A quantitative Lovász criterion for Property B

Asaf Ferber * Asaf Shapira †

Abstract

A well known observation of Lovász is that if a hypergraph is not 2-colorable, then at least one pair of its edges intersect at a single vertex. In this short paper we consider the quantitative version of Lovász's criterion. That is, we ask how many pairs of edges intersecting at a single vertex, should belong to a non 2-colorable n-uniform hypergraph? Our main result is an *exact* answer to this question, which further characterizes all the extremal hypergraphs. The proof combines Bollobás's two families theorem with Pluhar's randomized coloring algorithm.

1 Introduction

A hypergraph $\mathcal{H} = (V, E)$ consists of a vertex set V and a set of edges E where each $X \in E$ is a subset of V. If all edges of $\mathcal{H}$ have size n then $\mathcal{H}$ is called an n-uniform hypergraph, or n-graph for short. A hypergraph is 2-colorable if one can assign each vertex $v \in V$ one of two colors, say $Red/Blue$, so that each $X \in E$ contains vertices of both colors. Miller [6], and later Erdős in various papers, referred to this property as *Property B*, after F. Bernstein [2] who introduced it in 1907. Since deciding if a hypergraph is 2-colorable is NP-hard one cannot hope to find a simple characterization of all 2-colorable hypergraphs. Instead, one looks for general sufficient/necessary conditions for having this property. For example, a famous result of Seymour [8] states that if $\mathcal{H}$ is not 2-colorable then $|E| \geq |V|$. Probably the most well studied question of this type asks for the smallest number of edges in an n-graph that is not 2-colorable. The study of this quantity, denote $m(n)$, was popularized by Erdős, see [1] for a comprehensive treatment. Despite much effort by many researchers, even the asymptotic value of $m(n)$ has not been determined yet.

A pair of edges $X, Y \in E(\mathcal{H})$ is *simple* if $|X \cap Y| = 1$. Let $m_2(\mathcal{H})$ denote the number of ordered simple pairs of edges of $\mathcal{H}$. A well known observation of Lovász [5] states that if $\mathcal{H}$ is not 2-colorable then $m_2(\mathcal{H}) > 0$. Despite its simplicity, this observation underlies the best known bounds for $m(n)$, see [4, 7]. It is natural to ask if one can obtain a quantitative version of Lovász's observation, that is, estimate how small can $m_2(\mathcal{H})$ be in an n-graph not satisfying property B? Our main result in this paper states that (somewhat surprisingly), one can give an exact answer to the above extremal question as well as characterize the extremal n-graphs.

Let K^n_{2n-1} denote the complete n-graph on $2n-1$ vertices. It is easy to see that K^n_{2n-1} is not 2-colorable and that $m_2(K^n_{2n-1}) = n \cdot \binom{2n-1}{n}$. We first observe that this simple upper bound is tight.

Proposition 1.1. *If an n-graph is not 2-colorable then $m_2(\mathcal{H}) \geq n \cdot \binom{2n-1}{n}$.*

As with any extremal problem, one would like to know which graphs or hypergraphs are extremal with respect to this problem. For example, Turán's theorem states that among all n-vertex graphs not containing a complete t-vertex subgraph, there is only one graph maximizing the number of edges. In the setting of our

*Department of Mathematics, MIT, Cambridge, MA, USA. Email: ferbera@mit.edu. Supported in part by NSF grant 6935855.

†School of Mathematics, Tel Aviv University, Tel Aviv 69978, Israel. Email: asafico@tau.ac.il. Supported in part by ISF Grant 1028/16 and ERC Starting Grant 633509.

problem, it is easy to see that K^n_{2n-1} is not the only non 2-colorable n-graph satisfying $m_2(\mathcal{H}) = n \cdot \binom{2n-1}{n}$, since one can take a copy of K^n_{2n-1} and add to it more vertices and edges without increasing the number of simple pairs. Our main result in this paper characterizes the extremal n-graphs, by showing that this is in fact the only way to construct an n-graph meeting the bound of Proposition 1.1.

Theorem 1. *If a non 2-colorable n-graph $\mathcal{H}$ satisfies $m_2(\mathcal{H}) = n \cdot \binom{2n-1}{n}$ then it contains a copy of K^n_{2n-1}.*

While the proof of Proposition 1.1 is implicit in Pluhar's [7] argument for bounding $m(n)$, the proof of Theorem 1 is more intricate, relying on Bollobás's two families theorem [3] as well as on a refined analysis of Pluhar's randomized algorithm for 2-coloring n-graphs.

2 Proof of Proposition 1.1

In this section we describe several preliminary observations regarding a coloring algorithm introduced in [7], and use them to derive Proposition 1.1. The algorithm is the following:

Algorithm $\mathrm{Col}(\mathcal{H}, \pi)$. The input is a hypergraph $\mathcal{H} = (V, E)$ and an ordering $\pi : V \mapsto \{1, \ldots, |V|\}$ (that is, π is a bijection). The output is a 2-coloring of V (not necessarily a proper one). The algorithm runs in $|V|$ steps, where in each time step $1 \leq i \leq |V|$, the vertex $\pi^{-1}(i)$ is being colored *Blue* if this does not form any monochromatic *Blue* edge. Otherwise, $\pi^{-1}(i)$ is colored *Red*.

We now state an important property of $\mathrm{Col}(\mathcal{H}, \pi)$. For two disjoint subsets $X, Y \subseteq V$, we use the notation $\pi(X) < \pi(Y)$ whenever $\max_{x \in X} \pi(x) < \min_{y \in Y} \pi(y)$, that is, the elements of X precede all the elements of Y in the ordering π. Suppose (X, Y) is a simple pair of edges in $\mathcal{H}$ with[1] $X \cap Y = y$. We say that π *separates* (X, Y) if $\pi(X \setminus y) < \pi(y) < \pi(Y \setminus y)$.

Claim 2.1. *If $Col(\mathcal{H}, \pi)$ fails to properly color $\mathcal{H}$ then π separates at least one pair of simple edges.*

Proof. We first observe that (by definition) for every ordering π, the algorithm $\mathrm{Col}(H, \pi)$ does not produce monochromatic *Blue* edges. Suppose then it produced a *Red* edge $Y \in E$. Let y be the first vertex of Y according to the ordering π. If y was colored red, then there must have been an edge X so that $y \in X$, and all other vertices of X were already colored *Blue* (otherwise the algorithm would color y *Blue*). This means (X, Y) is simple and that π separates it. □

Note that the claim above already shows that if $\mathcal{H}$ is not 2-colorable then $m_2(\mathcal{H}) > 0$. For the proof of Proposition 1.1 we will also need the following simple fact.

Claim 2.2. *A random permutation separates any given simple pair with probability $1/n\binom{2n-1}{n}$.*

Proof. Let (X, Y) be a simple pair, and let $X \cap Y = y$. A permutation π separates (X, Y) if and only if $\pi(X \setminus y) < \pi(y) < \pi(Y \setminus y)$, and this happens with probability exactly

$$\frac{(n-1)!(n-1)!}{(2n-1)!} = \frac{1}{n\binom{2n-1}{n}}$$

as desired. □

The above claims suffice for proving Proposition 1.1.

Proof (of Proposition 1.1): Assume $m_2(\mathcal{H}) < n\binom{2n-1}{n}$. Suppose we pick a uniformly random π. Then by the union bound and Claim 2.2, we infer that with positive probability π does not separate any simple pair edges. Hence, there is a π not separating any simple pair. Claim 2.1 then implies that $\mathrm{Col}(\mathcal{H}, \pi)$ will produce a legal 2-coloring of $\mathcal{H}$. □

[1] Here, and in what follows, we slightly abuse notation by writing y instead of the more appropriate $\{y\}$.

3 Proof of Theorem 1

For the rest of this section fix some non 2-colorable n-graph $\mathcal{H} = (V, E)$ satisfying $m_2(\mathcal{H}) = n\binom{2n-1}{n}$. We need to show that $\mathcal{H}$ contains a copy of K^n_{2n-1}. We start with a few preliminary claims regarding $\mathcal{H}$.

First, we show that no π separates more than one simple pair.

Claim 3.1. *Every ordering π separates at most one simple pair.*

Proof. Suppose π separates two simple pairs. By Claim 2.2, the assumption on $m_2(\mathcal{H})$, and by linearity of expectation, the expected number of simple pairs separated by a random permutation is exactly 1. Hence, if π separates 2 simple pairs, then there must exist a permutation σ which separates less than 1, and therefore 0, simple pairs. Therefore, by Claim 2.1 we obtain that $\mathrm{Col}(\mathcal{H}, \sigma)$ produces a legal 2-coloring of $\mathcal{H}$, which is a contradiction to the assumption that $\mathcal{H}$ is not 2-colorable. □

Claim 3.2. *If (X, Y) and (X', Y) are simple pairs, then $X \cap Y \neq X' \cap Y$.*

Proof. We observe that if $X \cap Y = X' \cap Y = y$, then there is a π that separates both (X, Y) and (X', Y), and this will contradict Claim 3.1. Indeed, if (X, Y) and (X', Y) are simple pairs and $X \cap Y = X' \cap Y = y$, then $(X \cup X') \setminus y$ and Y are disjoint. Therefore, any π satisfying

$$\pi((X \cup X') \setminus y) < \pi(y) < \pi(Y \setminus y)$$

separates (X, Y) and (X', Y). This completes the proof. □

In addition to the above observations about $\mathcal{H}$, the last ingredient we will need is the following theorem of Bollobás [3].

Lemma 3.3. *Let I be an index set. For all $i \in I$, let A_i and B_i be subsets of a set V of p elements satisfying the following conditions:*

i. $A_i \cap B_i = \emptyset$ for all $i \in I$, and

ii. $A_j \not\subseteq A_i \cup B_i$ for all $i \neq j \in I$.

Then, we have

$$\sum_{i \in I} \frac{1}{\binom{p-|B_i|}{|A_i|}} \leq 1,$$

with equality if and only if $B_i = B$ for all $i \in I$ and the sets A_i are all the q-tuples of the set $P \setminus B$ for some value of q.

Let us now show how to use Lemma 3.3 in order to derive Theorem 1. Recall that V is the vertex set of $\mathcal{H}$ and set $p := |V|$. Let $M(\mathcal{H})$ be a collection of simple pairs (X, Y) defined as follows; out of all the simple pairs (X, Y) with the same "second" set Y, put in $M(\mathcal{H})$ one of these pairs. Observe that by Claim 3.2 each Y belongs to at most $|Y| = n$ simple pairs of the form (X, Y) (i.e, with Y as the second set), implying that $t := |M(\mathcal{H})| \geq \frac{1}{n} \cdot m_2(\mathcal{H}) = \binom{2n-1}{n}$. We now define a collection $\mathcal{F}$ consisting of pairs of subsets of V as follows: For every simple pair $s := (X, Y) \in M(\mathcal{H})$, define $A_s = X \setminus Y$ and $B_s = V \setminus (X \cup Y)$, and let $\mathcal{F} = \{(A_s, B_s) : s \in M(\mathcal{H})\}$. For convenience, let us rename the pairs in $\mathcal{F}$ as (A_i, B_i) with $1 \leq i \leq t$.

Now we wish to show that $\mathcal{F}$ satisfies the conditions in Lemma 3.3. Observe that if it does, then since

$$\sum_{i=1}^{t} \frac{1}{\binom{p-|B_i|}{|A_i|}} = \sum_{i=1}^{t} \frac{1}{\binom{2n-1}{n-1}} \geq 1,$$

it follows by the first part of Lemma 3.3 that the last inequality is in fact an equality. Therefore, by the second part of Lemma 3.3, we conclude that all the B_i's are the same set B, and the set of all the A_i's consists of all $n-1$ subsets of a ground set of size $2n-1$. That is, let $B = B_i$ and $U = V \setminus B$. Then we have that $|U| = 2n-1$, and that the sets A_i are all the $n-1$ subsets of U. Since by construction we have that $U \setminus A_i \in E(\mathcal{H})$ for all i, we conclude that $\mathcal{H}$ restricted to the set U is a copy of K^n_{2n-1} as desired. It thus remains to show the following:

Claim 3.4. *$\mathcal{F}$ satisfies the conditions in Lemma 3.3*

Proof. The first condition $A_i \cap B_i = \emptyset$ for all i is trivially satisfied by construction. For the second condition, let (A, B) and (A', B') be two elements in $\mathcal{F}$ coming from simple pairs (X, Y) and (X', Y') belonging to $M(\mathcal{H})$, respectively. Recall that by the way we defined $M(\mathcal{H})$ and $\mathcal{F}$ we have $Y \neq Y'$. Let us use y and y' to denote the unique elements in $X \cap Y$ and $X' \cap Y'$, respectively. We wish to show that $A \not\subseteq A' \cup B'$, which, by construction, is implied by $(X \setminus y) \cap Y' \neq \emptyset$. Assuming $(X \setminus y) \cap Y' = \emptyset$, we will derive a contradiction to Claim 3.1 by showing that there is a permutation π separating two distinct simple pairs.

Observe that it cannot be that $y \in Y'$. Indeed, if it was the case, then together with the assumption that $(X \setminus y) \cap Y' = \emptyset$ we would infer that (X, Y) and (X, Y') are both simple pairs intersecting at y (and distinct as $Y \neq Y'$), contradicting Claim 3.2. Assume then that $y \notin Y'$ (so in particular $y \neq y'$). We claim that we can find a π satisfying

$$\pi(X \setminus y) < \pi(y) < \pi((X' \setminus y') \setminus X) < \pi(y') < \pi((Y \cup Y') \setminus (X \cup X')).$$

Indeed, the only thing that needs to be justified is the ability to place y' as above, which follows from the fact that $y' \in Y'$ and the assumption $(X \setminus y) \cap Y' = \emptyset$ which together imply that $y' \notin X$. Observe that since π first places $X \setminus y$ and then y, the pair (X, Y) is separated by π. Such a π clearly places $X' \setminus y'$ before y' and the assumption $(X \setminus y) \cap Y' = \emptyset$ together with the fact that $y \notin Y'$ imply that such a π places all of $Y' \setminus y'$ after y', so it separates (X', Y') as well, giving us the desired contradiction. ☐

This completes the proof of Theorem 1.

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