Lower Envelopes of 3-Intersecting Surfaces in $\mathbb{R}^3$*

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Abstract

We show that the combinatorial complexity of the lower envelope of n continuous totally-defined bivariate functions in $\mathbb{R}^3$, such that any pair of them are either disjoint or intersect in a connected curve, and any triple of them intersect in at most three points, is $O(n^2\alpha(n))$, thus slightly improving the bound $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, due to Sharir [9][1].

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[1] A bound of the form $f(n) = O(n^{q+\varepsilon})$ means that, for any arbitrarily small $\varepsilon > 0$, there exists a constant c_ε, such that $f(n) \le c_\varepsilon n^{q+\varepsilon}$. Generally, c_ε tends to ∞ as ε decreases to 0.

1 Introduction

Let Σ be a collection of n surfaces in 3-space of constant description complexity[2], each of which is a graph of a totally defined continuous function. The lower envelope E_Σ of Σ is the *pointwise minimum* of these functions. That is, E_Σ is the graph of the following function:

$$E_\Sigma(\mathbf{x}) = \min_{\sigma \in \Sigma} \sigma(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \mathbb{R}^2.$$

The *combinatorial complexity* of E_Σ is the number of vertices, edges, and faces of the arrangement $\mathcal{A}(\Sigma)$ (see, e.g., [10]) that appear on E_Σ, where a face is a maximal connected portion of E_Σ contained in the graph of a single function, an edge is a maximal connected portion of E_Σ lying in the intersection of two distinct graphs and not meeting any other graph, and a vertex is an intersection point of three distinct graphs, which lies on E_Σ. The problem studied in this note is to obtain an upper bound on the combinatorial complexity of the lower envelope of n continuous totally-defined algebraic functions of constant description complexity in 3-space, such that each triple of their graphs intersect in at most three points.

Previous results. The problem of determining the combinatorial complexity of the lower envelope of surfaces in $\mathbb{R}^d$ has received considerable attention in the past two decades.

In two dimensions, the complexity of the lower envelope of n Jordan arcs, each pair of which intersect in at most some constant number s of points, is $O(\lambda_{s+2}(n))$, where $\lambda_q(m)$ is the maximum length of *Davenport-Schinzel sequences* of order q composed of m symbols, and is nearly-linear in m for any fixed q (see [10] for further details).

In three dimensions, it was shown by Halperin and Sharir [5] that the complexity of the lower envelope of n surface patches of constant description complexity, such that any triple of them intersect in at most two points, is $O\left(n^2 2^{c\sqrt{\log n}}\right)$, for some constant c that depends on the description comlpexity of the given surfaces.

In higher dimensions, the lower envelope of $(d-1)$-hyperplanes in $\mathbb{R}^d$ is the boundary of a convex polyhedron in d-space (bounded by at most n facets); thus, by the Upper Bound Theorem [7], its combinatorial complexity is $O\left(n^{\lfloor d/2 \rfloor}\right)$ (where the constant of proportionality depends on d). This bound is tight in the worst case. Using the above result, one can obtain a bound of $O\left(n^{\lceil d/2 \rceil}\right)$ on the lower envelope of n balls in d-space, by applying the so-called *lifting transformation*, in which each ball is mapped to a halfspace in $\mathbb{R}^{d+1}$. Pach and Sharir [8] showed that the combinatorial complexity of the lower envelope of n $(d-1)$-simplices in $\mathbb{R}^d$ is $O(n^{d-1}\alpha(n))$ (where $\alpha(\cdot)$ denotes the inverse Ackermann function). This bound is tight in the worst case. Finally, Sharir [9] presented a nearly-optimal bound of $O(n^{d-1+\varepsilon})$, for any $\varepsilon > 0$, for the complexity of the lower envelope of n surfaces or surface patches in d-space of constant description complexity. This result generalizes all previous results mentioned above, and, in particular, significantly extends the bound of Halperin and Sharir [5]. Nevertheless, obtaining a sharper (and more natural) upper bound of $O(n^{d-2}\lambda_q(n))$ (for some constant q that depends

[2]A set in $\mathbb{R}^d$ is said to have *constant description complexity* if it is a semi-algebraic set defined as a Boolean combination of a constant number of polynomial equalities and inequalities of constant maximum degree in a constant number of variables.

on the description complexity of the given surfaces) on this complexity, seems to be elusive, although it is strongly conjectured to be the correct bound.

Our result. In this note we show that the combinatorial complexity of the lower envelope of n functions in 3-space, satisfying the assumptions in the abstract, is $O(n^2\alpha(n))$, thus improving the bound $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, due to Sharir [9].

A simple extension of our analysis shows that the combinatorial complexity of the *union* of n three-dimensional regions of constant description complexity, such that any triple of their boundaries intersect in at most three points, is $O(n^2\alpha(n))$. We then derive, as an easy corollary, a similar bound on the complexity of the *sandwich region* between the lower envelope and the upper envelope of n such surfaces.

2 The Lower Envelope of 3-Intersecting Algebraic Surfaces in 3-Space

Preliminaries and overview. Let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a collection of n surfaces in 3-space that satisfy the following conditions (see [5, 9, 10] for a similar, albeit more general, set of conditions):

(i) Each σ_i is the graph of a continuous totally-defined algebraic function of constant description complexity.

(ii) Any pair of the given surfaces are either disjoint or intersect in a connected curve.

(iii) Any triple of the given surfaces intersect in at most three points.

(iv) The surfaces in Σ are in *general position*; for simplicity, we assume a strict version of this assumption, in which the coefficients of the polynomials defining the surfaces are algebraically independent over the rationals. This excludes degenerate configurations where four surfaces meet at a point, a pair of surfaces are tangent to each other, a singular point on one surface lies on the boundary of another surface or on an intersection curve between two other surfaces, etc.

Conditions (i)-(iii) are essential for the analysis, while condition (iv) is made to simplify the forthcoming arguments. It involves no real loss of generality, because, as can be shown (see [9, 10] for details), the maximum complexity of the envelope is achieved, up to a constant factor, when the given surfaces are in general position. This assumption implies that each vertex of the arrangement of the surfaces lies on exactly three surfaces, and is thus incident upon only a constant number of edges and faces. We also note that the number of edges on the lower envelope E_Σ that are not incident upon any vertex is only $O(n^2)$ (since each such edge is formed by the intersection of a pair of surfaces, and any such intersection has only $O(1)$ connected components). Thus our main goal is to bound the number of vertices of the lower envelope. Our main result is:

Theorem 2.1 *The combinatorial complexity of the lower envelope of a collection Σ of n functions that satisfy conditions (i)–(iii) is $O(n^2\alpha(n))$, where the constant of proportionality de-*

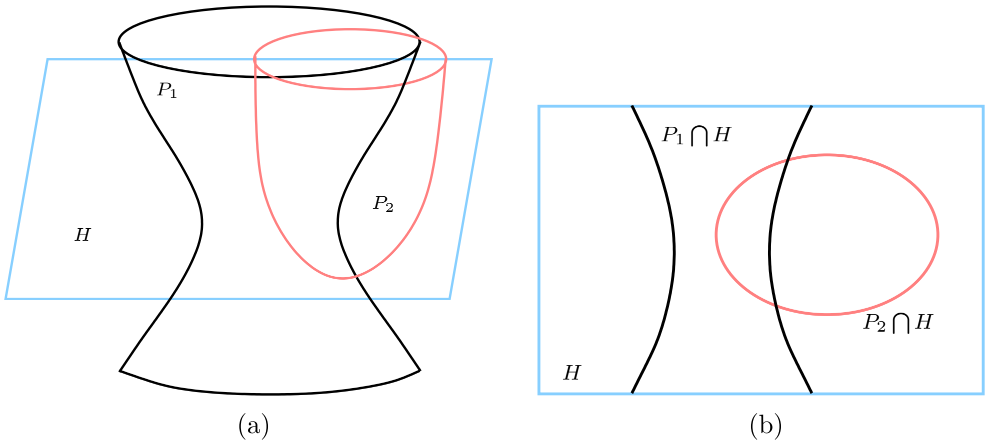

Figure 1: (a) The surface P_1 is a canonical hyperboloid, P_2 is a canonical paraboloid, and H is a slanted plane. These surfaces intersect in two points. (b) A view from above on H and the intersection curves $P_1 \bigcap H$, $P_2 \bigcap H$ that appear on it. The intersection curve $P_1 \bigcap H$ is unbounded, and the intersection curve $P_2 \bigcap H$ is closed.

pends on the description complexity of the functions in Σ*. The complexity can be* $\Omega(n^2)$ *in the worst case.*

A quadratic lower bound construction can be obtained using $n/2$ parabolic cylinders of the form $z := (x - a_i)^2$, and $n/2$ other parabolic cylinders of the form $z := (y - b_j)^2 + \xi$, with an appropriate choice of parameters (in this construction, any triple of surfaces intersect at most twice). We thus devote the remainder of this section to deriving the upper bound stated in Theorem 2.1. Specifically, in view of the preceding observations, we establish the asserted bound on the number of vertices of the envelope.

2.1 The Number of Vertices on the Lower Envelope

For simplicity of exposition, we replace each surface $\sigma \in \Sigma$ with the three-dimensional (semi-unbounded) region bounded from below by σ; we refer to these regions as "halfspaces", and denote by $\mathcal{S}$ the collection of these halfspaces. We now fix a halfspace $s \in \mathcal{S}$, and intersect its boundary ∂s with all the other halfspaces in $\mathcal{S}$, obtaining a set of planar regions on ∂s of the form $\gamma = \partial s \cap s'$, for each $s' \neq s$, $s' \in \mathcal{S}$. Due to condition (i), each region in γ has at most a constant number of connected components (which depends on the description complexity of the input functions), and we consider each of them as a separate region; the overall number of these regions is thus $O(n)$. Each region γ is bounded by (a connected component of) the intersection curve $\partial s \cap \partial s'$, $s' \neq s$, $s' \in \mathcal{S}$. Let Γ be the set of these regions; as just noted, $|\Gamma| = O(n)$.

We show below that the combinatorial complexity of the *union* $\bigcup_{\gamma \in \Gamma} \gamma$ (i.e., the number of vertices and edges of the arrangement $\mathcal{A}(\Gamma)$, induced by the regions in Γ (along $\sigma = \partial s$), which are not contained in the relative interior of any region in Γ) is $O(n\alpha(n))$, thereby showing that the combinatorial complexity of the union of the regions in $\mathcal{S}$ is $O(n^2\alpha(n))$, from which the theorem readily follows.

We thus fix a halfspace $s \in \mathcal{S}$, and denote ∂s by σ. First, due to condition (i), the intersection curve of σ and $\partial s'$, for each $s' \in \mathcal{S}$, $s' \neq s$, is a Jordan curve of constant description complexity, which is either closed or unbounded; see Figure 1. We partition the set of the planar regions along ∂s into two subcollections R and B, also referred to as the "red" collection and the "blue" collection, respectively, where the red (resp., blue) collection consists of those regions whose boundary is an unbounded (resp., closed) Jordan curve. Each intersection vertex v appearing on the boundary of the union $\bigcup_{\gamma\in\Gamma}\gamma$ is consequently classified as being either (i) *red-red*, if the two regions that contain v on their boundary are red; (ii) *blue-blue*, if these two regions are blue; (iii) *red-blue*, if one of these regions is red and the other is blue.

Let us first bound the number of red-red vertices of the union. Since any triple of the original surfaces intersect in at most three points (condition (ii)), it follows that the boundaries of any pair of regions in R intersect in at most three points as well. An easy adaption of the analysis of Edelsbrunner *et al.* [4] shows that the complexity of the union of these regions is $O(n\alpha(n))$. More precisely, the analysis in [4] assumes that each of the given regions is bounded (from below or above) by a portion of the x-axis and a Jordan arc that starts and ends on the x-axis, and that each pair of these arcs intersect at most three times. This implies that the overall number of red-red vertices of the union (of the regions in Γ) is at most $O(n\alpha(n))$.

We next bound the number of blue-blue vertices of the union. Since the regions in B are bounded by closed Jordan curves, and we assume general position, it follows that each pair of their boundaries must intersect an even number of times, so, due to condition (ii), the boundaries of any pair of regions in B intersect at most twice. Hence B is a collection of *pseudo-discs*, which implies that the combinatorial complexity of their union is only linear [6], and thus the overall number of blue-blue vertices of the union is $O(n)$.

It therefore remains to show that the number of red-blue vertices appearing on the boundary of the union is $O(n\alpha(n))$.

We use a variant of the technique of [1], where it is shown that the number of tangencies in a family of n *pairwise intersecting pseudo-parabolas* in the plane (that is, graphs of totally-defined continuous univariate functions, each pair of which either intersect twice, or have a single point of non-crossing tangency), is at most $2n-4$ (for $n \geq 3$). Specifically, let us denote the set of all red edges appearing on the boundary of the (red) union $\bigcup_{\gamma\in R}\gamma$ by Ξ_R, and the set of all blue edges appearing on the boundary of the (blue) union $\bigcup_{\gamma\in B}\gamma$ by Ξ_B. We construct a bipartite graph $G = (\Xi_R \cup \Xi_B, E)$, where the set E of edges consists of all pairs (ξ, ξ'), $\xi \in \Xi_R$, $\xi' \in \Xi_B$, such that at least one of the intersection points of ξ, ξ' lies on the boundary of the union. Note that a region $\gamma \in \Gamma$ that does not appear on the boundary of the union does not contribute any feature to G. We realize G as a topological graph drawn in the plane as follows. Each edge $\xi \in \Xi_R \cup \Xi_B$ is mapped to some arbitrary point $p_\xi \in \xi$ that lies on the boundary of the union of all the regions and does not lie on the boundary of any other region. For each pair $\xi \in \Xi_R$, $\xi' \in \Xi_B$, which intersect in a red-blue vertex w that appears on the boundary of the union, we draw the corresponding edge e of G so that it connects the pair of the representing points p_ξ, $p_{\xi'}$, by first connecting p_ξ to w along ξ (in any of the two possible ways), and then connecting w to $p_{\xi'}$ along ξ'; see Figure 2. We draw only one edge between

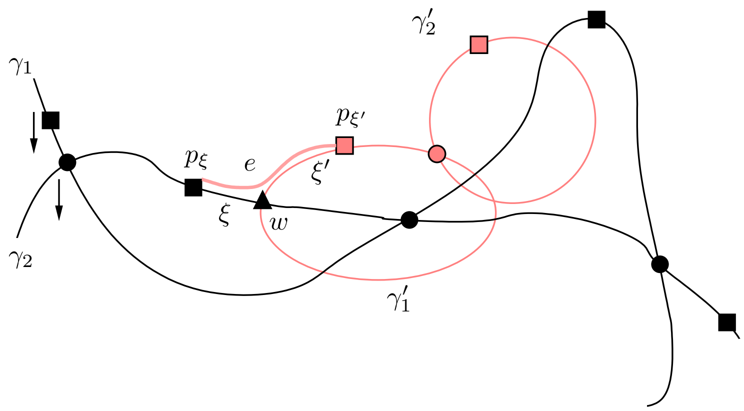

Figure 2: Drawing G in the plane.

any pair ξ, ξ', even though ξ and ξ' may intersect in more than one point on the boundary of the union. Still, the number of red-blue vertices is at most twice the number of edges of G (due to condition (ii) and the fact that the blue regions are closed, and also to the general position assumption).

The number of vertices in G is $|\Xi_R| + |\Xi_B| = O(n\alpha(n))$, and, by construction, the number of the edges in G is at least half the number of red-blue vertices of the union. We next claim that G is a planar graph, which will imply that the number of its edges is $O(n\alpha(n))$, and will thus yield a similar bound on the number of red-blue vertices of the union on ∂s.

We use the Hanani-Tutte Theorem [11] (see also [3]), which states that a graph that can be drawn in the plane, such that each pair of its edges, which do not share a common vertex, cross an *even* number of times, is a planar graph.

We split each edge e of G at the red-blue vertex w that induces it, and obtain two "half-edges", where the first half-edge is red and the second one is blue. We then show that any pair of red and blue half-edges, which do not share a common vertex, are either disjoint or cross twice. (Note that, by construction, any pair of red half-edges, which do not share a vertex, are disjoint, and the same holds for pairs of blue half-edges.) It will then clearly follow that each pair of edges of G cross an even number of times.

Let $e = (p_\xi, w)$ be a red half-edge that connects the red vertex p_ξ of G to the red-blue vertex w of the union, and let $e' = (p_{\xi'}, w')$ be a blue half-edge, defined in an analogous manner. Let $\gamma \in R$, $\gamma' \in B$ be the regions whose boundaries contain e, e', respectively. Clearly, e, e' intersect at most twice (since this property holds for the entire boundaries of γ, γ'), it suffices to show that e and e' cannot have a single crossing. Suppose, to the contrary, that e and e' intersect in a single point q. Since the two vertices p_ξ and w that e connects lie on the boundary of the union, they cannot lie in the interior of γ', which implies that e intersects $\partial\gamma'$ in two points, q and another point q'. Symmetrically, e' intersects $\partial\gamma$ in two points as well, q and another point q''. By construction, we must have $q' \neq q''$, since, otherwise, one of p_ξ, w, $p_{\xi'}$, w' would have to lie in the interior of the other region, and thus could not appear on the boundary of the union, contrary to assumption. This implies that $\partial\gamma$, $\partial\gamma'$ intersect in at least

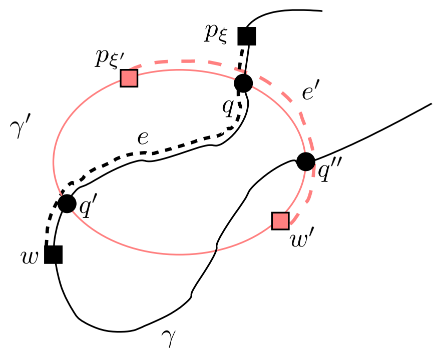

Figure 3: The proof of the planarity of G.

three points, contradicting the fact (already noted above) that they intersect at most twice; see Figure 3. This completes the proof of Theorem 2.1. □

3 Extensions

Unions. We can extend the analysis of Section 2 to the case where some of the surfaces are boundaries of compact regions in $\mathbb{R}^3$, and where we want to bound the complexity of the *union* of these regions. (In the analysis of Section 2, the corresponding regions are unbounded "upper halfspaces", and the lower envelope is indeed the boundary of their union.) In this case we replace condition (i) by

(i') Each σ_i is either a closed connected surface, homeomorphic to a 2-sphere, or the graph of a continuous totally-defined algebraic function, of constant description complexity,

and leave conditions (ii), (iii) as above.

It is easy to check that the preceding analysis carries over, more or less verbatim, to the current, more general, scenario, and we thus obtain:

Theorem 3.1 *The combinatorial complexity of the union of n regions in 3-space, bounded by surfaces σ that satisfy conditions (i'),(ii) (and (iii)), is $O(n^2\alpha(n))$, where the constant of proportionality depends on the description complexity of the bounding surfaces. The bound is almost tight in the worst case.*

Sandwiches. With a slight modification of the analysis, one can show that the combinatorial complexity of the *sandwich region* between the upper envelope and the lower envelope of two respective families of n functions each, satisfying conditions (i)-(iii), is $O(n^2\alpha(n))$ as well. Indeed, the sandwich region is the complement of the union of the $2n$ respective halfspaces, where we take the upper halfspace of each function participating in the lower envelope, and the lower halfspace of each function participating in the upper envelope. Thus, a variant of Theorem 2.1 yields:

Corollary 3.2 *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two respective collections of n functions each, which satisfy conditions (i)-(iii). Then the combinatorial complexity of the sandwich region between the upper envelope of the functions in $\mathcal{F}_1$ and the lower envelope of the functions in $\mathcal{F}_2$ is $O(n^2\alpha(n))$, where the constant of proportionality depends on the description complexity of the functions in $\mathcal{F}_1$, $\mathcal{F}_2$. The bound is almost tight in the worst case.*

Concluding remarks and open problems. We have presented an upper bound of $O(n^2\alpha(n))$ on the combinatorial complexity of the lower envelope of n functions in 3-space, satisfying conditions (i)–(iii). An open problem that this note raises is to tighten the small remaining gap between our upper bound and the quadratic lower bound, which we conjecture to be the actual bound.

Another major and challenging open problem is to improve the bound $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, of Halperin and Sharir [5], [9], on the complexity of the lower envelope of *any* family of n surface patches in 3-space of constant description complexity. In this case, we support the prevailing conjecture that the actual bound is $O(n\lambda_s(n))$, where s is a constant that depends on the description complexity of the given surfaces.

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