Small Complete Minors Above the Extremal Edge Density

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Abstract

A fundamental result of Mader from 1972 asserts that a graph of high average degree contains a highly connected subgraph with roughly the same average degree. We prove a lemma showing that one can strengthen Mader's result by replacing the notion of high connectivity by the notion of vertex expansion.

Another well known result in graph theory states that for every integer t there is a smallest real $c(t)$, such that every n-vertex graph with $c(t)n$ edges contains a K_t-minor. Fiorini, Joret, Theis and Wood asked whether every n-vertex graph G that has at least $(c(t)+\epsilon)n$ edges, must contain a K_t-minor of order at most $C(\epsilon)\log n$. We use our extension of Mader's theorem to prove that such a graph G must contain a K_t-minor of order at most $C(\epsilon)\log n\log\log n$. Known constructions of graphs with high girth show that this result is tight up to the $\log\log n$ factor.

1   Introduction

1.1   Graph minors and the main result

All graphs considered here are finite and have no loops or parallel edges. The order of a graph is the number of its vertices. A graph H is a minor of a graph G if H can be obtained from G by a sequence of edge deletions, vertex deletions and edge contractions. In this case we say that G has an H-minor. Let K_t denote the complete graph on t vertices. Since we will be mainly interested in K_t-minors it will be easier for us to use the following equivalent definition of a K_t-minor. A graph G has a K_t-minor if G contains t vertex disjoint connected subgraphs $S_1,\ldots,S_t$ and $\binom{t}{2}$ paths $(P_{i,j})_{1\le i<j\le t}$, such that $P_{i,j}$ connects S_i to S_j, each path $P_{i,j}$ is disjoint from all sets S_k with $k\ne i,j$, and the paths $P_{i,j}$ are internally vertex disjoint, that is, $P_{i,j}$ can only intersect with $P_{i,j'}$ or $P_{i',j}$ at its endpoint vertices. This K_t-minor is called topological if each subgraph S_i consists of a single vertex.

The notion of minor is undoubtedly one of the most well studied topics in graph theory. A central result in this area states that a linear number of edges is enough to force the appearance of a K_t-minor. Formally, for every integer $t\ge 3$ define

$$c(t)=\min\{c:d(G)\ge c\text{ implies that }G\text{ has a }K_t\text{-minor}\}\,,\qquad(1)$$

where $d(G)=|E(G)|/|V(G)|$. Mader [16] has shown that the displayed set does indeed have a minimum (that is, its infimum is a member of the set) and that $c(t)\le 2^{t-3}$. He later [17] obtained

the improved bound $c(t) \leq 16t \log t$ (all logarithms in this paper are base 2). His result was improved by Kostochka [12] and Thomason [22] who proved that $c(t) = \Theta(t\sqrt{\log t})$. Thomason [23] later proved an even sharper bound, showing that $c(t) = (\alpha + o(1))t\sqrt{\log t}$ for some explicit constant α, with the $o(1)$ term going to 0 as $t \to \infty$.

Fiorini, Joret, Theis and Wood [6] raised the following problem: how many edges suffice to guarantee that a graph contains not only a K_t-minor, but one which has *few vertices*? Observe that graphs with logarithmic girth (which can be constructed by deleting short cycles from random graph $G(n, p), p = c/n$ or explicitly see, e.g., [15]) show that for any constant C, there is a graph G with $d(G) \geq C$ and no K_3 minor of order $o(\log n)$. We also need $d(G) \geq c(t) = (\alpha + o(1))t\sqrt{\log t}$ to guarantee *some* K_t-minor. So the question boils down to finding the smallest constant $c > c(t)$, such that any graph G with $d(G) \geq c$ contains a K_t-minor of order $O(\log n)$. Fiorini et al. [6] proved that if $d(G) \geq 2^{t-2} + \epsilon$ then G has a K_t-minor of order $C(\epsilon) \log n$. Note that the average degree here is exponentially larger than the one needed to guarantee a K_t-minor. This motivated Fiorini et al. [6] to ask if in fact any graph G with $d(G) \geq c(t) + \epsilon$ contains a K_t-minor of order $C(\epsilon) \log n$. That is, while $c(t)n$ edges are sufficient (and necessary) to guarantee *some* K_t-minor, adding only $o(n)$ additional edges should force the appearance of the (asymptotically) smallest K_t-minor one can force even with Cn edges, for any constant C. Our main result in this paper comes very close to answering their question positively.

Theorem 1 *For every $\epsilon > 0$ and integer $t \geq 3$ there exist $n_0 = n_0(\epsilon, t)$, such that every n-vertex graph G with $n \geq n_0$ and $d(G) \geq c(t) + \epsilon$ contains a K_t-minor of order $O(\frac{c(t)t^2}{\epsilon} \log n \log \log n)$.*

As we mentioned above, Thomason [23] has shown that $c(t) = (\alpha + o(1))t\sqrt{\log t}$. The easier half of his result shows that a random graph (on an appropriate number of vertices) with $d(G) = (1 - \epsilon)c(t)$ has no K_t-minor. Myers [18] has later strengthened his result (see [18] for the precise condition where this fact holds) by showing that n-vertex graphs of density p that do not contain a K_t-minor larger than the one we expect to find in a random graph $G(n, p)$ must be quasi-random. So roughly speaking, random graphs are extremal with respect to the critical density where one expects to find a K_t-minor in *arbitrary* graphs. A positive answer to the problem of Fiorini et al. [6] would thus show that random graphs are also in some sense extremal with respect to the actual order of the K_t-minor we expect to find. Specifically, it is possible to show that the smallest K_t-minor in a random graph $G(n, p), p = c/n$ such that $d(G) = c/2 > c(t)$ has order $O(\log n)$. The problem if [6] can thus be phrased as asking if in fact, appropriate random graphs maximize the order of the smallest K_t-minor among all graphs of density $c(t) + \epsilon$.

Let us finally mention an old conjecture of Erdős, stating that a graph with $n^{1+\epsilon}$ edges contains a non-planar subgraph of size $C(\epsilon)$. This conjecture was confirmed (in a very strong sense) by Kostochka and Pyber [11] who proved that any graph with $4^{t^2} n^{1+\epsilon}$ edges contains a topological K_t-minor of size $O(t^2 \log t/\epsilon)$. So the problem of [6] that we study here is in some sense a strengthening of the conjecture of Erdős for $\epsilon = 1/\log n$. Furthermore, as noted in [6], one can adapt the argument of [11] to show that a graph G with $d(G) \geq 16^t$ contains a K_t-minor of size $O(\log n)$.

1.2 Expansion in graphs, proof overview and the key lemma

We believe that an important aspect of this paper is the proof technique we employ here which relies on the notion of expansion in graphs and might be applicable in other settings. Perhaps a good perspective on our approach comes from dense graphs, that is graphs with cn^2 edges. Probably the most powerful tool one has at his disposal when studying dense graphs is Szemerédi's regularity lemma [21], which asserts that *any* dense graph can be approximated by a graph consisting of a bounded number of quasi-random graphs. Since quasi-random graphs are much easier to work with, this lemma allows one to reduce a problem on arbitrary graphs to the same problem on quasi-random graphs. We refer the reader to [10] for more details on the regularity lemma and its applications.

When it comes to sparse graphs, there is no analogue of the regularity lemma. But in recent years, a parallel paradigm has emerged, the underlying idea of which can be thought of as stating that *any* sparse graph is close to being the disjoint union of expander graphs. While the regularity lemma supplies one notion of approximation/quasi-randomness for all applications involving dense graphs, it seems like for sparse graphs different applications call for different notions of approximation and expansion. We refer the reader to [2, 7, 24] for some examples where this paradigm was applied.

Just like graph minors, expansion is one of the most well studied topics in graph theory, with a remarkable number of applications in diverse areas such as theoretical computer science, additive number theory and information theory (just to name a few). We will thus refrain from giving a detailed account and instead refer the reader to the surveys [8, 13] for more details. There are several known results connecting expansion and existence of K_t-minors in graphs, see, e.g., [1, 19, 9, 14]. In all these papers the goal was to maximize the value of t. Our task here is quite different, we want to minimize the number of vertices in the minor, keeping t fixed.

For the proof of Theorem 1 we will need a very strong notion of expansion. The price will be that we will ask for a very weak notion of approximation[1], which will turn out to be sufficient for proving Theorem 1. In what follows, for a set of vertices S we use $N(S)$ to denote the neighborhood of S, that is the set of vertices not in S that are connected to at least one vertex in S. The notion of expansion we will use is the following:

Definition 1.1 (δ-Expander) *An m-vertex graph H is said to be a δ-expander if for every integer $0 \leq d \leq \log \log m - 1$ and $S \subseteq V(H)$ of order $|S| \leq m/2^{2^d}$ we have*

$$|N(S)| \geq \frac{\delta 2^d}{\log m (\log \log m)^2}|S| \tag{2}$$

Observe that (disregarding the $(\log \log m)^2$ term) if G is an m-vertex δ-expander then sets of vertices of size cm have vertex expansion about $1/\log m$ while sets of vertices of size m^c have vertex expansion $\Theta(1)$. The following lemma shows that we can indeed find a δ-expander in any graph with sufficiently many edges.

Lemma 1.2 (Key Lemma) *If G satisfies $d(G) = c$, then for every $0 < \delta \leq \frac{1}{256}$ we can find in G a subgraph H, such that $d(H) \geq (1 - \delta)c$ and H is a δ-expander.*

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[1]In fact, what we will ask for in Lemma 1.2 is just one subgraph with very good expansion properties, so this can hardly be called an approximation. In Section 5 we will suggest a possible strengthening of Lemma 1.2, involving a stronger notion of approximation, and the possible applications of such a lemma.

Remark 1.3 *Note that the only lower bound on the order of H supplied by the lemma is $2(1-\delta)c$ which follows from the fact that $|V(H)| \geq 2d(H) \geq 2(1-\delta)c$. Up to the δ error this is all that one can hope for since the graph might be a disjoint union of cliques of order $2c$.*

In a nutshell, the proof of Theorem 1 proceeds by first invoking Lemma 1.2 on the graph G thus obtaining a graph H satisfying the expansion properties of Definition 1.1. We then show (see Lemma 3.2) how one can find a small K_t-minor inside H, a task which is much easier given the fact that H has strong expansion properties. As we noted above, one can come up with different notions of expansion when studying sparse graphs. And indeed, in order to prove Theorem 1 we will actually have to prove another variant of Lemma 1.2, which uses a slightly different notion of expansion than the one defined in (2). It might very well be possible to prove other variants of Lemma 1.2, suitable for tackling other problems. See the concluding remarks for another possible variant which might have interesting applications.

As we mentioned in the abstract, Lemma 1.2 can be thought of as a strengthening of Mader's Theorem (see [4], Theorem 1.4.3). Indeed, Mader's Theorem states that any graph G with $d(G) \geq 2k$ has a k-connected subgraph H satisfying $d(H) \geq k$. So Lemma 1.2 gives a similar conclusion only it replaces the notion of k-connectivity with the stronger notion of vertex expansion.

1.3 Organization

The rest of the paper is organized as follows. In Section 2 we prove Lemma 1.2. As we mentioned earlier, the proof can be easily adapted to other settings, and indeed in Section 4 we will prove another variant of this lemma. In Section 3 we prove a weaker version of Theorem 1, giving a bound of order $O(\log n(\log\log n)^3)$. Then, in Section 4, we "boost" this weaker bound by removing a $(\log\log n)^2$ factor thus obtaining the bound stated in Theorem 1. Section 5 contains some concluding remarks and open problems.

2 Proof of Key Lemma

In this section we prove Lemma 1.2. Given a graph G and a subset of its vertices U, $G[U]$ denotes the subgraph of G induced by the set U. We start with the following simple yet crucial observation.

Claim 2.1 *If G is an n-vertex graph satisfying $d(G) = c$, and $S \subset V(G)$ is such that $|N(S)| < \gamma|S|$, then either $d(G[V \setminus S]) \geq c$ or $d(G[S \cup N(S)]) \geq (1-\gamma)c$.*

Proof: Indeed, if $d(G[V \setminus S]) < c$, $d(G[S \cup N(S)]) < (1-\gamma)c$ and $|N(S)| < \gamma|S|$ then we can bound the number of edges in G as follows,

$$e(G) < c(n - |S|) + (1-\gamma)c(|S| + |N(S)|) \leq cn - \gamma c|S| + c|N(S)| \leq cn \ ,$$

contradicting the assumption that G has cn edges. ∎

In what follows, we say that a graph $G = (V, E)$ fails to be a δ-expander at scale d if there is a vertex set $S \subseteq V$ of size at most $m/2^{2^d}$ violating (2).

Proof of Lemma 1.2: Set $G_0 = G$ and consider the following iterative process; if either $|G_t| \le 256$ or if G_t is a δ-expander the process stops. Otherwise, set $m = |G_t|$ and $d_t = d(G_t)$. Since G_t is not a δ-expander, there must be some $0 \le d \le \log \log m - 1$ and a set $S_t \subseteq V(G_t)$ of size at most $m/2^{2^d}$ violating (2). We can deduce via Claim 2.1 (with $c = d_t$ and $\gamma = \frac{\delta 2^d}{\log m (\log \log m)^2}$) that in this case either $d(G[V(G_t) \setminus S_t]) \ge d_t$ or $d(G[S_t \cup N(S_t)]) \ge (1 - \frac{\delta 2^d}{\log m (\log \log m)^2}) d_t$. In the former case we set $G_{t+1} = G[V(G_t) \setminus S_t]$ and in the latter we set $G_{t+1} = G[S_t \cup N(S_t)]$. For brevity, in what follows we call the above two cases, Case 1 and Case 2. Let G' be the graph returned at the end of the process. We shall show that $d(G') \ge (1 - \delta)c$, but we first note that this will finish the proof. Indeed, if $|G'| > 256$ then by definition of the process we get that G' must be a δ-expander, so G' satisfies the requirements of the lemma. If $|G'| \le 256$ then one of its connected components, call it G'', must also satisfy $d(G'') \ge (1 - \delta)c$. Since $\delta \le 1/256$ and G'' is connected, we get that G'' is (trivially) a δ-expander, so we can return G''.

We now show that $d(G') \ge (1 - \delta)c$. Recall that for each of the graphs G_t in the process $d_t = d(G_t)$ (so $d_0 = c$). Note that at each iteration either $d_{t+1} \ge d_t$ (in Case 1) or $d_{t+1} = \gamma_t d_t$ for some $\gamma_t > 0$. Fix some $4 \le k \le \log \log n$ and suppose $r < r'$ are such that G_r is the first graph in the process whose order is in the range $[2^{2^{k-1}}, 2^{2^k}]$ and $G_{r'}$ is the last such graph in the process. We wish to compute a lower bound on $d_{r'+1}/d_r$, that is, a lower bound on the fraction of edge loss that can occur when "passing" through the interval $[2^{2^{k-1}}, 2^{2^k}]$. Observe that if at some iteration $r \le t \le r'$ of the process, the graph G_t fails to be a δ-expander at scale d, then either $\gamma_t \ge 1$ (Case 1), or $|G_{t+1}| \le |G_t|/2^{2^d}$. For each $0 \le d \le k$ let a_d be the number of times Case 2 occurred due to G_t failing to be a δ-expander at scale d. Then we have

$$1 \le |G_{r'+1}| \le |G_r| / \prod_{d=0}^{k} 2^{a_d 2^d} \le 2^{2^k} / \prod_{d=0}^{k} 2^{a_d 2^d} \ ,$$

implying that

$$\sum_{d=0}^{k} a_d 2^d \le 2^k \ . \tag{3}$$

As noted in the first paragraph of the proof, if at some iteration $r \le t \le r'$ Case 2 happened due to the fact that G_t failed to be a δ-expander at scale d, then we know that $\gamma_t \ge 1 - \delta 2^d / \log m (\log \log m)^2$ (recall that m denotes the order of G_t). Since we are considering the case where $2^{2^{k-1}} \le m \le 2^{2^k}$ this means that in this range we have

$$\gamma_t \ge 1 - \delta 2^d / 2^{k-1} (k-1)^2 \ . \tag{4}$$

Using that a_d was the number of times γ_t satisfied (4), we have that

$$d_{r'+1}/d_r = \prod_{t=r}^{r'} \gamma_t \ \ge \ \prod_{d=0}^{k} \left(1 - \frac{\delta 2^d}{2^{k-1}(k-1)^2}\right)^{a_d} \ge \left(1 - \delta \sum_{d=0}^{k} \frac{a_d 2^d}{2^{k-1}(k-1)^2}\right)$$

$$\ge \ \left(1 - \frac{2\delta}{(k-1)^2}\right) \ ,$$

where the last inequality follows from (3). We have thus established that d_t goes down by a factor of at most $1 - 2\delta/(k-1)^2$ when the process passes through the interval $[2^{2^{k-1}}, 2^{2^k}]$. Since we stop

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the process when $|G_t| \leq 256$, we only need to consider $k \geq 4$. Hence, altogether d_t can decrease by a factor of at most

$$\prod_{k=4}^{\log\log n} \left(1 - \frac{2\delta}{(k-1)^2}\right) \geq 1 - 2\delta \sum_{k=4}^{\log\log n} \frac{1}{(k-1)^2} \geq 1 - \delta. \tag{5}$$

So when the process ends we indeed obtain a δ-expander $G_{t'}$ satisfying $d(G_{t'}) \geq c(1 - \delta)$. ∎

3 A Weaker Bound

As mentioned above, our goal in this section is to prove the following slightly weaker version of Theorem 1.

Lemma 3.1 *The following holds for every $\epsilon > 0$, integer $t \geq 3$ and large enough $n \geq n_1(\epsilon, t)$. Every n-vertex graph satisfying $d(G) \geq c(t) + \epsilon$ has a K_t-minor of order $O(\frac{c(t)t^2}{\epsilon} \log n (\log\log n)^3)$.*

The main lemma we will use to prove Lemma 3.1 is the following result, which shows that one can find small K_t-minors in δ-expanders.

Lemma 3.2 *The following holds for all $\delta > 0$ integer t and large enough $m \geq m_0(\delta, t)$. If H is a δ-expander on m vertices, then it has a K_t-minor of order $O(\frac{t^2}{\delta} \log m (\log\log m)^3)$.*

Let us first show how can one derive Lemma 3.1 from Lemma 3.2

Proof of Lemma 3.1: We can clearly assume that $\epsilon < c(t)$ (otherwise we can replace ϵ with (say) $1/2$, which is smaller than $c(t)$ for all $t \geq 3$). Let G be an n-vertex graph satisfying $d(G) = c(t) + \epsilon$. Applying Lemma 1.2 to G with $\delta = \epsilon/3c(t)$ we obtain a subgraph H satisfying

$$d(H) \geq (1 - \delta)(c(t) + \epsilon) > c(t) .$$

Hence, if $|V(H)| < m_0(\delta, t)$ we get from the definition of $c(t)$ in (1) that H has a K_t-minor of order at most[2] $m_0(\delta, t) = m_0(\epsilon/3c(t), t) \leq \log(n)$. If $|V(H)| \geq m_0(\delta, t)$ then we get from the second assertion of Lemma 1.2 that H must be an $\frac{\epsilon}{3c(t)}$-expander. Hence, we can apply Lemma 3.2 to find in H a K_t-minor of order $O(\frac{c(t)t^2}{\epsilon} \log n (\log\log n)^3)$. ∎

We now turn to prove Lemma 3.2. For a subset of vertices U of a graph G we denote by $B_k(U)$ the ball of radius k around U, i.e., the set of all vertices of G which can be reached by a path of length at most k from some vertex in U. We start with the following simple observation:

Claim 3.3 *Suppose U and V are two vertex sets in an m-vertex graph G such that the following condition holds for every integer $0 \leq d \leq \log\log m - 1$; whenever $|B_k(U)| \leq m/2^{2^d}$ we have*

$$|N(B_{k+1}(U))| \geq \frac{\delta 2^d}{10 \log m (\log\log m)^2} |B_k(U)| , \tag{6}$$

and a similar condition holds with respect to V. Then there is a path connecting U to V of length at most $\frac{20}{\delta} \log m (\log\log m)^3$.

Proof: It is clearly enough to show that $|B_k(U)| > m/2$ for some $k \leq \frac{10}{\delta} \log m (\log \log m)^3$. So suppose $m/2^{2^d} \leq |B_k(U)| \leq m/2^{2^{d-1}}$. Then by (6) we either have $|B_{k+t}(U)| > m/2^{2^{d-1}}$ or

$$m/2^{2^{d-1}} \geq |B_{k+t}(U)| \geq |B_k(U)| \left(1 + \frac{\delta 2^{d-1}}{10 \log m (\log \log m)^2}\right)^t \geq m/2^{2^d - \frac{\delta t 2^{d-1}}{10 \log m (\log \log m)^2}} . \tag{7}$$

It is easy to check that the RHS of (7) is larger than the LHS when $t \geq \frac{10}{\delta} \log m (\log \log m)^2$. In other words, within at most $\frac{10}{\delta} \log m (\log \log m)^2$ steps, the neighborhood around U jumps from size larger than $m/2^{2^d}$ to size larger than $m/2^{2^{d-1}}$. Since there are only $\log \log m$ intervals $[m/2^{2^d}, m/2^{2^{d-1}}]$ we see that $|B_k(U)| > m/2$ for some $k \leq \frac{10}{\delta} \log m (\log \log m)^3$. ∎

Definition 3.4 (γ-Expanding Ball) *Let v be a vertex in a graph G. We say that $B_k(v)$ is a γ-expanding ball if for every $1 \leq i \leq k-1$ we have $|B_{i+1}(v)| \geq |B_i(v)|(1+\gamma)$. We also call v the center of the ball and k the radius. A set of vertices is said to be a γ-expanding ball if it is equal to an expanding ball $B_k(v)$ for some center v and radius k.*

Claim 3.5 *The following holds for all $\delta > 0$ integer t and large enough $m \geq m_0(\delta, t)$. If G is a δ-expander on m vertices then one of the following holds;*

1. *G has t vertices $v_1, \ldots, v_t$ each of degree at least $\log^4 m$.*

2. *Set $\gamma = \delta/5(\log \log m)^2$. Then G contains t disjoint vertex sets $S_1, \ldots, S_t$ such that $G[S_i]$ is a γ-expanding ball, $m^{1/5} \leq |S_i| \leq m^{1/4}$ and every vertex in $G[S_i]$ has degree at most $\log^4 m$.*

Proof: If G has t vertices $v_1, \ldots, v_t$ each of degree larger than $\log^4 m$ then we have the first case of the lemma. So suppose for the rest of the proof that G has at most t vertices of degree larger than $\log^4 m$. We need to show that we can pick sets $S_1, \ldots, S_t$ satisfying the second condition of the lemma. Let T be the set containing the vertices of degree larger than $\log^4 m$. Let us also say that a set S is *nice* if $G[S]$ is a γ-expanding ball and $m^{1/5} \leq |S| \leq m^{1/4}$. Note that a nice set in $G \setminus T$ satisfies the second condition of the lemma.

It is clearly enough to show that for any set $W \supseteq T$ of at most $m^{1/3}$ vertices, we can find in $G \setminus W$ a nice set S. Once we know this, we can simply iteratively pick the sets S_i one after the other where at iteration i we will pick S_i from $G \setminus W_i$ with $W_i = (\bigcup_{j<i} S_j) \cup T$. Since $|T|, |S_1|, \ldots, |S_{i-1}| \ll m^{1/3}$ this set has size smaller than $m^{1/3}$.

So suppose to the contrary that there is a $W \supseteq T$ of size $m^{1/3}$ such that $G \setminus W$ has no nice set S and set $G_0 = G \setminus W$. As we assume that $W \supseteq T$, all the vertices in G_0 have degree at most $\log^4 m$. So pick a vertex $v_1 \in G_0$ and let $B_k(v_1)$ be the vertices at distance at most k from v_1 in G_0. Let k_1 be the smallest integer such that $|B_{k_1+1}(v)| < |B_{k_1}(v)|(1+\gamma)$. We claim that $|B_{k_1+1}(v)| < m^{1/4}$. To see this observe that the fact that all the vertices in G_0 have degree at most $\log^4 m$ implies that for every $k \leq k_1$ we have $|B_{k-1}(v)| \geq |B_k(v)|/\log^4 m$ so if $|B_{k_1+1}(v)| \geq m^{1/4}$ then there must be a k_1', such that $m^{1/5} \leq |B_{k_1'}(v)| \leq m^{1/4}$ and $|B_{i+1}(v)| \geq |B_i(v)|(1+\gamma)$ for all $1 \leq i \leq k_1'$. But in this case $B_{k_1'}(v)$ would be a nice set. So setting $T_1 = B_{k_1}(v_1)$ we have $|T_1| < m^{1/4}$ and $|N_{G_0}(T_1)| < \gamma|T_1|$.

Let $G_1 = G_0 \setminus T_1$. Take now another vertex $v_2 \in G_1$ and repeat the above process. We will eventually end up with a set $T_2 = B_{k_2}(v_2)$ of size smaller than $m^{1/4}$ satisfying $|N_{G_1}(T_2)| < \gamma|T_2|$.

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Define $G_2 = G_1 \setminus T_2$. We continue picking sets $T_i \subset G_{i-1}$ until the first time $|\bigcup_i T_i| > \sqrt{m}/2$. Since each set T_i is of size at most $m^{1/4} \leq \sqrt{m}/2$ this means that $\sqrt{m}/2 \leq |\bigcup_i T_i| \leq \sqrt{m}$. Now, the fact that each of the sets T_i satisfies $|N_{G_{i-1}}(T_i)| < \gamma|T_i|$ along with the facts that $G_i = G_{i-1} \setminus T_i$ and $|\bigcup_i T_i| \leq \sqrt{m}$ implies that

$$\left| N_{G_0}\left(\bigcup_i T_i\right) \right| \leq \sum_i |N_{G_{i-1}}(T_i)| < \sum_i \gamma|T_i| \leq \gamma\sqrt{m} = \delta\sqrt{m}/5(\log\log m)^2 . \tag{8}$$

Now recall that G was assumed to be a δ-expander and since $\sqrt{m}/2 \leq |\cup T_i| \leq \sqrt{m}$ we must have (using $d = \log\log m - 1$ in (2))

$$\left| N_G\left(\bigcup_i T_i\right) \right| \geq \delta\left|\bigcup_i T_i\right|/(\log\log m)^2 \geq \delta\sqrt{m}/4(\log\log m)^2 . \tag{9}$$

Let us now recall that G_0 was obtained from G by removing a set W of no more than $m^{1/3}$ vertices. Hence, recalling (8) we see

$$\left| N_G\left(\bigcup_i T_i\right) \right| \leq |W| + \left| N_{G_0}\left(\bigcup_i T_i\right) \right| \leq m^{1/3} + \delta\sqrt{m}/5(\log\log m)^2 < \delta\sqrt{m}/4(\log\log m)^2 ,$$

which contradicts (9). ∎

Claim 3.6 *The following holds for all $\delta > 0$ integer t and large enough $m \geq m_0(\delta, t)$. If G is a δ-expander of order m that has t vertices as in the first case of Claim 3.5 then it has a K_t-minor of order $O(\frac{t^2}{\delta}\log m(\log\log m)^3)$.*

Proof: Let $v_1, \ldots, v_t$ be the vertices satisfying the first assertion of Claim 3.5. We will show that in this case G has a topological K_t-minor of the required order. To do this, we show that we can find $\binom{t}{2}$ paths $P_{i,j}$, where each path connects v_i to v_j, has length at most $\frac{20}{\delta}\log m(\log\log m)^3$ and is internally disjoint from all other paths. Our plan is to successively find these paths by invoking Claim 3.3. All we need to do is show that for any v_i, v_j and any set W of size at most $\log^2 m$ the sets $\{v_i\}, \{v_j\}$ satisfy (6) with respect to the graph $G \setminus W$. Once we establish this fact, we will be able successively pick the paths $P_{i,j}$ via Claim 3.3 where at each iteration we will take W to consist of the vertices v_t, $t \neq i, j$, together with the internal vertices of the paths $P_{i',j'}$ we have already picked.

We turn to show that $\{v_i\}$ satisfies (6) in the graph $G' = G \setminus W$ (the proof for $\{v_j\}$ is identical). We first recall the assumption of the lemma that v_i has degree $\log^4 m$ in G. Since we assume that $|W| \leq \log^2 m$ we clearly have

$$N_{G'}(v_i) \geq N_G(v_i) - |W| \geq \log^4 m - \log^2 m \geq \frac{1}{2}\log^4 m .$$

Hence, for all $k \geq 1$ we have

$$|B_k(v_i)| \geq |B_1(v_i)| \geq \frac{1}{2}\log^4 m . \tag{10}$$

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Suppose now that $|B_k(v_i)| \leq m/2^{2^d}$. Since we assume that G is a δ-expander we deduce that

$$
\begin{aligned}
|N_{G'}(B_k(v_i))| &\geq |N_G(B_k(v_i))| - |W| \geq \frac{\delta 2^d}{\log m (\log \log m)^2} |B_k(v_i)| - \log^2 m \\
&\geq \frac{\delta 2^d}{2 \log m (\log \log m)^2} |B_k(v_i)| \,,
\end{aligned}
$$

where the last inequality follows from (10). We thus get that G' satisfies (6) with $U = \{v_i\}$ and $V = \{v_j\}$ so we can indeed find a path connecting v_i to v_j of length at most $\frac{20}{\delta} \log m (\log \log m)^3$. ∎

Claim 3.7 *The following holds for all $\delta > 0$ integer t and large enough $m \geq m_0(\delta, t)$. If G is a δ-expander of order m that has t sets as in the second case of Claim 3.5 then it has a K_t-minor of order $O(\frac{t^2}{\delta} \log m (\log \log m)^3)$.*

Proof: Let $S_1, \ldots, S_t$ be the sets satisfying the second condition of Claim 3.5. Recall that in this case each set S_i is a γ-expanding ball around a vertex v_i in the induced subgraph $G[S_i]$. Also recall that $m^{1/5} \leq |S_i| \leq m^{1/4}$ and that all the vertices of $G[S_i]$ have degree at most $\log^4 m$. This means that there is some k_i, such that the ball around v_i in the graph $G[S_i]$ satisfies $\log^4 m \leq |B_{k_i}(v_i)| \leq \log^8 m$. For every $1 \leq i \leq t$ set $C_i = B_{k_i}(v_i)$ and note that C_i is also a γ-expanding ball in $G[S_i]$ around the same center v_i.

We will shortly show that one can find $\binom{t}{2}$ internally vertex disjoint paths $P_{i,j}$, where each $P_{i,j}$ connects C_i to C_j, has length at most $\frac{20}{\delta} \log m (\log \log m)^3$ and avoids all the sets $C_\ell, \ell \neq i, j$. But let us first observe why this will conclude the proof. For each $1 \leq i \neq j \leq t$ let $Q_{i,j}$ be some path connecting v_i to the unique vertex of $P_{i,j}$ that belongs to C_i. For each $1 \leq i \leq t$ set $V_i = \cup_j Q_{i,j}$. Then each set V_i is connected in G and the paths $P_{i,j}$ are internally vertex disjoint and avoid the sets $V_\ell, \ell \neq i, j$ so contracting the sets V_i indeed gives us a K_t-minor in G. As to the order of this minor, note that since C_i is a γ-expanding-ball (with $\gamma = \delta/5(\log \log m)^3$) of size at most $\log^8 m$ its radius is bounded by

$$
\log_{1+\gamma} |C_i| \leq \frac{2 \log(\log^8 m)}{\gamma} \leq 80 (\log \log m)^4 / \delta \,,
$$

so each of the paths $Q_{i,j}$ is of length $o(\log m)$. Hence the total size of the sets V_i is much smaller than $\log m$. Thus, once we show how to find the above mentioned paths $P_{i,j}$ with lengths bounded by $\frac{20}{\delta} \log m (\log \log m)^3$ we obtain a K_t-minor of order at most

$$
\log m + \frac{20 t^2}{\delta} \log m (\log \log m)^3 = O\Big(\frac{t^2}{\delta} \log m (\log \log m)^3 \Big) \,.
$$

Our plan is to show that one can simply iteratively pick the paths $P_{i,j}$ by successive applications of Claim 3.3. To this end, we need to show that after picking some of the paths, we can still find in the remaining graph another path. Since each of the paths is of length at most $\frac{20}{\delta} \log m (\log \log m)^2 \ll \log^2 m$, it is enough to show that for any set of vertices W of size at most $\log^2 m$, that is disjoint from each of the sets $C_1, \ldots, C_t$, and such that for any $i \neq j$, the sets C_i and C_j satisfy (6) with respect to the graph

$$
G' = G \setminus (W \cup \bigcup_{\ell \neq i,j} C_\ell) \,. \tag{11}
$$

Once we know this, we will iteratively pick the paths $P_{i,j}$, where at each iteration we will take W to be the union of the internal vertices of the paths we have already picked. To establish (6) we will need to consider two "regimes" of growth; the first is when C_i grows within the set S_i, and the second, when it grows out of S_i.

Let us first make a simple (but crucial) observation about the sets C_i. Recall that $S_i = B_{k_i'}(v_i)$ in $G[S_i]$ for some vertex v_i and k_i', that $|S_i| \geq m^{1/5}$ and that each vertex in S_i has degree at most $\log^4 m$. This means that $k_i' \geq \log m / 20 \log \log m$. We also recall that $|C_i| \leq \log^8 m$ and since S_i is a γ-expanding ball with $\gamma \geq \delta/5(\log \log m)^3$ this means that in the induced subgraph $G[S_i]$ we have that $C_i = B_{k_i}(v_i)$ for some $k_i \leq \frac{80}{\delta}(\log \log m)^4$. Combining the above facts about k_i and k_i' and setting $r_i = \log m / 30 \log \log m$ we have in graph $G[S_i]$

$$B_{r_i}(C_i) \subseteq S_i . \tag{12}$$

We now turn to show that C_i satisfies (6) with respect to G' defined in (11). Since S_i is a γ-expanding ball we get from (12) that $|N_{G[S_i]}(C_i)| \geq \delta|C_i|/5(\log \log m)^2$. Recall now that S_i is disjoint from $\bigcup_{r \neq i,j} C_r$, and that (12) implies that $N_{G[S_i]}(C_i) \subseteq S_i$. Hence

$$
\begin{aligned}
|N_{G'}(C_i)| &\geq |N_{G[S_i]}(C_i)| - |W| \geq \delta|C_i|/5(\log \log m)^2 - \log^2 m \\
&\geq \delta|C_i|/10(\log \log m)^2 ,
\end{aligned}
$$

where the last inequality uses the fact that $|C_i| \geq \log^4 m$. In other words, we have in G'

$$|B_1(C_i)| \geq \left(1 + \frac{\delta}{10(\log \log m)^2}\right)|C_i| . \tag{13}$$

Recalling (12) we can continue inductively and get that for all $k \leq r_i$ we have

$$
\begin{aligned}
|N_{G'}(B_k(C_i))| &\geq |N_{G[S_i]}(B_k(C_i))| - |W| \geq \delta|B_k(C_i)|/5(\log \log m)^2 - \log^2 m \\
&\geq \delta|B_k(C_i)|/10(\log \log m)^2 ,
\end{aligned}
\tag{14}
$$

where in the last inequality we use the fact that $|B_k(C_i)| \geq |C_i| \geq \log^4 m$. It follows that C_i satisfies (6) for all $k \leq r_i$ (in the graph G'). To see this, note that by (12) $B_k(C_i) \subset B_{r_i}(C_i) \subseteq S_i$ implying that $|B_k(C_i)| \ll \sqrt{m}$. This means that in (6) the relevant d is $\log \log m - 1$, implying that we should show that $|N_{G'}(B_k(C_i))| \geq \delta|B_k(C_i)|/20(\log \log m)^2$ as we indeed derived in (14). From (14) we also have

$$|B_{r_i}(C_i)| \geq |C_i|\left(1 + \frac{\delta}{10(\log \log m)^2}\right)^{r_i} \geq \left(1 + \frac{\delta}{10(\log \log m)^2}\right)^{\frac{\log m}{30 \log \log m}} \geq \log^{10} m . \tag{15}$$

Consider now some $k > r_i$ and suppose $|B_k(C_i)| \leq m/2^{2^d}$. Then since G is assumed to be a δ-expander we have

$$
\begin{aligned}
|N_{G'}(B_k(C_i))| &\geq |N_G(B_k(C_i))| - |W| - \sum_\ell |C_\ell| \geq \delta 2^d |B_k(C_i)|/(\log \log m)^2 - \log^9 m \\
&\geq \delta 2^d |B_k(C_i)|/2(\log \log m)^2 ,
\end{aligned}
$$

where the last inequality follows from (15) which tells us that for any $k \geq r_i$ we have $|B_k(C_i)| \geq |B_{r_i}(C_i)| \geq \log^{10} m$. So C_i satisfies (6) for all d. ∎

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Proof of Lemma 3.2: Immediate from Claims 3.5, 3.6 and 3.7. ∎

4 Proof of Theorem 1

As discussed in Section 1, one can come up with different variants of Lemma 1.2 involving different notions of expansion. And indeed, to prove Theorem 1 we will need to redo most of Section 3 with a different notion of expansion, defined as follows.

Definition 4.1 ((δ, n)-Expander) *An m-vertex graph H is said to be a (δ, n)-expander if for every integer $0 \le d \le \log \log m - 1$ and $S \subseteq V(H)$ of size $|S| \le m/2^{2^d}$ we have*

$$|N(S)| \ge \frac{\delta 2^d}{\log n}|S| \tag{16}$$

To apply the above notion of expansion we will need the following two lemmas, which are appropriate variants of Lemmas 1.2 and 3.2.

Lemma 4.2 *Let G be an n-vertex graph satisfying $d(G) = c$. Then for every $\delta > 0$, the graph G has a subgraph G', such that $d(G') \ge (1 - 2\delta)c$ and G' is a (δ, n)-expander*

Lemma 4.3 *The following holds for all $0 < \delta < 1/4$ integer t and large enough $m \ge m_0(\delta, t)$. If G is an m-vertex (δ, n)-expander with $2^{\log n/(\log \log n)^2} \le m \le n$, then G contains a K_t-minor with at most $O(\frac{c(t)t^2}{\epsilon} \log n \log \log n)$ vertices.*

Let us first show how to derive Theorem 1 from the above two lemmas and the result of the previous section.

Proof of Theorem 1: We can clearly assume that $\epsilon < c(t)$ (otherwise we can replace ϵ with (say) $1/2$, which is smaller than $c(t)$ for all $t \ge 3$). Let G be an n-vertex graph satisfying $d(G) = c(t) + \epsilon$. Apply Lemma 4.2 on G with $\delta = \epsilon/8c(t)$. Suppose first that the graph G' returned by the lemma is of order $m \le 2^{\log n/(\log \log n)^2}$. Then the assertion of the lemma guarantees that $d(G') \ge (1 - 2\delta)(c(t) + \epsilon) \ge c(t) + \epsilon/2$. If $m \le n_1(\epsilon/2, t)$ we return G'. The definition of $c(t)$ then guarantees that G' has a K_t-minor, whose order is at most[3] $n_1(\epsilon/2, t) \le \log n$ and we are done. If $n_1(\epsilon/2, t) \le m \le 2^{\log n/(\log \log n)^2}$ then Lemma 3.1 guarantees that G' contains a K_t-minor of order at most $O(\frac{c(t)t^2}{\epsilon} \log m(\log \log m)^3) = O(\frac{c(t)t^2}{\epsilon} \log n \log \log n)$ as needed. Finally, if $m \ge 2^{\log n/(\log \log n)^2}$ then the fact that G' must be a (δ, n) expander guarantees, together with Lemma[4] 4.3, that G' contains a K_t-minor of order $O(\frac{c(t)t^2}{\epsilon} \log n \log \log n)$ thus completing the proof. ∎

Let us now turn to prove Lemmas 4.2 and 4.3. In what follows, we say that a graph $G = (V, E)$ fails to be a (δ, n)-expander at scale d if there is a vertex set $S \subseteq V$ of size at most $m/2^{2^d}$ violating (16).

---

[3]Here we use the assumption that $n \ge n_0(\epsilon, t)$. Specifically, we take $n_0(\epsilon, t) = 2^{n_1(\epsilon/2, t)}$, where $n_1(\epsilon, t)$ is the function used in Lemma 3.1.

[4]Here we again use the assumption that $n \ge n_0(\epsilon, t)$. Specifically, it is enough to take $n_0(\epsilon, t) = (m_0(\delta, t))^{\log m_0(\delta, t)}$, where $\delta = \epsilon/8c(t)$ and $m_0(\delta, t)$ is the function in Lemma 4.3

Proof of Lemma 4.2: Set $G_0 = G$ and consider the following iterative process; if either $|G_t| \leq 4$ or if G_t is a (δ, n)-expander the process returns G_t. Otherwise, set $m = |G_t|$ and $d_t = d(G_t)$. Since G_t is not a (δ, n)-expander, there must be some $0 \leq d \leq \log \log m - 1$ and a set $S_t \subseteq V(G_t)$ of size at most $m/2^{2^d}$ violating (16). We can deduce via Claim 2.1 (with $c = d_t$ and $\gamma = \delta 2^d / \log n$) that in this case either $d(G[V(G_t) \setminus S_t]) \geq d_t$ or $d(G[S_t \cup N(S_t)]) \geq (1 - \delta 2^d / \log n) d_t$. In the former case we set $G_{t+1} = G[V(G_t) \setminus S_t]$ and in the latter we set $G_{t+1} = G[S_t \cup N(S_t)]$. For brevity, in what follows we call the above two cases, Case 1 and Case 2. Let G' be the graph returned at the end of the process. We will now show that $d(G') \geq (1 - 2\delta)c$. Note that this assertion implies that if $|V(G')| > 4$ then the definition of the process implies that G' is a (δ, n)-expander and if $|V(G')| \leq 4$, then since $\delta < 1/4$ one of G' connected components, say G'', is (trivially) a δ-expander satisfying $d(G'') \geq (1 - 2\delta)c$.

Recall that for each of the graphs G_t in the process $d_t = d(G_t)$ (so $d_0 = c$). Note that at each iteration either $d_{t+1} \geq d_t$ (in Case 1) or $d_{t+1} = \gamma_t d_t$ for some $\gamma_t > 0$. Fix some $1 \leq k \leq \log \log n$ and suppose $r < r'$ are such that G_r is the first graph in the process whose size is in the range $[2^{2^{k-1}}, 2^{2^k}]$ and $G_{r'}$ is the last such graph in the process. We wish to compute a lower bound on $d_{r'+1}/d_r$, that is, a lower bound on the fraction of edge loss that can occur when "passing" through the interval $[2^{2^{k-1}}, 2^{2^k}]$. Observe that if at some iteration $r \leq t \leq r'$ of the process, the graph G_t fails to be a (δ, n)-expander at scale d, then either $\gamma_t \geq 1$ (Case 1), or $|G_{t+1}| \leq |G_t|/2^{2^d}$. For each $0 \leq d \leq k$ let a_d be the number of times Case 2 occurred due to G_t failing to be a (δ, n)-expander at scale d. Then we have

$$1 \leq |G_{r'+1}| \leq |G_r| / \prod_{d=0}^{k} 2^{a_d 2^d} \leq 2^{2^k} / \prod_{d=0}^{k} 2^{a_d 2^d} \ ,$$

implying that

$$\sum_{d=0}^{k} a_d 2^d \leq 2^k \ . \tag{17}$$

Now, if at some iteration $r \leq t \leq r'$ Case 2 happened due to the fact that G_t failed to be a (δ, n)-expander at scale d, then (as noted above) we know that in this case

$$\gamma_t \geq 1 - \delta 2^d / \log n \ . \tag{18}$$

Using that a_d was the number of times γ_t satisfied (18), we have that

$$d_{r'+1}/d_r = \prod_{t=r}^{r'} \gamma_t \ \geq \ \prod_{d=0}^{k} \left(1 - \frac{\delta 2^d}{\log n}\right)^{a_d} \geq \left(1 - \delta \sum_{d=0}^{k} \frac{a_d 2^d}{\log n}\right)$$

$$\geq \ \left(1 - \frac{\delta 2^k}{\log n}\right) \ ,$$

where the last inequality follows from (17). We have thus established that d_t goes down by a factor of at most $1 - \delta 2^k / \log n$ when the process passes through the interval $[2^{2^{k-1}}, 2^{2^k}]$. Hence, altogether d_t can decrease by a factor of at most

$$\prod_{k=1}^{\log \log n} \left(1 - \frac{\delta 2^k}{\log n}\right) \geq 1 - \frac{\delta}{\log n} \sum_{k=1}^{\log \log n} 2^k \geq 1 - 2\delta. \tag{19}$$

So when the process ends we indeed obtain a (δ, n)-expander $G_{t'}$ satisfying $d(G_{t'}) \geq c(1 - 2\delta)$. ∎

As it turns out the proof of Lemma 4.3 can be obtained by repeating almost verbatim the proofs of Claims 3.3, 3.5, 3.6 and 3.7, while replacing the notion of δ-expander with (δ, n)-expander, and making some minor adaptations to the calculations. Hence we only state the appropriate variants of these claims. The detailed proofs of these claims can be found in the appendix of the arxiv version of this paper. The proof of Lemma 4.3 follows immediately from Claims 4.5, 4.6 and 4.7.

Claim 4.4 *Suppose U and V are two vertex sets in an m-vertex graph G such that the following condition holds for every integer $0 \leq d \leq \log \log m - 1$; whenever $|B_k(U)| \leq m/2^{2^d}$ we have*

$$|N(B_{k+1}(U))| \geq \frac{\delta 2^d}{10 \log n}|B_k(U)| \,, \tag{20}$$

and a similar condition holds with respect to V. Then there is a path connecting U to V of length at most $\frac{20}{\delta} \log n (\log \log n)$.

Claim 4.5 *The following holds for all $\delta > 0$ integer t and large enough $n \geq n_0(\delta, t)$. If G is a (δ, n)-expander on m vertices with $2^{\log n/(\log \log n)^2} \leq m \leq n$ then one of the following holds;*

1. *G has t vertices $v_1, \ldots, v_t$ each of degree at least $\log^4 m$.*

2. *Set $\gamma = \frac{\delta \log m}{5 \log n}$. Then G contains t disjoint vertex sets $S_1, \ldots, S_t$ such that $G[S_i]$ is a γ-expanding ball, $m^{1/5} \leq |S_i| \leq m^{1/4}$ and every vertex in $G[S_i]$ has degree at most $\log^4 m$.*

Claim 4.6 *The following holds for all $\delta > 0$ integer t and large enough $n \geq n_0(\delta, t)$. If G is a (δ, n)-expander on $2^{\log n/(\log \log n)^2} \leq m \leq n$ vertices which has t vertices as in the first case of Claim 4.5 then it has a K_t-minor of order $O(\frac{t^2}{\delta} \log n \log \log n)$.*

Claim 4.7 *The following holds for all $\delta > 0$ integer t and large enough $n \geq n_0(\delta, t)$. If G is a (δ, n)-expander on $2^{\log n/(\log \log n)^2} \leq m \leq n$ vertices which has t sets as in the second case of Claim 4.5 then it has a K_t-minor of order $O(\frac{t^2}{\delta} \log n \log \log n)$.*

5   Concluding Remarks and Open Problems

- Of course, the obvious open problem is to close the $O(\log \log n)$ gap between the upper bound of Theorem 1 and the simple lower bound mentioned in Section 1. It seems that we have pushed our approach to the limit, and that in order to obtain a K_t-minor of order $C(\epsilon) \log n$ some new ideas are needed.

  We note that the proof of [6] showing that $(2^{t-2} + \epsilon)n$ edges suffice to guarantee a K_t-minor of order $C(\epsilon) \log n$ proceeds by applying the well known argument showing that $c(t) \leq 2^{t-3}$. In some sense, we are able to prove our bound already for $d(G) \geq c(t) + \epsilon$ since we do *not* implicitly prove any bound on $c(t)$ but instead rely on it as a black-box. Having said that, it might very well be possible to use one of the proofs that give a tight bound on $c(t)$ to improve our lower bound.

- A natural variant of the problem studied here concerns topological K_t-minors. As noted in [6], the method of [11] can probably be used to show that a graph with $4^{t^2}n$ edges has a topological K_t-minor of order $O(\log n)$. However, it is known (see [3] and [4]) that there is a smallest $c'(t) = \Theta(t^2)$ such that any graph with $c'(t)n$ edges has a topological K_t-minor. It thus seems reasonable to conjecture that if G is an n-vertex graph with $(c'(t) + \epsilon)n$ edges then G must contain a topological K_t-minor of order at most $C(\epsilon) \log n$. We suspect that the ideas and tools in the present paper should allow one to prove (at least) a weaker version of this conjecture, giving a topological K_t-minor of order at most $C(\epsilon) \log n \, \text{poly}(\log \log n)$.

- Let us say that a graph G is ϵ-far from being K_t-minor free if one should remove from G at least ϵn edges in order to turn G into a K_t-minor free graph. We conjecture that if G is ϵ-far from being K_t-minor free then G has a K_t-minor of order $C(\epsilon) \log n$. At the moment, we are unable to prove even a weaker result in which the K_t-minor is of order $C(\epsilon) \log n \, \text{poly}(\log \log n)$. One approach to proving such a result would be to prove a strengthened version of Lemma 1.2 stating that for any $\delta > 0$, and large enough graph G one can remove from G at most δn edges and thus obtain a graph G' in which every connected component is a δ-expander. Note that Lemma 3.2 together with the above variant of Lemma 1.2 would immediately resolve the above conjecture (with the slightly weaker bound $O(\log n \, \text{poly}(\log \log n))$). Finally, observe that any graph with $(c(t) + \epsilon)n$ edges is by definition ϵ-far from being K_t-minor free, hence this conjecture strengthens the problem raised in [6]. One can of course raise the same conjecture with respect to topological minors.

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6 Missing Proofs from Section 4

Proof of Claim 4.4: It is clearly enough to show that $|B_k(U)| > m/2$ for some $k \leq \frac{10}{\delta} \log n \log \log n$. So suppose $m/2^{2^d} \leq |B_k(U)| \leq m/2^{2^{d-1}}$. Then by (20) we either have $|B_{k+t}(U)| > m/2^{2^{d-1}}$ or

$$m/2^{2^{d-1}} \geq |B_{k+t}(U)| \geq |B_k(U)| \left(1 + \frac{\delta 2^{d-1}}{10 \log n}\right)^t \geq m/2^{2^d - \frac{\delta t 2^{d-1}}{10 \log n}} . \tag{21}$$

It is easy to check that the RHS of (21) is larger than the LHS when $t \geq \frac{10}{\delta} \log n$. In other words, within at most $\frac{10}{\delta} \log n$ steps, the neighborhood around U jumps from size larger than $m/2^{2^d}$ to size larger than $m/2^{2^{d-1}}$. Since there are only $\log \log m \leq \log \log n$ intervals $[m/2^{2^d}, m/2^{2^{d-1}}]$ we see that $|B_k(U)| > m/2$ for some $k \leq \frac{10}{\delta} \log n \log \log n$. ∎

Proof of Claim 4.5: If G has t vertices $v_1, \ldots, v_t$ each of degree larger than $\log^4 m$ then we have the first case of the lemma. So suppose for the rest of the proof that G has at most t vertices of degree larger than $\log^4 m$. We need to show that we can pick sets $S_1, \ldots, S_t$ satisfying the second condition of the lemma. Let T be the set containing the vertices of degree larger than $\log^4 m$. Let us also say that a set S is *nice* if $G[S]$ is a γ-expanding ball and $m^{1/5} \leq |S| \leq m^{1/4}$. Note that a nice set in $G \setminus T$ satisfies the second condition of the lemma.

It is clearly enough to show that for any set $W \supseteq T$ of at most $m^{1/3}$ vertices, we can find in $G \setminus W$ a nice set S. Once we know this, we can simply iteratively pick the sets S_i one after the other where at iteration i we will pick S_i from $G \setminus W_i$ with $W_i = (\bigcup_{j<i} S_j) \cup T$. Since $|T|, |S_1|, \ldots, |S_{i-1}| \ll m^{1/3}$ this set has size smaller than $m^{1/3}$.

So suppose to the contrary that there is a $W \supseteq T$ of size $m^{1/3}$ such that $G \setminus W$ has no nice set S and let $G_0 = G \setminus W$. As we assume that $W \supseteq T$, all the vertices in G_0 have degree at most $\log^4 m$. So pick a vertex $v_1 \in G_0$ and let $B_k(v_1)$ be the vertices at distance at most k from v_1 in G_0. Let k_1 be the smallest integer such that $|B_{k_1+1}(v)| < |B_{k_1}(v)|(1+\gamma)$. We claim that $|B_{k_1+1}(v)| < m^{1/4}$. To see this observe that the fact that all the vertices in G_0 have degree at most $\log^4 m$ implies that for every $k \leq k_1$ we have $|B_{k-1}(v)| \geq |B_k(v)|/\log^4 m$ so if $|B_{k_1+1}(v)| \geq m^{1/4}$ then there must be a k_1', such that $m^{1/5} \leq |B_{k_1'}(v)| \leq m^{1/4}$ and $|B_{i+1}(v)| \geq |B_i(v)|(1+\gamma)$ for all $1 \leq i \leq k_1'$. But in this case $B_{k_1'}(v)$ would be a nice set. So setting $T_1 = B_{k_1}(v_1)$ we have $|T_1| < m^{1/4}$ and $|N_{G_0}(T_1)| < \gamma|T_1|$.

Let $G_1 = G_0 \setminus T_1$. Take now another vertex $v_2 \in G_1$ and repeat the above process. We will eventually end up with a set $T_2 = B_{k_2}(v_2)$ of size smaller than $m^{1/4}$ satisfying $N_{G_1}|T_2| < \gamma|T_2|$. Let $G_2 = G_1 \setminus T_2$. We continue picking sets $T_i \subset G_{i-1}$ until the first time $|\bigcup_i T_i| > \sqrt{m}/2$. Since each set T_i is of size at most $m^{1/4} \leq \sqrt{m}/2$ this means that $\sqrt{m}/2 \leq |\bigcup_i T_i| \leq \sqrt{m}$. Now, the fact that each of the sets T_i satisfies $|N_{G_{i-1}}(T_i)| < \gamma|T_i|$ along with the facts that $G_i = G_{i-1} \setminus T_i$ and $|\bigcup_i T_i| \leq \sqrt{m}$ implies that

$$\left| N_{G_0}\left(\bigcup_i T_i\right) \right| \leq \sum_i |N_{G_{i-1}}(T_i)| < \sum_i \gamma|T_i| \leq \gamma\sqrt{m} = \delta\sqrt{m} \log m / 5 \log n . \tag{22}$$

Now recall that G was assumed to be a (δ, n)-expander and since $\sqrt{m}/2 \leq |\cup T_i| \leq \sqrt{m}$ we must have

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(using $d = \log \log m - 1$ in (16))

$$\left| N_G \left(\bigcup_i T_i \right) \right| \geq \delta \left| \bigcup_i T_i \right| \frac{\log m}{2 \log n} \geq \delta \sqrt{m} \log m / 4 \log n \ . \tag{23}$$

Note that G_0 was obtained from G by removing a set W of no more than $m^{1/3}$ vertices and that by our assumption $m \gg \log^6 n$. Hence, using (22) we have

$$\left| N_G \left(\bigcup_i T_i \right) \right| \leq |W| + \left| N_{G_0} \left(\bigcup_i T_i \right) \right| \leq m^{1/3} + \delta \sqrt{m} \log m / 5 \log n < \delta \sqrt{m} \log m / 4 \log n \ ,$$

which contradicts (23). ∎

Proof of Claim 4.6: Let $v_1, \ldots, v_t$ be the vertices satisfying the first assertion of Claim 3.5. We will show that in this case G has a topological K_t-minor of the required order. To do this we show that we can find $\binom{t}{2}$ paths $P_{i,j}$, where each path connects v_i to v_j, has length at most $\frac{20}{\delta} \log n \log \log n$ and is internally disjoint from all other paths. Our plan is to successively find these paths by invoking Claim 4.4. All we need to do is show that for any v_i, v_j and any set W of size at most $\log^2 n$ the sets $\{v_i\}, \{v_j\}$ satisfy (20) with respect to the graph $G \setminus W$. Once we establish this fact, we will be able successively pick the paths $P_{i,j}$ via Claim 4.4 where at each iteration we will take W to consist of the vertices v_t, $t \neq i, j$, together with the internal vertices of the paths $P_{i',j'}$ we have already picked.

We turn to show that $\{v_i\}$ satisfies (20) in the graph $G' = G \setminus W$ (the proof for $\{v_j\}$ is identical). We first recall the assumption of the lemma that v_i has degree $\log^4 m$ in G. Since we assume that $|W| \leq \log^2 n$ and $\log m \geq \log^{1-o(1)} n$ we have

$$N_{G'}(v_i) \geq N_G(v_i) - |W| \geq \log^4 m - \log^2 n \geq \frac{1}{2} \log^4 m \ .$$

Hence, for all $k \geq 1$

$$|B_k(v_i)| \geq |B_1(v_i)| \geq \frac{1}{2} \log^4 m \gg \log^3 n \ . \tag{24}$$

Suppose now that $|B_k(v_i)| \leq m/2^{2^d}$. Since we assume that G is a (δ, n)-expander we deduce that

$$
\begin{aligned}
|N_{G'}(B_k(v_i))| & \geq \ |N_G(B_k(v_i))| - |W| \geq \frac{\delta 2^d}{\log n} |B_k(v_i)| - \log^2 n \\
& \geq \ \frac{\delta 2^d}{2 \log n} |B_k(v_i)| \ ,
\end{aligned}
$$

where the last inequality follows from (24) and the assumptions that $n \geq n_0(\delta, t)$. We thus get that G' satisfies (20) with $U = \{v_i\}$ and $V = \{v_j\}$ so we can indeed find a path connecting v_i to v_j of length at most $\frac{20}{\delta} \log n \log \log n$. ∎

Proof of Claim 4.7: Let $S_1, \ldots, S_t$ be the sets satisfying the second condition of Claim 4.5. Recall that in this case each set S_i is a γ-expanding ball around a vertex v_i in the induced subgraph $G[S_i]$. Also recall that $m^{1/5} \leq |S_i| \leq m^{1/4}$ and that all the vertices of $G[S_i]$ have degree at most $\log^4 m$. This means that there is some k_i, such that the ball around v_i in the graph $G[S_i]$ satisfies $\log^4 m \leq |B_{k_i}(v_i)| \leq \log^8 m$. For every $1 \leq i \leq t$ set $C_i = B_{k_i}(v_i)$ and note that C_i is also a γ-expanding ball in $G[S_i]$ around the same center v_i.

We will shortly show that one can find $\binom{t}{2}$ internally vertex disjoint paths $P_{i,j}$, where each $P_{i,j}$ connects C_i to C_j, has length at most $\frac{20}{\delta} \log n \log \log n$ and avoids all the sets $C_\ell, \ell \neq i, j$. But let us first observe why this will conclude the proof. For each $1 \leq i \neq j \leq t$ let $Q_{i,j}$ be some path connecting v_i to the unique vertex of $P_{i,j}$ that belongs to C_i. For each $1 \leq i \leq t$ set $V_i = \cup_j Q_{i,j}$. Then each set V_i is connected in G and the paths $P_{i,j}$ are internally vertex disjoint and avoid the sets $V_\ell, \ell \neq i, j$ so contracting the sets V_i indeed gives us a K_t-minor in G. As to the order of this minor, note that since C_i is a γ-expanding-ball (with $\gamma = \delta \log m / 5 \log n$) of size at most $\log^8 m$ its radius is bounded by

$$\log_{1+\gamma} |C_i| \leq \frac{2 \log(\log^8 m)}{\gamma} \leq \frac{80 \log n \log \log m}{\delta \log m} \,,$$

so each of the paths $Q_{i,j}$ is of length $o(\log n)$ (here we rely on the lemma's assumption that $\log^{1-o(1)} n \leq \log m \leq \log n$). Hence the total size of the sets V_i is smaller than $\log n$. Thus, once we show how to find the above mentioned paths $P_{i,j}$ with lengths bounded by $\frac{20}{\delta} \log n \log \log n$ we obtain a K_t-minor of order at most

$$\log n + \frac{20t^2}{\delta} \log n \log \log n = O\Big(\frac{t^2}{\delta} \log n \log \log n \Big).$$

Our plan is to show that one can simply iteratively pick the paths $P_{i,j}$ by successive applications of Claim 4.4. To this end, we need to show that after picking some of the paths, we can still find in the remaining graph another path. Since each of the paths is of length at most $\frac{20}{\delta} \log n \log \log n \ll \log^2 n$, it is enough to show that for any set of vertices W of size at most $\log^2 n$, that is disjoint from each of the sets $C_1, \ldots, C_t$, and such that for any $i \neq j$, the sets C_i and C_j satisfy (20) with respect to the graph

$$G' = G \setminus \Big(W \cup \bigcup_{\ell \neq i,j} C_\ell \Big). \tag{25}$$

Once we know this, we will iteratively pick the paths $P_{i,j}$, where at each iteration we will take W to be the union of the internal vertices of the paths we have already picked. To establish (20) we will need to consider two "regimes" of growth; the first is when C_i grows within the set S_i, and the second, when it grows out of S_i.

Let us first make a simple (but crucial) observation about the sets C_i. Recall that $S_i = B_{k'_i}(v_i)$ in $G[S_i]$ for some vertex v_i and k'_i, that $|S_i| \geq m^{1/5}$ and that each vertex in S_i has degree at most $\log^4 m$. This means that $k'_i \geq \log m / 20 \log \log m$. We also recall that $|C_i| \leq \log^8 m$ and since S_i is a γ-expanding ball with $\gamma \geq \delta \log m / 5 \log n$ this means that in the induced subgraph $G[S_i]$ we have that $C_i = B_{k_i}(v_i)$ for some

$$k_i \leq \frac{2 \log(\log^8 m)}{\gamma} \leq \frac{80 \log n \log \log m}{\delta \log m} \leq (\log \log m)^4 \,,$$

where the second inequality relies on the lemma's assumption that $\log m \geq \log^{1-o(1)} n$. Combining the above facts about k_i and k_i' we get that setting $r_i = \log m / 30 \log \log m$ we have in the graph $G[S_i]$

$$B_{r_i}(C_i) \subseteq S_i \,. \tag{26}$$

We now turn to show that C_i satisfies (20) with respect to G' defined in (25). Since S_i is a γ-expanding ball we get from (12) that $|N_{G[S_i]}(C_i)| \geq \delta |C_i| \log m / 5 \log n$. Recall now that S_i is disjoint from $\bigcup_{r \neq i,j} C_r$, and that (26) implies that $N_{G[S_i]}(C_i) \subseteq S_i$. Hence

$$
\begin{aligned}
|N_{G'}(C_i)| &\geq |N_{G[S_i]}(C_i)| - |W| \geq \delta |C_i| \log m / 5 \log n - \log^2 n \\
&\geq \delta |C_i| \log m / 10 \log n \,,
\end{aligned}
$$

where the last inequality uses the fact that $|C_i| \geq \log^4 m$ and the assumption $\log m \geq \log^{1-o(1)} n$. In other words, we have in G'

$$|B_1(C_i)| \geq \left(1 + \frac{\delta \log m}{10 \log n}\right) |C_i| \,. \tag{27}$$

Recalling (26) we can continue inductively and get that for all $k \leq r_i$ we have

$$
\begin{aligned}
|N_{G'}(B_k(C_i))| &\geq |N_{G[S_i]}(B_k(C_i))| - |W| \geq \delta |B_k(C_i)| \log m / 5 \log n - \log^2 n \\
&\geq \delta |B_k(C_i)| \log m / 10 \log n \,,
\end{aligned} \tag{28}
$$

where in the last inequality we again used the facts that $|B_k(C_i)| \geq |C_i| \geq \log^4 m$ and that $\log m \geq \log^{1-o(1)} n$. We get that C_i satisfies (20) for all $k \leq r_i$. To see this, note that by (26) $B_k(C_i) \subset B_{r_i}(C_i) \subseteq S_i$ implying that $|B_k(C_i)| \ll \sqrt{m}$. This means that in (20) the relevant d is $\log \log m - 1$, implying that we should show that $|N_{G'}(B_k(C_i))| \geq \delta |B_k(C_i)| \log m / 20 \log n$ as we indeed derive in (28). ¿From (28) we also have

$$|B_{r_i}(C_i)| \geq |C_i| \left(1 + \frac{\delta \log m}{10 \log n}\right)^{r_i} \geq \left(1 + \frac{\delta \log m}{10 \log n}\right)^{\frac{\log m}{30 \log \log m}} \geq \log^{11} m \,, \tag{29}$$

where the last inequality uses the the assumption that $\log m \geq \log^{1-o(1)} n$.

Consider now some $k > r_i$ and suppose $|B_k(C_i)| \leq m / 2^{2^d}$. Then since G is assumed to be a (δ, n)-expander we have

$$
\begin{aligned}
|N_{G'}(B_k(C_i))| &\geq |N_G(B_k(C_i))| - |W| - \sum_\ell |C_\ell| \geq \delta 2^d |B_k(C_i)| / \log n - \log^9 m \\
&\geq \delta 2^d |B_k(C_i)| / 2 \log n \,,
\end{aligned}
$$

where the last inequality follows by combining the assumption that $\log m \geq \log^{1-o(1)} n$ with (29) which tells us that for any $k \geq r_i$ we have $|B_k(C_i)| \geq |B_{r_i}(C_i)| \geq \log^{11} m$. So C_i satisfies (20) for all d. $\blacksquare$