# On the Union of Cylinders in Three Dimensions*

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## Abstract

*We show that the combinatorial complexity of the union of $n$ infinite cylinders in $\mathbb{R}^3$, having arbitrary radii, is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$; the bound is almost tight in the worst case, thus settling a conjecture of Agarwal and Sharir [3], who established a nearly-quadratic bound for the restricted case of nearly congruent cylinders. Our result extends, in a significant way, the result of Agarwal and Sharir [3], in particular, a simple specialization of our analysis to the case of nearly congruent cylinders yields a nearly-quadratic bound on the complexity of the union in that case, thus significantly simplifying the analysis in [3]. Finally, we extend our technique to the case of "cigars" of arbitrary radii (that is, Minkowski sums of line-segments and balls), and show that the combinatorial complexity of the union in this case is nearly-quadratic as well. This problem has been studied in [3] for the restricted case where all cigars are (nearly) equal-radii. Based on our new approach, the proof follows almost verbatim from the analysis for infinite cylinders, and is significantly simpler than the proof presented in [3].*

## 1 Introduction

Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a collection of $n$ infinite cylinders in $\mathbb{R}^3$. Denote the boundary of $K_i$ by $C_i$, for $i = 1, \ldots, n$, and put $\mathcal{C} = \{C_1, \ldots, C_n\}$; with a slight abuse of notation, we also refer to the elements of $\mathcal{C}$ as cylinders. Let $\mathcal{A}(\mathcal{C})$ denote the three-dimensional arrangement induced by the cylinders in $\mathcal{C}$, i.e., the decomposition of 3-space into vertices, edges, faces, and three-dimensional cells, each being a maximal connected set contained in the intersection of a fixed subcollection of the cylinders of $\mathcal{C}$ and not meeting any other cylinder. Let $\mathcal{U} = \bigcup_{i=1}^{n} K_i$ denote the union of $\mathcal{K}$. The *combinatorial complexity* of $\mathcal{U}$ is the number of vertices, edges and faces of the arrangement $\mathcal{A}(\mathcal{C})$ appearing on the union boundary. The problem studied in this paper is to obtain a nearly-quadratic upper bound on the combinatorial complexity of the union $\mathcal{U}$.

**Previous results.** The problem of determining the combinatorial complexity of the union of geometric objects has received considerable attention in the past twenty years, although most of the earlier work has concentrated on the planar case. See [1] for a recent comprehensive survey of the area.

The case involving *pseudodiscs* (that is, a collection of simply connected planar regions, where the boundaries of any two distinct objects intersect at most twice), arises for Minkowski sums of a fixed convex object with a set of pairwise disjoint convex objects (which is the problem one faces in translational motion planning of a convex robot), and has been studied by Kedem *et al.* [17]. In this case, the union has only linear complexity. Matoušek *et al.* [19, 20] proved that the union of $n$ $\alpha$-*fat* triangles (where each of their angles is at least $\alpha$) in the plane has only $O(n)$ holes, and its combinatorial complexity is $O(n \log\log n)$. The constant of proportionality, which depends on the fatness factor $\alpha$, has later been improved by Pach and Tardos [23]. Extending the study to the realm of curved objects, Efrat and Sharir [13] studied the union of planar convex fat objects. Here we say that a planar convex object $c$ is $\alpha$-*fat*, for some fixed $\alpha > 1$, if there exist two concentric disks, $D \subseteq c \subseteq D'$, such that the ratio between the radii of $D'$ and $D$ is at most $\alpha$. In this case, the combinatorial complexity of the union of $n$ such objects, such that the boundaries of each pair of objects intersect in a constant number of points, is $O(n^{1+\varepsilon})$, for any $\varepsilon > 0$. See also Efrat and Katz [11] and Efrat [10] for related (and slightly sharper) nearly-linear bounds.

In three and higher dimensions, it was shown by Aronov *et al.* [6] that the complexity of the union of $k$ convex polyhedra with a total of $n$ facets in $\mathbb{R}^3$ is $O(k^3 + nk \log k)$, and it can be $\Omega(k^3 + nk\alpha(k))$ in the worst case. The bound was improved by Aronov and Sharir [5] to $O(nk \log k)$ (and $\Omega(nk\alpha(k))$) when the given polyhedra

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are Minkowski sums of a fixed convex polyhedron with $k$ pairwise-disjoint convex polyhedra. (This problem arises in the case of a translating convex polyhedral robot in $\mathbb{R}^3$ amid a collection of polyhedral obstacles.) Boissonnat *et al.* [7] proved that the maximum complexity of the union of $n$ axis-parallel hypercubes in $\mathbb{R}^d$ is $\Theta\left(n^{\lceil d/2 \rceil}\right)$, and that the bound improves to $\Theta\left(n^{\lfloor d/2 \rfloor}\right)$ if all hypercubes have the same size. Pach *et al.* [22] showed that the combinatorial complexity of the union of $n$ nearly congruent arbitrarily oriented cubes in three dimensions is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$ (see also [21] for a subcubic bound on the complexity of the union of fat wedges in 3-space). Agarwal and Sharir [3] have shown that the complexity of the union of $n$ congruent infinite cylinders is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. In fact, the more general problem studied in [3] involves the union of the Minkowski sums of $n$ pairwise disjoint triangles with a ball (where congruent infinite cylinders are obtained when the triangles become lines), and the nearly quadratic bound is extended in [3] to this case as well. Aronov *et al.* [4] showed that the union complexity of $n$ $\kappa$-*round* objects in $\mathbb{R}^3$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where an object $c$ is $\kappa$-round if for each $p \in \partial c$ there exists a ball $B \subset c$ that touches $p$ and its radius is at least $\kappa \cdot diam(c)$. The bound is $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$, for $\kappa$-round objects in $\mathbb{R}^4$. Finally, Ezra and Sharir [14] have recently shown that the complexity of the union of $n$ $\alpha$-*fat* tetrahedra (that is, tetrahedra, each of whose four solid angles at its four respective apices is at least $\alpha$) of arbitrary sizes in $\mathbb{R}^3$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. This result immediately yields a nearly-quadratic bound on the complexity of the union of arbitrary cubes, and thus generalizes the result of Pach *et al.* [22], who showed this bound only for the case where the cubes have nearly equal size lengths. Each of the above known nearly-quadratic bounds (for the three-dimensional case) is almost tight in the worst case.

To recap, all of the above results indicate that the combinatorial complexity of the union in these cases is roughly "one order of magnitude" smaller than the complexity of the arrangement that they induce. While considerable progress has been made on the analysis of unions in three dimensions, the case of the union of infinite cylinders of arbitrary radii has so far been remained elusive.

**Our result.** In this paper we make a significant progress on the problem of bounding the complexity of the union of infinite cylinders of arbitrary radii in 3-space, and show a nearly-quadratic bound on this complexity, thus settling a conjecture of Agarwal and Sharir [3], who showed this bound only for the case where the cylinders are (nearly) congruent. Our bound, which is the first known non-trivial bound for this general problem, is almost tight in the worst case.

Specifically, we show that the complexity of the union of $n$ cylinders as above is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on $\varepsilon$. The analysis is based on some of the ideas presented in [3, 14], and on "$(1/r)$-cuttings", in order to partition space into triangular-prism subcells, so that, on average, the overwhelming majority of the cylinders intersecting a subcell $\Delta$ are "good", in the sense that they behave as *functions* within $\Delta$ with respect to some direction $\rho$. Thus vertices of the union that are incident (only) to good cylinders appear on the boundary of the "sandwich region" enclosed between the "$\rho$-lower" envelope of a subset of these functions and the "$\rho$-upper" envelope of another subset of such functions, and, as shown in [2, 18], the complexity of this region is nearly-quadratic. It then only remains to analyze the number of other types of vertices (incident to some of the few "bad" cylinders that cross $\Delta$), a task which is handled by the divide-and-conquer mechanism based on our cutting (see below for details).

The problem studied in this paper is a generalization of the case where all the cylinders are equal radii - a problem that has been studied by Agarwal and Sharir [3]. We show that a simple specialization of our analysis to that case yields the same asymptotic bound on the complexity of the union as above. The analysis, based on our approach, is significantly simpler than the analysis in [3], and can thus replace that of [3]. (Note that we use a variant of some of the ideas given in [3], however, most of the analysis steps taken in [3] are no longer needed.)

We extend our analysis to the case of "cigars" of arbitrary radii, that is, Minkowski sums of line-segments and balls, and show that the bound on the combinatorial complexity of the union is nearly-quadratic in this case as well. This problem has been studied in [3] for the restricted case where the cigars are equal-radii. Here too, our analysis is significantly simpler than that of [3], and, in particular, the original problem is much easier to extend to this case using our new approach than the approach of [3].

## 2 The Complexity of the Union

### 2.1 Preliminaries and Overview

We first assume that the cylinders in $\mathcal{K}$ are in general position, that is, no axes of two of them are parallel or intersect, no two cylinders are tangent to each other, no curve of intersection of the boundaries of any two cylinders is tangent to a third one, and no four cylinder boundaries meet. Following the arguments in [3], this assumption involves no loss of generality. This general position assumption implies that each vertex of the arrangement of the cylinders lies on exactly three cylindrical boundaries, and is thus incident upon only a constant number of edges and faces. The number of edges and faces on the boundary of the union $\partial\mathcal{U}$ that are not incident upon any vertex is $O(n^2)$. Thus the combinatorial complexity of $\mathcal{U}$ is $O(n^2 + |V(\mathcal{C})|)$, where $V(\mathcal{C})$ is the set of vertices of $\mathcal{A}(\mathcal{C})$ that appear on $\partial U$. Our main result is:

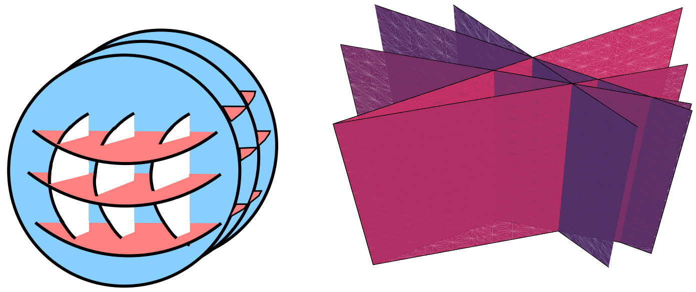



**Figure 1.** (a) Coins in 3-space arranged in a grid-like form. (b) The decomposition into vertical prism cells.

**Theorem 2.1** *For any set $\mathcal{C}$ of $n$ infinite cylinders in $\mathbb{R}^3$, $|V(\mathcal{C})| = O(n^{2+\varepsilon})$, where the constant of proportionality depends on $\varepsilon$. The bound is almost tight in the worst case.*

It is relatively easy (using standard techniques; see, e.g., [24]) to construct a set of $n$ infinite cylinders that yield $\Omega(n^2)$ vertices on the boundary of their union (see also [3] for further details). We thus devote the remainder of this section to deriving the upper bound stated in Theorem 2.1.

We note that it is crucial to assume that the cylinders are infinite. Otherwise, the combinatorial complexity of their union is $\Omega(n^3)$ in the worst case. Indeed, suppose we have a set of $n$ cylinders, each of which with a sufficiently large radius and height that is arbitrarily close to 0. We can now arrange these cylinders in a (three-dimensional) grid-like structure, resulting in $\Omega(n^3)$ holes in the union; see Figure 1(a).

## 2.2 The problem decomposition—an overview

We use a divide-and-conquer approach, based on $(1/r)$-cuttings [9, 8]. Specifically, we project all the cylinders in $\mathcal{K}$ onto the $xy$-plane, thereby obtaining a set $\mathcal{K}^0$ of $n$ infinite strips. Let $\mathcal{L}$ denote the set of their bounding lines. (We assume that the coordinate system is generic, so none of these lines projects to a single point.)

We construct a $(1/r_0)$-cutting of the planar arrangement $\mathcal{A}(L)$, by taking a random sample $R_0$ of $O(r_0 \log r_0)$ lines of $\mathcal{L}$, for some sufficiently large constant parameter $r_0$, constructing the planar arrangement $\mathcal{A}(R_0)$, and triangulating each of its cells, using, e.g., bottom-vertex triangulation. The number of simplices is proportional to the overall complexity of $\mathcal{A}(R_0)$, and is thus $O({r_0}^2 \log^2 r_0)$. The $\varepsilon$-net theory [9, 16] implies that, with high probability, each simplex of the resulting decomposition is crossed by at most $n/r_0$ lines of $\mathcal{L}$. We pick one sample $R_0$ for which this property holds, and fix it in the throughout analysis.

We next lift the sub-triangles of each of the cells of $\mathcal{A}(R_0)$ into vertical prisms. Let $\Xi$ denote the collection of these prism-cells; see Figure 1(b) for an example.

Our goal is to bound the number of intersection vertices of the union in each cell of $\Xi$ separately, and then sum these bounds over all these cells. Fix a cell $\Delta$ of $\Xi$. We classify each cylinder $K \in \mathcal{K}$ that intersects $\Delta$ as being either *wide* in $\Delta$, if the radius $r$ of $K$ satisfies $r \geq w/2$, where $w$ is the *width* of $\Delta$ (that is, the minimum distance between any pair of parallel (vertical) planes that bound $\Delta$), or *narrow*, otherwise. See Figure 2 for an example.

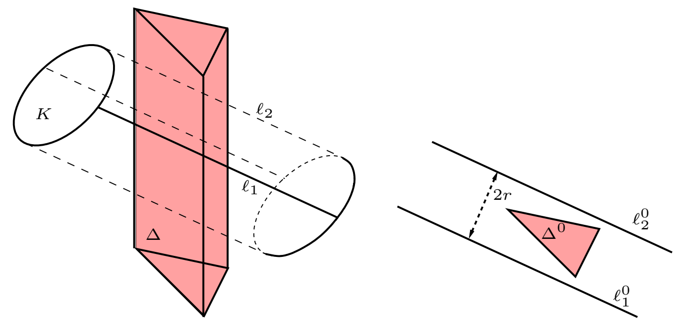
As a consequence, each intersection vertex $v$ of the union that appears in $\Delta$ is classified as either *WWW*, if all three cylinders that are incident to $v$ are wide in $\Delta$, *WWN*, if two of them are wide and one is narrow, *WNN*, if one of them is wide and two are narrow, or *NNN*, if all of them are narrow in $\Delta$. In all four cases, the three relevant cylinders are distinct.

Let $\Delta$ be a prism-cell of $\Xi$. We first observe that if the radius $r$ of a cylinder $K$ that meets $\Delta$ is smaller than $w/2$ (that is, $K$ is narrow in $\Delta$), at least one of the two vertical tangency generator lines $\ell$ (also known as the *silhouette*) of $K$ must intersect $\Delta$. Indeed, if both lines do not meet $\Delta$, the diameter $d := 2r$ of $K$ must be larger than $w$, as is easily verified, contrary to the assumption that $K$ is narrow in $\Delta$. We thus charge the crossing of $K$ and $\Delta$ to that of $\ell$ and $\Delta$, and since $\mathcal{L}$ is in fact the set of the projections of the silhouette lines onto the $xy$-plane, we conclude that $\Delta$ meets at most $\frac{n}{r_0}$ narrow cylinders; see once again Figure 2.

In what follows, we denote by $\Delta$ a fixed prism-cell of $\Xi$, and by $\mathcal{W} = \mathcal{W}^\Delta$ (resp., $\mathcal{N} = \mathcal{N}^\Delta$) the set of the wide (resp., narrow) cylinders within $\Delta$. Put $M_W = M_W^\Delta := |\mathcal{W}^\Delta|$, and $M_N = M_N^\Delta := |\mathcal{N}^\Delta|$. As just discussed, $M_W \leq n$, $M_N \leq \frac{n}{r_0}$.

## 2.3 The overall recursive analysis

We now present a recursive decomposition, where in each step, we are given a (parent) cell $\Delta_0$ and a set $\mathcal{N}^{\Delta_0}$ of narrow cylinders that meet $\Delta_0$, and we construct $O({r_0}^2 \log^2 r_0)$ prism-subcells $\Delta$ of $\Delta_0$ as above. At the first




(a) (b)

**Figure 2.** The cylinder $K$ is wide within the prism-cell $\Delta$. (a) The pair of the silhouette lines $\ell_1$, $\ell_2$ of $K$ do not meet $\Delta$. (b) The projection of this scene onto the $xy$-plane. The lines $\ell_1^0$, $\ell_2^0$ are the respective projections of $\ell_1$, $\ell_2$, $\Delta^0$ is the projection of $\Delta$, and $r$ is the radius of $K$. The strip bounded by $\ell_1^0$, $\ell_2^0$ contains $\Delta^0$ in its interior.

recursive step, $\Delta_0$ equals to the entire three-dimensional space, and $\mathcal{N}^{\Delta_0}$ is the entire set of the input cylinders (and thus $|\mathcal{N}^{\Delta_0}| = n$ at that step). As the space is progressively cut up into subcells, more and more narrow cylinders become wide, and the size of the set $\mathcal{N}^{\Delta_0}$ keeps decreasing. During each step of the recursion, we immediately dispose of any *new* WWW-, WWN-, and WNN-vertices within each subcell $\Delta$ (that is, vertices that were NNN in the parent cell $\Delta_0$), and continue to bound the number of the remaining NNN-vertices recursively. In particular, this implies that we dispose of all the *new* wide cylinders (that is, cylinders that were narrow in $\Delta_0$) in a single step, and thus we do not keep processing them in any further recursive step, but process only the (remaining) narrow ones. The recursion bottoms out when $M_N \le c$, for some absolute constant $c \ge 3$. In this case the number of the remaining intersection vertices of the union within the current cell $\Delta_0$ is $O(1)$.

We show below that the overall number of vertices of the first three types in a subcell $\Delta$ is $O\left((M_N^\Delta + M_W^\Delta)^{2+\varepsilon}\right)$, for any $\varepsilon > 0$. This implies that the overall number of new WWW-, WWN-, and WNN-vertices generated within each subcell $\Delta$ of $\Delta_0$ is $O\left((M_N^{\Delta_0})^{2+\varepsilon}\right)$ (since $M_W^\Delta \le M_N^{\Delta_0}$, $M_N^\Delta \le \frac{M_N^{\Delta_0}}{r_0}$), for a total of $O\left({r_0}^2 \log^2 r_0 (M_N^{\Delta_0})^{2+\varepsilon}\right) = O\left((M_N^{\Delta_0})^{2+\varepsilon}\right)$ such vertices (recall that $r_0$ is constant), over all subcells of $\Delta_0$. Let $U_0(M_N)$ denote the maximum number of intersection vertices that appear on the boundary of the union at a recursive step involving up to $M_N$ narrow cylinders. Then $U_0$ satisfies the following recurrence:

$$U_0(M_N) \le \begin{cases} O\left(M_N^{2+\varepsilon}\right) + O({r_0}^2 \log^2 r_0) U_0(M_N/r_0), & \text{if } M_N > c, \\ O(1), & \text{if } M_N \le c, \end{cases}$$

where $\varepsilon > 0$ is arbitrary, $c \ge 3$ is an appropriate constant as above, and the constant of proportionality in the non-recursive term depends on $r_0$ (and on $\varepsilon$).

Using induction on $M_N$, it is easy to verify that the solution of this recurrence is $U_0(M_N) = O(M_N^{2+\varepsilon})$, for any $\varepsilon > 0$, slightly larger than the $\varepsilon$ in the non-recursive term, but still arbitrarily close to 0, with a constant of proportionality that depends on $\varepsilon$. Substituting the initial value $M_N = n$, we conclude that the overall number of vertices of the union is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, as asserted.

We thus devote the remainder of this section to deriving the bound, stated above, on the number of vertices of the first three types, which is the major part of the analysis.

**The number of WWW-vertices of the union.** Our next goal is to show that the number of WWW-vertices of the union, contained in the fixed prism-cell $\Delta$, is only nearly-quadratic.

Let $H$, $H'$ be the pair of (parallel and vertical) planes that determine the width $w$ of $\Delta$. We use a simple variant of the analysis given in [3], in order to show that the WWW-vertices of $\Delta$ (in fact, it is sufficient to assume that they lie in the slab bounded by $H$, $H'$) appear on the boundary of the *sandwich region* enclosed between a lower envelope of a collection of portions of the wide cylinders and an upper envelope of another such collection. The analysis of [2] (see also [18]) implies that this complexity is only nearly-quadratic. For the sake of completeness, we repeat some of the details given in [3], and modify them to fit our setup.

Let $\kappa$ be a sufficiently large constant, whose value will be fixed shortly. We partition each of the wide cylinder boundaries into $\kappa$ canonical strips (bounded by generator lines, parallel to the cylinder axis), each with an angular span of $2\pi/\kappa$ in the cylindrical coordinate frame induced by the cylinder.

Let $\mathbb{S}^2$ be the unit sphere of directions in $\mathbb{R}^3$. We say that a direction $\rho \in \mathbb{S}^2$ is *good* for a strip $\tau$ if the following two conditions hold.

**(i)** The angle between $\rho$ and the (outer) normal $\mathbf{n}$ of either of the planes $H$, $H'$ is smaller than $\pi/2 - \pi/\kappa$.

**(ii)** Each line tangent to (the relative interior of) $\tau$ forms an angle of at least $\pi/\kappa$ with $\rho$.

In other words, the first condition implies that the inner product $\rho \cdot \mathbf{n}$ is sufficiently large (in particular, they are not orthogonal), and the second condition implies that when we enter into the cylinder $K$, which contains $\tau$ as a bounding strip, in the $\rho$-direction from a point on $\tau$, we do not remain close to the boundary of $K$ (and, in particular, to $\tau$). Intuitively, these two conditions imply that when we move in the $\rho$-direction, we expect to leave the strip enclosed between $H$, $H'$ relatively soon, but make a sufficiently long way inside $K$.

We say that $\rho$ is a *good direction* for a vertex $v$ of the union, incident upon three canonical strips, if it is good for each of these strips; see Figure 3. As observed in [3], the set $B_\tau$ of bad directions for a fixed strip $\tau$ is the union $B_1 \cup B_2$, where

**(a)** $B_1$ is contained in a spherical band consisting of all points lying at spherical distance at most $\pi/\kappa$ from the great circle on $\mathbb{S}^2$ normal to $\mathbf{n}$ (in fact, this circle is obtained by intersecting $\mathbb{S}^2$ with a plane parallel to $H$ (or $H'$) through its center). The area of $B_1$ is $4\pi \sin(\pi/\kappa)$.

**(b)** $B_2$ is contained in a spherical band consisting of all points lying at spherical distance at most $2\pi/\kappa$ from a great circle on $\mathbb{S}^2$; see [3] for the easy proof. The area of $B_2$ is thus $4\pi \sin(2\pi/\kappa)$.

It thus follows (see [3] for further details) that the area of the set of good directions for $v$ is at least $4\pi[1 - 3\sin(2\pi/\kappa) - \sin(\pi/\kappa)]$.

As observed in [3], the set of the good directions contains a spherical cap of some constant opening angle $\delta$, if $\kappa$ is sufficiently large. It then implies that each vertex $v$ on the boundary of the union has at least one good direction, taken from a set $\mathcal{Z}$ of $O\left(1/\delta^2\right)$ points on $\mathbb{S}^2$, for some sufficiently small value of $\delta$ that depends on $\kappa$. Note that the above considerations do not assume that $v$ is a vertex of type WWW. For each fixed $\rho \in \mathcal{Z}$, let $V_\Delta(\rho)$ be the subset of all vertices (of any of the above four types) of the union that lie inside $\Delta$, for which $\rho$ is a good direction.

We now partition the slab $\sigma$ (and thus $\Delta$) enclosed between $H$, $H'$ into $t > \frac{1}{\sin^2(\pi/\kappa)}$ equal subslabs $\sigma'$ by adding $t$ planes parallel to $H$ inside $\sigma$ at equal distances. Our next goal is to show that the strips $\tau$, for which $\rho$ is a good direction, behave as functions (with respect to that direction) within $\sigma'$. The following lemma is a simple variant of [3, Lemma 2.9].

**Lemma 2.2** *Let $\sigma'$ be a subslab of $\sigma$, and let $v$ be a WWW-vertex of the union, contained in $\sigma' \cap \Delta$ and incident upon three strips $\tau_1$, $\tau_2$, $\tau_3$. Let $\rho \in \mathcal{Z}$ be a good direction for $v$. Then any line parallel to $\rho$ intersects $\tau_1$ in at most one point. Moreover, for any point $u \in \tau_1 \cap \sigma'$, in which we enter into the cylinder $K_1$ bounded by $\tau_1$ in the direction parallel to $\rho$, we reach $\partial\sigma'$ before exiting $K_1$. Similar properties hold for $\tau_2$, $\tau_3$.*

**Proof:** Suppose first that there is a line parallel to $\rho$ that intersects $\tau_1$ twice. This would imply that $\tau_1$ must have a tangent in the $\rho$-direction, as is easily verified, contrary to the assumption that $\rho$ is a good direction.

As to the second assertion of the lemma, let $u'$ be the other intersection point of $\partial K_1$ and the line passing through $u$ and parallel to $\rho$. It is easy to verify that $|uu'|$ is minimized when $uu'$ is orthogonal to the axis of $K_1$ and forms an angle $\pi/\kappa$ with the tangent plane to $K_1$ at $u$. It thus follows that $|uu'| \geq 2r_1 \sin(\pi/\kappa)$, where $r_1$ is the radius of $K_1$. On the other hand, since $uu'$ forms an angle of at most $\pi/2 - \pi/\kappa$ with the normal $\mathbf{n}$ (to either of the two bounding planes of $\sigma'$), it follows that the length of the projection of $uu'$ on $\mathbf{n}$ is at least $|uu'|\sin(\pi/\kappa) \geq 2r_1 \sin^2(\pi/\kappa) \geq w \sin^2(\pi/\kappa)$, where $w$ is the width of $\sigma$, as above. Thus if we choose $t > \frac{1}{\sin^2(\pi/\kappa)}$, $u'$ must lie outside $\sigma'$. This completes the proof of the lemma. □

We now denote by $\mathcal{T}_{\sigma'}(\rho)$, the set of canonical strips $\tau$ that cross the subslab $\sigma'$ and have at least one vertex in $V_\Delta(\rho)$. We clip all the strips to $\Delta \cap \sigma'$. A (clipped) strip $\tau$ in $\mathcal{T}_{\sigma'}(\rho)$ is *$\rho$-upper* (resp., *$\rho$-lower*) if, for any point $u \in \tau$, the point $u + \alpha\rho$ lies in the exterior (resp., interior) of the cylinder $K$ whose boundary contains $\tau$, for sufficiently small values of $\alpha > 0$. Let $\mathcal{T}^+_{\sigma'}(\rho)$ (resp., $\mathcal{T}^-_{\sigma'}(\rho)$) be the set of the $\rho$-upper (resp., $\rho$-lower) strips of $\mathcal{T}_{\sigma'}(\rho)$. The *$\rho$-upper envelope* of $\mathcal{T}^+_{\sigma'}(\rho)$ is the set of points $u$ on the strips of $\mathcal{T}^+_{\sigma'}(\rho)$, such that a ray from $u$ in the $(+\rho)$-direction does not meet any other strip in $\mathcal{T}^+_{\sigma'}(\rho)$. The *$\rho$-lower envelope* of $\mathcal{T}^-_{\sigma'}(\rho)$ is defined analogously.

Following the analysis of [3] and Lemma 2.2, each WWW-vertex $v \in V_\Delta(\rho)$ must lie on the boundary of the sandwich region enclosed between the $\rho$-upper envelope of the strips in $\mathcal{T}^+_{\sigma'}(\rho)$ and the $\rho$-lower envelope of the strips in $\mathcal{T}^-_{\sigma'}(\rho)$, and, according to the results of [2, 18], the number of these vertices $v$ is $O\left(M_W^{2+\varepsilon}\right)$, for any $\varepsilon > 0$, with a constant of proportionality that depends on $\kappa$ and $\varepsilon$. Repeating the analysis for all subslabs $\sigma'$, and all good directions $\rho \in \mathcal{Z}$, we obtain a similar bound on the overall number of WWW-vertices of the union that are contained in $\Delta$.

**Remark:** We note that it is crucial to process only the strips that have at least one vertex $v$ of the union for which $\rho$ is a good direction, and construct the $\rho$-lower and $\rho$-upper envelopes only with respect to these strips. All the remaining strips should not be considered at that step, as they may violate the envelope properties. In particular, we ignore wide cylinders whose corresponding strips do not contain any such vertices $v$.

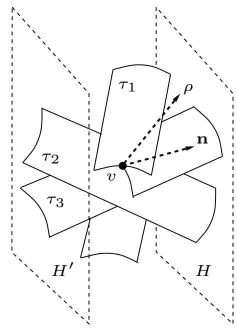



**Figure 3.** The vector $\rho$ is a good direction for the vertex $v$ of the union, incident upon three cylinder strips $\tau_1$, $\tau_2$, $\tau_3$, where $\mathbf{n}$ is the normal to the planes $H$, $H'$.

**The number of WWN-vertices of the union.** We next establish a nearly-quadratic bound on the number of WWN-vertices of the union, with respect to a fixed prism-cell $\Delta$. Let us first fix a good direction $\rho \in \mathcal{Z}$ and a subslab $\sigma'$. We bound the number of WWN-vertices in each subcell $\Delta' = \Delta \cap \sigma'$ separately, and then sum these bounds over all these subcells. Note that some of the narrow cylinders within $\Delta$ may become wide in $\Delta'$, nevertheless, we regard them as narrow; this does not effect the analysis. We thus have, at each step of the analysis, a prism-cell $\Delta'$ (confined to a slab $\sigma'$), a set $\mathcal{W} = \mathcal{W}^{\Delta'}$ of $M_W$ wide cylinders, and a set $\mathcal{N} = \mathcal{N}^{\Delta'}$ of $M_N$ narrow cylinders.

Since each WWN-vertex $v \in V_{\Delta'}(\rho)$ (where $V_{\Delta'}(\rho)$ is the subset of all vertices of the union that lie inside $\Delta'$ for which $\rho$ is a good direction) involves a wide cylinder, it must lie on the boundary of the sandwich region enclosed between the $\rho$-upper and $\rho$-lower envelopes of the strips in $\mathcal{T}^+_{\sigma'}(\rho)$ and $\mathcal{T}^-_{\sigma'}(\rho)$, respectively. Moreover, $v$ is obtained as the intersection of an edge of the sandwich region and a narrow cylinder in $\Delta'$.

We now apply a second decomposition step, in which we partition $\Delta'$ into subprisms $\Delta''$ (clipped to within $\Delta'$), so that each of them contains a relatively small number of both narrow and wide cylinders, as follows.

We draw two random samples $R^+(\rho) \subseteq \mathcal{T}^+_{\sigma'}(\rho)$, $R^-(\rho) \subseteq \mathcal{T}^-_{\sigma'}(\rho)$ of $O(r \log r)$ strips each, for some sufficiently large constant parameter $r$, and construct the overlay $\mathcal{M}_\rho$ of the respective minimization diagrams of the $\rho$-upper and $\rho$-lower envelopes of the strips in $R^+(\rho)$ and $R^-(\rho)$; these diagrams are obtained on a plane $H_{\rho^\perp}$ perpendicular to the $\rho$-direction. By the results of [2, 18], the overall complexity of $\mathcal{M}_\rho$ is $O(r^{2+\varepsilon})$, for any $\varepsilon > 0$. We next refine the overlay $\mathcal{M}_\rho$ by projecting the narrow cylinders in $\Delta'$ onto the plane $H_{\rho^\perp}$, taking a random sample $R$ of $O(r \log r)$ of their bounding lines[1], and then overlaying $\mathcal{M}_\rho$ with the lines in $R$; we thus draw three random samples in total. Let $\mathcal{M}^R_\rho$ be the refined decomposition. We first show:

**Lemma 2.3** *The overall number of vertices in $\mathcal{M}^R_\rho$ is $O(r^{2+\varepsilon})$, for any $\varepsilon > 0$.*

**Proof:** The overall complexity of the arrangement $\mathcal{A}(R)$ of the lines in $R$ is $O\left(r^2 \log^2 r\right)$, and, as noted above, the overall complexity of $\mathcal{M}_\rho$ is $O(r^{2+\varepsilon})$, for any $\varepsilon > 0$. We next show that the overall number of edge-crossings between $\mathcal{M}_\rho$ and $\mathcal{A}(R)$ is $O(r \log r \lambda_6(r \log r)) = O(r^2 \text{polylog} r)$, where $\lambda_s(q)$ is the maximal length of Davenport-Schinzel sequences of order $s$ on $q$ symbols (see [24]). This will assert the bound in the lemma.

Indeed, let $\mathcal{H}_R$ be the set of the $O(r \log r)$ planes containing the lines in $R$ and parallel to the $\rho$-direction. Each plane $H \in \mathcal{H}_R$ intersects each of the strips in $R^+(\rho)$, $R^-(\rho)$ in an elliptic arc (clipped to within $\Delta'$). Let $\Gamma^+_H$ be the set of these $\rho$-upper arcs, and let $\Gamma^-_H$ be the set of these $\rho$-lower arcs. Since $H$ is parallel to the $\rho$-direction, it must meet each of the edges $e$ on the boundary of the sandwich region, enclosed between the $\rho$-upper envelope of the strips in $R^+(\rho)$ and the $\rho$-lower envelope of the strips in $R^-(\rho)$, in a vertex that appears along the boundary of the sandwich region enclosed between the upper envelope of the arcs in $\Gamma^+_H$ and the lower envelope of the arcs in $\Gamma^-_H$ (in the coordinate frame of $H$). The overall number of these vertices is $O(\lambda_6(r \log r))$, since each pair of such arcs intersect in at most four points (see, e.g., [24] for details). Each of these vertices is projected on the plane $H_{\rho^\perp}$, perpendicular to the $\rho$-direction, to an edge-crossing between $\mathcal{M}_\rho$ and $\mathcal{A}(R)$. In fact, by similar arguments, we can conclude that the overall number of edges $e$ of either of the $\rho$-upper envelope of the strips in $R^+(\rho)$, or $\rho$-lower envelope of the strips in $R^-(\rho)$ that meet $H$ is also $O(\lambda_6(r \log r))$

[1] From technical reasons, at the *first* step of this decomposition we need to ignore the cylinders that were narrow in $\Delta$ and became wide in $\Delta'$. This does not affect the analysis, as these cylinders are wide at the next decomposition step.

(for technical reasons, we need that latter bound, since $\mathcal{M}_\rho$ corresponds to the overlay of these two envelopes, which may somewhat be different from the sandwich region enclosed between these two envelopes). Summing this bound over all planes $H \in \mathcal{H}_R$ yields the asserted bound. □

We next triangulate each of the cells of $\mathcal{M}_\rho^R$ (in the coordinate frame of $H_{\rho^\perp}$), and lift each of these simplices in the $\rho$-direction, thereby obtaining a collection $\Xi$ of $O(r^{2+\varepsilon})$ triangular prism-subcells. We collect all these prisms that lie in the sandwich region enclosed between the $\rho$-upper envelope of the strips in $R^+(\rho)$ and the $\rho$-lower envelope of the strips in $R^-(\rho)$. Each of these prisms is bounded in the $\rho^-$- and $\rho^+$-directions (either by the boundary of $\Delta'$ or) by strips $\tau^+ \in R^+(\rho)$ appearing on the $\rho$-upper envelope, and $\tau^- \in R^-(\rho)$ appearing on the $\rho$-lower envelope, respectively. We thus obtain a decomposition of the above sandwich region (within $\Delta'$) into $O(r^{2+\varepsilon})$ triangular prisms.

We observe that each cylinder $K$ that is wide in $\Delta'$ continues to be wide in each clipped prism-subcell $\Delta''$, since the width of $\Delta''$ cannot exceed that of $\Delta'$ (or of $\Delta$). By the cutting properties, with high probability, each prism-subcell $\Delta''$ of the resulting decomposition is crossed by at most $O(M_W/r)$ strips in $\mathcal{T}^+_{\sigma'}(\rho)$ and by at most $O(M_W/r)$ strips in $\mathcal{T}^-_{\sigma'}(\rho)$, with a constant of proportionality that depends linearly on $\kappa$. We pick a pair of samples $R^+(\rho)$, $R^-(\rho)$ for which this property holds, and fix them in the throughout analysis.

We next bound the number of wide cylinders within a prism-subcell $\Delta''$. The strips in $\mathcal{T}^+_{\sigma'}(\rho) \cup \mathcal{T}^-_{\sigma'}(\rho)$ are said to be *good*, and all the remaining strips (of the wide cylinders within $\Delta'$) are said to be *bad*. Clearly, the number of wide cylinders $K$ in $\Delta''$ that have at least one good strip that meets $\Delta''$ is at most $M_W/r$, since we can charge $K$ to that good strip. If $K$ has only bad strips that meet $\Delta''$, we ignore $K$ in the subsequent analysis of $\Delta''$. This does not violate the bound on the number of (original) WWN-vertices $v$ in $\Delta''$, since each such vertex is obtained on the boundary of the sandwich region enclosed between the $\rho$-upper and $\rho$-lower envelopes of the good strips, and thus $v$ does not lie on $\partial K \cap \Delta''$. We thus continue the subsequent analysis within $\Delta''$ with at most $M_W/r$ wide cylinders.

Each cylinder $K$ that is narrow in $\Delta'$ either becomes wide in $\Delta''$, or continues to be narrow in that cell. Let $\mathcal{L}_\rho$ be the set of the $\rho$-*silhouette* lines of the narrow cylinders within $\Delta'$ (that is, generator lines in the $\rho$-direction). Applying similar arguments as above, a narrow cylinder $K$ within a prism-subcell $\Delta''$ of the decomposition must have a $\rho$-silhouette line $\ell$, whose projection on $H_{\rho^\perp}$ meets the projection of $\Delta''$ on that plane. We thus charge the crossing of $K$ and $\Delta''$ to that of the respective projections of $\ell$ and $\Delta''$, and conclude that the number of narrow cylinders within $\Delta''$ is at most $M_N/r$, with high probability. We pick one sample $R$ for which this property holds, and fix it in the throughout analysis. We have thus shown:

**Lemma 2.4** *The number of wide cylinders within a subcell $\Delta''$ of $\Delta'$ that contribute WWN-vertices to the union is at most $M_W/r$, and the number of narrow cylinders in that subcell is at most $M_N/r$.*

We bound the number of WWN-vertices of the union by applying a recursive decomposition scheme, where in each step we construct $O(r^{2+\varepsilon})$ prism-cells $\Delta''$, using $(1/r)$-cutting of the good strips (taken from the current set of wide cylinders that we process in $\Delta'$) and the boundary projections of the narrow cylinders, as above, where each subproblem consists of at most $M_W/r$ wide cylinders, and at most $M_N/r$ narrow cylinders. In each step we dispose immediately of all the *new* WWW-vertices in $\Delta''$ (that is, vertices that were WWN in the parent cell $\Delta'$, and have just became WWW) and continue to process in recursion the remaining WWN-vertices. Thus cylinders that were narrow in the parent cell of $\Delta''$ and became wide in $\Delta''$ need not be processed in any further recursive step, the only wide cylinders that we process are those of the original problem. The recursion bottoms out when either $M_W \le c$, or $M_N \le c$, for some absolute constant $c \ge 3$. We then bound by brute-force the number of the (remaining) WWN-vertices, and thus obtain an overall bound of $O(M_W^2 M_N) = O(M_W^2 + M_N)$ on the number of these vertices.

The number of new WWW-vertices in $\Delta''$ is $O\left((M_N + M_W)M_W^{1+\delta}\right)$, for any $\delta > 0$. Indeed, the number of new wide cylinders within $\Delta''$ is at most $M_N$, and the number of the old wide cylinders (cylinders that were wide in the parent cell of $\Delta''$) is at most $M_W$. Assume first that $M_N \le M_W$. Following the considerations given in the WWW-case, we conclude that the number of new WWW-vertices in $\Delta''$ is $O(M_W^{2+\delta})$, for any $\delta > 0$. If $M_N > M_W$, we partition the set of the narrow cylinders into $\left\lceil \frac{M_N}{M_W} \right\rceil$ (roughly) equal subsets, each of which containing at most $M_W$ elements. Each new WWW-vertex $v$ involves a pair of old wide cylinders and a new wide cylinder within $\Delta''$. We thus apply the nearly-quadratic bound in $\Delta''$ on the set of the old wide cylinders and each of the subsets of new wide cylinders separately, and then sum these bounds over all subsets, thereby obtaining an overall bound of $O(M_N M_W^{1+\delta})$, for any $\delta > 0$ (see also [3, 14] for similar considerations). The bound now follows. Since $r$ is constant, we obtain that the overall number of new WWW-vertices, over all prism-subcells $\Delta''$, is also $O\left((M_N + M_W)M_W^{1+\delta}\right)$.

Let $U_1(M_N, M_W)$ denote the maximum number of WWN-vertices that appear on the boundary of the union at a recursive step involving up to $M_N$ narrow cylinders, and $M_W$ wide cylinders. Then $U_1$ satisfies the following recurrence:

$$U_1(M_N, M_W) \le \begin{cases} O\left((M_N + M_W)M_W^{1+\delta}\right) \quad + \\ \quad O(r^{2+\varepsilon})U_1\left(\frac{M_N}{r}, \frac{M_W}{r}\right), \\ \qquad\qquad\qquad \text{if } \min\{M_N, M_W\} > c, \\ \\ O(M_N + M_W^2), \\ \qquad\qquad\qquad \text{if } \min\{M_N, M_W\} \le c, \end{cases}$$

where $\varepsilon > 0$, $\delta > 0$ are arbitrary, $c \ge 3$ is an appropriate constant, and where the constant of proportionality in the first expression depends on ($\varepsilon$, $\delta$, $\kappa$ and on) $r$. It is straightforward to verify (see also [15], [14]), that the solution of this recurrence is $U_1(M_N, M_W) = O((M_N + M_W)M_W^{1+\varepsilon})$, for any $\varepsilon > 0$ (slightly larger than $\delta$ in the non-recursive term and the $\varepsilon$ in the recursive term, but still arbitrarily close to 0), with a constant of proportionality that depends on $\varepsilon$ and on $\kappa$.

Summing this bound over all $\Delta' = \Delta \cap \sigma'$, and over all good directions $\rho \in \mathcal{Z}$, we obtain that the overall number of WWN-vertices in $\Delta$ is $O\left((M_N + M_W)M_W^{1+\varepsilon}\right)$, for any $\varepsilon > 0$, with a constant of proportionality that depends on $\varepsilon$ and $\kappa$.

**Remark:** The recursive mechanism is such that we sample the strips that belong to the wide cylinders and not the actual wide cylinders, since we have to take only those strips that have vertices $v$ of the union, for which $\rho$ is a good direction, and then recursively construct out of them the sandwich region that they induce. The role of the wide cylinders is in bounding the number of new WWW-vertices created in a subcell $\Delta''$ of the current cell $\Delta'$. We thus recurse with the wide cylinders, but construct the cutting with their good strips.

**The number of WNN-vertices of the union.** As in the WWN-case, we fix a good direction $\rho \in \mathcal{Z}$ and a subslab $\sigma'$, and bound the number of WNN-vertices in each subcell $\Delta' = \Delta \cap \sigma'$ separately. We then sum these bounds over all these subcells, and over all $\rho$. Here too, we observe that since each WNN-vertex $v \in V_\Delta(\rho)$ involves a wide cylinder, it must lie on the boundary of the sandwich region enclosed between the $\rho$-upper and $\rho$-lower envelopes of the strips in $\mathcal{T}^+_{\sigma'}(\rho)$ and $\mathcal{T}^-_{\sigma'}(\rho)$, respectively.

Using similar notations as in the WWN-case, we apply a similar decomposition, where we recursively construct $O(r^{2+\varepsilon})$ cells $\Delta''$, each of which meets at most $M_W/r$ wide cylinders and at most $M_N/r$ narrow cylinders. At each recursive step, we dispose immediately of all the *new* WWW- and WWN-vertices (that is, vertices $v$ that were WNN-vertices at the parent cell of $\Delta''$, and have become either WWW- or WWN-vertices at $\Delta''$), and continue to process in recursion the (remaining) WNN-vertices. Applying similar considerations as above, the overall number of new WWW- and WWN-vertices in the subcells $\Delta''$, over all $\Delta''$, is $O\left((M_N + M_W)M_N^{1+\delta}\right)$, for any $\delta > 0$. At the bottom of the recursion (when either $M_W \le c$, or $M_N \le c$, for some absolute constant $c \ge 3$), we bound by brute-force the number of the (remaining) WNN-vertices, obtaining an overall bound of $O(M_W M_N^2) = O(M_W + M_N^2)$ on the number of these vertices. Letting $U_2(M_N, M_W)$ denote the maximum number of WNN-vertices that appear on the boundary of the union at a recursive step involving up to $M_N$ narrow cylinders and $M_W$ wide cylinders, we have:

$$U_2(M_N, M_W) \le \begin{cases} O\left((M_N + M_W)M_N^{1+\delta}\right) \quad + \\ \quad O(r^{2+\varepsilon})U_2\left(\frac{M_N}{r}, \frac{M_W}{r}\right), \\ \qquad\qquad\qquad \text{if } \min\{M_N, M_W\} > c, \\ \\ O(M_N^2 + M_W), \\ \qquad\qquad\qquad \text{if } \min\{M_N, M_W\} \le c, \end{cases}$$

where, as above, $\varepsilon > 0$, $\delta > 0$ are arbitrary, $c \ge 3$ is an appropriate constant, and where the constant of proportionality in the first expression depends on ($\varepsilon$, $\delta$, $\kappa$ and on) $r$. Here too, it is easy to verify that the solution of this recurrence is $U_2(M_N, M_W) = O\left((M_N + M_W)M_N^{1+\varepsilon}\right)$, for any $\varepsilon > 0$ (slightly larger than $\delta$ in the non-recursive term and the $\varepsilon$ in the recursive term, but still arbitrarily close to 0), with a constant of proportionality that depends on $\varepsilon$ and on $\kappa$.

**Remarks:** (1) The WWN- and WNN-cases, albeit being similar, cannot be handled at the same recursive step, and need to be analyzed separately. When a WNN-vertex $v$ becomes WWN in a subcell $\Delta''$, we need to consider the envelopes with respect to a (possibly, new) good direction $\rho'$, not necessarily equal to $\rho$. These envelopes correspond to the old and the *new* wide cylinders in $\Delta''$, and we need to continue with the recursive decomposition described for the WWN-case, according to the direction $\rho'$, with the set of these wide cylinders and the narrow ones that have survived when passing from $\Delta'$ to $\Delta''$.

(2) The only part of the analysis in which we explicitly use the geometric structure of the cylinders is in the WWW-case. The analysis for the WWN- WNN-cases only uses the property that the vertices lie on the boundary of the sandwich region enclosed between two envelopes of the good strips.

**The case of congruent cylinders.** We now study the case where all the cylinders have equal radii, and show that a simple specialization of our approach leads to a nearly-quadratic bound on the complexity of their union.

We only need to make the following simple observation. We apply the decomposition described in Section 2.2, and observe that, since all cylinders are equal radii, *all* of them are either wide or narrow in a fixed cell $\Delta$ of the decomposition. Thus all vertices of the union inside $\Delta$ are either of type WWW or of type NNN. We thus construct this decomposition recursively, where in each step we dispose of all the new WWW-vertices within each subcell $\Delta$, and continue to bound the number of NNN-vertices recursively. Here we do not need to apply the second decomposition step described for the WWN- and WNN-cases.

We note that these arguments can easily be extended to the case of nearly-congruent cylinders, that is, when the radii of the cylinders are different, but vary between some constant $\alpha < 1$ and 1. We omit the easy proof in this version.

## 3 An extension to the case of cigars.

We now extend the analysis to the following case. We are given a set $\mathcal{S} = \{s_1, \ldots, s_n\}$ of $n$ line-segments in 3-space and a set $\mathcal{B} = \{B_1, \ldots, B_n\}$ of $n$ balls of arbitrary radii. The set $\mathcal{C} = \{s_1 \oplus B_1, \ldots, s_n \oplus B_n\}$ consisting of the *Minkowski sums* of $s_i \in S$ and $B_i \in B$ (that is, $s_i \oplus B_i = \{x + y \mid x \in s_i, y \in B_i\}$), for $i = 1, \ldots, n$, is the input set to our problem. The elements of $\mathcal{C}$ are also referred to as *cigars* in this case. We show:

**Theorem 3.1** *The combinatorial complexity of the union of $n$ cigars of arbitrary radii in 3-space is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on $\varepsilon$. The bound is almost tight in the worst case.*

We devote this section to the proof of Theorem 3.1.

We first assume that the cigars in $\mathcal{C}$ are in general position. As discussed in [3], this excludes the cases where no pair of the segments in $\mathcal{S}$ are parallel or intersect, that no two cigars in $\mathcal{C}$ are tangent to each other, that no curve of intersection of the boundaries of any two cigars is tangent to a third one, that no triple intersection of the boundaries of the cigars lie on any circle separating the cylindrical and spherical portions of one of them, and that no four boundaries meet at a point.

As a result, each vertex $v$ of the union $\mathcal{U} = \bigcup_{C \in \mathcal{C}} C$ is the intersection point of either (i) three cylindrical boundaries, (ii) a pair of cylindrical boundaries and one spherical boundary, (iii) a cylindrical boundary and a pair of spherical boundaries, or (iv) three spherical boundaries. We refer to each of these vertices as being of type *CCC*, *CCS*, *CSS*, or *SSS*, respectively. In the analysis of [3], these four cases are handled separately, however, using our new approach, we show that all these cases can be analyzed simultaneously. Note that in the SSS-case, each vertex $v$ of the union lies on the boundary of the union of three distinct spheres, and thus the problem is reduced in this case to bounding the complexity of the union of $2n$ balls in 3-space, which is known to be $O(n^2)$ (see, e.g., [24]), however, this case is subsumed by our analysis.

We now apply a similar decomposition scheme to that of Section 2.2. That is, we project all the cigars in $\mathcal{C}$ onto the $xy$-plane, and construct a $(1/r_0)$-cutting for the boundaries of these projections, for some sufficiently large constant parameter $r_0$, thereby obtaining a set of $O({r_0}^2 \log^2 r_0)$ pseudo-triangles, which we lift into vertical prisms.

We next use a similar definition of wideness as in the case of cylinders. That is, a cigar $C \in \mathcal{C}$ that intersects a prism-cell $\Delta$ is wide in $\Delta$, if its radius $r$ is at least $w/2$ (where $w$ is the width of $\Delta$), and narrow, otherwise. Here too, it is easy to verify that the silhouette of a narrow cigar in a cell $\Delta$ must cross that cell.

We apply a similar recursive mechanism as in Section 2.3. The major difference in the analysis with respect to the case of cylinders is in bounding the number of WWW-vertices. We use here similar ideas to those presented in [3] for bounding the complexity of cigars of equal radii. Let $\kappa$ be the same constant as in Section 2.3. We partition each of the cylindrical boundaries of the cigars $C$ into $\kappa$ canonical strips as before, and we cover (both) hemispherical portions of $C$ by $O(\kappa^2)$ spherical caps, each of opening angle at most $\pi/\kappa$, so that no point lies in more than a constant number of caps. We define a good direction for a cap in a similar manner as for strips (see conditions (i)–(ii) in the WWW-case in Section 2.3). The set of bad directions for such a spherical cap $\tau$ is again the union $B_1 \cup B_2$, where $B_1$ is defined as in the case of cylinders, and $B_2$ is defined as the spherical band consisting of all points at spherical distance at most $2\pi/\kappa$ from a great circle $B_\tau$ on $\mathbb{S}^2$, where $B_\tau$ corresponds to the plane parallel to the tangent plane of the cap $\tau$ at its center.

Following the reasonings in Section 2.3, each vertex $v$ of the union has at least one good direction, taken from a set $\mathcal{Z}$ of $O(1)$ directions. It is now easy to verify that the analog to Lemma 2.2 in this extended case continues to hold as well.

Let us fix a prism-cell $\Delta$ of the decomposition and a good direction $\rho$, and let $\sigma$, $\sigma'$, $V_\Delta(\rho)$, $\mathcal{T}^+_{\sigma'}(\rho)$, $\mathcal{T}^-_{\sigma'}(\rho)$ be as in Section 2.3. Following these notations, let $\mathcal{S}^+_{\sigma'}(\rho)$ be the set of the $\rho$-upper spherical caps that cross the subslab $\sigma'$ and contribute at least one vertex to $V_\Delta(\rho)$. We define $\mathcal{S}^-_{\sigma'}(\rho)$ in an analogous manner. It now follows that each vertex $v \in V_\Delta(\rho)$ must appear on the boundary of the sandwich region enclosed between the $\rho$-upper envelope of the strips and caps in $\mathcal{T}^+_{\sigma'}(\rho) \cup \mathcal{S}^+_{\sigma'}(\rho)$ and the $\rho$-lower envelope of the strips and caps in $\mathcal{T}^-_{\sigma'}(\rho) \cup \mathcal{S}^-_{\sigma'}(\rho)$. Repeating the analysis for all good directions $\rho$, we obtain a nearly-quadratic bound on the overall number of WWW-vertices of the union in a prism-cell $\Delta$.

The remaining steps of the analysis follow almost verbatim from the analysis for the case of cylinders. This concludes the proof of Theorem 3.1.

## 4 Concluding Remarks and Open Problems

The only part of the analysis in which we have explicitly used the geometric structure of the cylinders is in the WWW-case. The analysis in the WWN- WNN-cases only uses the property that the vertices lie on the boundary of the sandwich region enclosed between two envelopes of the good strips, where the major step of the analysis in these cases relies on the decomposition that we apply. Thus our machinery can be extended to any set of bodies in 3-space, for which there is a natural definition for wideness (reps., narrowness), from which one can derive similar properties to those of Lemma 2.2. One such problem concerns the union of cones. Another related problem is to extend our machinery from cigars to *kreplach*, where in this setting, we modify the definitions from Section 3, so that now $\mathcal{S}$ consists of $n$ pairwise-disjoint triangles. This problem was addressed by Agarwal and Sharir [3], who presented a nearly-quadratic bound on the complexity of the union where the balls in the Minkowski sum have the same radii. An open problem is extend this bound for the case where the balls have arbitrary radii.

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## References

[1] P. K. Agarwal, J. Pach and M. Sharir. State of the union, of geometric objects: A review, to appear in *Proc. Joint Summer Research Conference on Discrete and Computational Geometry—Twenty Years Later*, Contemp. Math, AMS, Providence, RI.

[2] P. K. Agarwal., O. Schwarzkopf, and M. Sharir. The overlay of lower envelopes and its applications. *Discrete Comput. Geom.*, 15(1):1-13, 1996.

[3] P. K. Agarwal and M. Sharir. Pipes, cigars, and kreplach: The union of Minkowski sums in three dimensions. *Discrete Comput. Geom.*, 24:645–685, 2000.

[4] B. Aronov, A. Efrat, V. Koltun, and M. Sharir. On the union of $\kappa$-round objects in three and four dimensions. *Discrete Comput. Geom.*, 36:511–526, 2006.

[5] B. Aronov and M. Sharir. On translational motion planning of a convex polyhedron in 3-space. *SIAM J. Comput.*, 26:1785–1803, 1997.

[6] B. Aronov, M. Sharir, and B. Tagansky. The union of convex polyhedra in three dimensions. *SIAM J. Comput.*, 26:1670–1688, 1997.

[7] J. D. Boissonnat, M. Sharir, B. Tagansky, and M. Yvinec. Voronoi diagrams in higher dimensions under certain polyhedral distance functions. *Discrete Comput. Geom.*, 19:485–519, 1998.

[8] B. Chazelle and J. Friedman. A deterministic view of random sampling and its use in geometry. *Combinatorica*, 10:229–249, 1990.

[9] K. L. Clarkson. New applications of random sampling in computational geometry. *Discrete Comput. Geom.*, 2:195–222, 1987.

[10] A. Efrat. The complexity of the union of $(\alpha, \beta)$-covered objects. *SIAM J. Comput.*, 34:775–787, 2005.

[11] A. Efrat and M. Katz. On the union of $\alpha$-curved objects. *Comput. Geom. Theory Appl.*, 14:241–254, 1999.

[12] A. Efrat G. Rote. and M. Sharir. On the union of fat wedges and separating a collection of segments by a line. *Computat Geom: Theory Appl.*, 3:277–288, 1994.

[13] A. Efrat and M. Sharir. On the complexity of the union of fat objects in the plane. *Discrete Comput. Geom.*, 23:171–189, 2000.

[14] E. Ezra and M. Sharir. Almost tight bound for the union of fat tetrahedra in three dimensions. In *Proc. 48nd Annu. IEEE Sympos. Found. Comput. Sci.*, pages 525–535, 2007.

[15] E. Ezra and M. Sharir. Counting and representing intersections among triangles in three dimensions. *Comput. Geom. Theory Appl.*, 32:196–215, 2005.

[16] D. Haussler and E. Welzl. $\varepsilon$-nets and simplex range queries. *Discrete Comput. Geom.*, 2:127–151, 1987.

[17] K. Kedem, R. Livne, J. Pach, and M. Sharir. On the union of Jordan regions and collision-free translational motion amidst polygonal obstacles. *Discrete Comput. Geom.*, 1:59–71, 1986.

[18] V. Koltun and M. Sharir. The partition technique for overlays of envelopes. *SIAM J. Comput.*, 32(4):841–863, 2003.

[19] J. Matoušek, N. Miller, J. Pach, M. Sharir, S. Sifrony, and E. Welzl. Fat triangles determine linearly many holes. In *Proc. 32nd Annu. IEEE Sympos. Found. Comput. Sci.*, pages 49–58, 1991.

[20] J. Matoušek, J. Pach, M. Sharir, S. Sifrony, and E. Welzl. Fat triangles determine linearly many holes. *SIAM J. Comput.*, 23:154–169, 1994.

[21] J. Pach. On the complexity of the union of geometric objects. *Lecture Notes in Computer Science*, 2098:292–307, 2001.

[22] J. Pach, I. Safruti, and M. Sharir. The union of congruent cubes in three dimensions. *Discrete Comput. Geom.*, 30:133–160, 2003.

[23] J. Pach and G. Tardos. On the boundary complexity of the union of fat triangles. *SIAM J. Comput.*, 31:1745–1760, 2002.

[24] M. Sharir and P. K. Agarwal. *Davenport-Schinzel Sequences and their Geometric Applications*. Cambridge University Press, New York, 1995.