\mathcal{F}$  is a family of  $k + l$  sets of size  $k + l$ , then it is possible to partition  $\mathcal{F}$  into a family  $\mathcal{F}_1$  of  $k$  sets and a family  $\mathcal{F}_2$  of  $l$  sets, to choose for each set  $S \in \mathcal{F}_1$  a subset  $S' \subseteq S$  of size  $k$ , and to choose for each set  $T \in \mathcal{F}_2$  a subset  $T' \subseteq T$  of size  $l$ , so that  $S' \cap T' = \emptyset$  for every  $S \in \mathcal{F}_1$  and  $T \in \mathcal{F}_2$ .*

**Proof** Suppose that  $\mathcal{F} = \{C_1, \dots, C_{k+l}\}$ , and define  $C = \cup_{i=1}^{k+l} C_i$ . For every partition  $\pi$  of  $C$  into the two subsets  $A$  and  $B$ , we denote  $\mathcal{R}(\pi) = \{V \in \mathcal{F} : |V \cap A| > k\}$ ,  $\mathcal{L}(\pi) = \{V \in \mathcal{F} : |V \cap B| > l\}$ , and  $\mathcal{M}(\pi) = \{V \in \mathcal{F} : |V \cap A| = k \text{ and } |V \cap B| = l\}$ . We now start with the partition of  $C$  into the two subsets  $A = C$  and  $B = \emptyset$ , and start moving one element at a time from  $A$  to  $B$  until we obtain a partition  $\pi_1$  of  $C$  into the two subsets  $A$  and  $B$  and a partition  $\pi_2$  into the two subsets  $A' = A - \{c\}$  and  $B' = B \cup \{c\}$ , such that  $|\mathcal{R}(\pi_1)| > k$  and  $|\mathcal{R}(\pi_2)| \leq k$ . It is easy to see that  $\mathcal{L}(\pi_2) \subseteq \mathcal{L}(\pi_1) \cup \mathcal{M}(\pi_1)$ , and therefore  $|\mathcal{L}(\pi_2)| < l$ . We now partition  $\mathcal{M}(\pi_2)$  into two subsets  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , such that  $\mathcal{F}_1 = \mathcal{R}(\pi_2) \cup \mathcal{M}_1$  has size  $k$  and  $\mathcal{F}_2 = \mathcal{L}(\pi_2) \cup \mathcal{M}_2$  has size  $l$ . For every set  $S \in \mathcal{F}_1$  we choose a subset  $S' \subseteq S \cap A'$  of size  $k$ , whereas for every  $T \in \mathcal{F}_2$  we choose a subset  $T' \subseteq T \cap B'$  of size  $l$ . Since  $A'$  and  $B'$  are disjoint, we have that  $S' \cap T' = \emptyset$  for every  $S \in \mathcal{F}_1$  and  $T \in \mathcal{F}_2$ .  $\square$

**Lemma 7.2** *Suppose that  $k, m \geq 1$ . If  $\mathcal{F}$  is a family of  $km$  sets of size  $km$ , then it is possible to partition  $\mathcal{F}$  into the  $m$  subsets  $\mathcal{F}_1, \dots, \mathcal{F}_m$ , each of size  $k$ , and to choose for each set  $S \in \mathcal{F}$  a subset  $S' \subseteq S$  of size  $k$ , so that  $S' \cap T' = \emptyset$  for every  $i \neq j$ ,  $S \in \mathcal{F}_i$  and  $T \in \mathcal{F}_j$ .*

**Proof** By induction on  $m$ . For  $m = 1$  the result is trivial. Assuming that the result is true for  $m$ ,  $m \geq 1$ , we prove it is true for  $m + 1$ . Let  $\mathcal{F}$  be a family of  $k(m + 1)$  sets of size  $k(m + 1)$ . By lemma 7.1, it is possible to partition  $\mathcal{F}$  into a family  $\mathcal{F}_1$  of  $k$  sets and a family  $\mathcal{F}_2$  of  $km$  sets, to choose for each  $S \in \mathcal{F}_1$  a subset  $S' \subseteq S$  of size  $k$ , and to choose for each set  $T \in \mathcal{F}_2$  a subset  $T' \subseteq T$  of size  $km$ , so that  $S' \cap T' = \emptyset$  for every  $S \in \mathcal{F}_1$  and  $T \in \mathcal{F}_2$ . The proof is completed by applying the induction hypothesis on  $\mathcal{F}_2$ .  $\square$

**Proof of Theorem 1.20** Let  $G = (V, E)$  be a strongly  $k$ -choosable graph. Let  $V_1, \dots, V_r$  be pairwise disjoint subsets of  $V$ , each of size at most  $km$ . Let  $H$  be the graph  $[G, V_1, \dots, V_r]$ . To complete the proof, it suffices to show that  $H$  is  $km$ -choosable. For each  $v \in V$ , let  $S(v)$  be a set

of size  $km$ . By lemma 7.2, for every  $i$ ,  $1 \leq i \leq r$ , is it possible to partition  $V_i$  into the  $m$  subsets  $V_{i,1}, \dots, V_{i,m}$ , each of size at most  $k$ , and to choose for each vertex  $v \in V_i$  a subset  $C(v) \subseteq S(v)$  of size  $k$ , so that  $C(u) \cap C(v) = \emptyset$  for every  $p \neq q$ ,  $u \in V_{i,p}$  and  $v \in V_{i,q}$ . Since the graph  $[G, V_{1,1}, \dots, V_{r,m}]$  is  $k$ -choosable, we can obtain a proper vertex-coloring of  $H$  assigning to each vertex a color from its set.  $\square$

**Proof of Theorem 1.21** Apply lemma 7.2 as in proof of theorem 1.20.  $\square$

**Proof of Corollary 1.22** It is proved in [11] that if  $G$  is a 4-regular graph on  $3n$  vertices and  $G$  has a decomposition into a Hamiltonian circuit and  $n$  pairwise vertex disjoint triangles, then  $ch(G) = 3$ . The result follows from theorem 1.21.  $\square$

**Proof of Theorem 1.23** Since  $s\chi(1) = 2$ , we can assume that  $d > 1$ . Suppose first that  $d$  is even, and denote  $d = 2r$ . Construct a graph  $G$  with  $12r - 3$  vertices, partitioned into 8 classes, as follows. Let these classes be  $A, B_1, B_2, C_1, C_2, D_1, D_2, E$ , where  $|A| = |D_1| = |D_2| = 2r$ ,  $|B_1| = |B_2| = r$ ,  $|C_1| = |C_2| = r - 1$ , and  $|E| = 2r - 1$ . Each vertex in  $A$  is joined by edges to each member of  $B_1$  and each member of  $B_2$ . Each member of  $D_1$  is adjacent to each member of  $D_2$ . Consider the following partition of the vertex set of  $G$  into three classes of cardinality  $4r - 1$  each:

$$V_1 = B_1 \cup C_1 \cup D_1, V_2 = B_2 \cup C_2 \cup D_2, V_3 = A \cup E.$$

We claim that  $H = [G, V_1, V_2, V_3]$  is not  $(4r - 1)$ -colorable. In a proper  $(4r - 1)$ -vertex coloring of  $H$ , every color used for coloring the vertices of  $A$  must appear on a vertex of  $C_1 \cup D_1$  and on a vertex of  $C_2 \cup D_2$ . Since  $|C_1 \cup C_2| < |A|$ , there is a color used for coloring the vertices of  $A$  which appears on both  $D_1$  and  $D_2$ . But this is impossible as each vertex in  $D_1$  is adjacent to each member of  $D_2$ . Thus  $s\chi(G) > 4r - 1$  and as the maximum degree in  $G$  is  $2r$ , this shows that  $s\chi(2r) \geq 4r$ .

Suppose next that  $d$  is odd, and denote  $d = 2r + 1$ . Construct a graph  $G$  with  $12r + 3$  vertices, partitioned into 8 classes, as follows. Let these classes be named as before, where  $|A| = |D_1| = |D_2| = 2r + 1$ ,  $|B_1| = r + 1$ ,  $|C_1| = r - 1$ ,  $|B_2| = |C_2| = r$ , and  $|E| = 2r$ . In the same manner we can prove that  $[G, V_1, V_2, V_3]$  is not  $(4r + 1)$ -colorable. Thus  $s\chi(G) > 4r + 1$  and as the maximum degree in  $G$  is  $2r + 1$ , this shows that  $s\chi(2r + 1) \geq 4r + 2$ , completing the proof.  $\square$

**Acknowledgment** We would like to thank Noga Alon for helpful discussions.

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