Exact Limit Theorems for Restricted Integer Partitions

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Abstract

For a set of positive integers A, let $p_A(n)$ denote the number of ways to write n as a sum of integers from A, and let $p(n)$ denote the usual partition function. In the early 40s, Erdős extended the classical Hardy–Ramanujan formula for $p(n)$ by showing that A has density α if and only if $\log p_A(n) \sim \log p(\alpha n)$. Nathanson asked if Erdős's theorem holds also with respect to A's lower density, namely, whether A has lower-density α if and only if $\log p_A(n)/\log p(\alpha n)$ has lower limit 1. We answer this question negatively by constructing, for every $\alpha > 0$, a set of integers A of lower density α, satisfying

$$\liminf_{n\to\infty} \frac{\log p_A(n)}{\log p(\alpha n)} \geq \left(\frac{\sqrt{6}}{\pi} - o_\alpha(1) \right) \log(1/\alpha) .$$

We further show that the above bound is best possible (up to the $o_\alpha(1)$ term), thus determining the exact extremal relation between the lower density of a set of integers and the lower limit of its partition function. We also prove an analogous theorem with respect to the upper density of a set of integers, answering another question of Nathanson.

1   Introduction

A partition of an integer n is a sequence of positive integers $a_1 \leq a_2 \leq \ldots$ whose sum is n. The classical partition function $p(n)$ denotes the number of partitions of n. More generally, for a set of positive integers A, we denote by $p_A(n)$ the number of partitions of n using integers taken from A. The study of various properties of these restricted partition functions is amongst the oldest topics in mathematics. Some classical examples are Euler's Pentagonal Numbers Theorem and the Rogers–Ramanujan identities. The reader is referred to [1, 2, 3] for a more thorough background on this topic. Our goal in this paper is to obtain asymptotic estimates for such restricted partition functions. Arguably, the most well known result of this type is the classical Hardy–Ramanujan formula [10] (discovered independently by Uspensky [21]) stating[1] that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}} . \tag{1}$$

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[1]The results of [10, 21] actually give much more accurate asymptotic estimates for $p(n)$.

Following [10], asymptotic estimates for $p_A(n)$ were obtained for various sets A. For example, already Hardy and Ramanujan [11] obtained bounds analogous to (1) when A is the set of primes, when A is the set of odd integers, and when A is the set of kth powers of positive integers. Szekeres [19, 20] obtained tight asymptotic bounds for partitions avoiding large numbers, that is, when[2] $A = [1, m(n)]$ for various functions $m(n)$, see also [4, 18]. In the other direction, Diximier and Nicolas [5] studied partitions avoiding small integers, namely, when $A = [m(n), n]$ for various functions $m(n)$, see also [13, 14]. Finally, Nathanson [16] and Erdős and Lehner [7] studied the case of A of fixed size.

In a remarkable paper from the early 40's, Erdős [6] gave an elementary proof of a slightly weaker version of (1). He further extended (1) by showing that if A is a set of density α with[3] $\gcd(A) = 1$, then $p_A(n)$ behaves like $p(\alpha n)$, more precisely [4]

$$\lim_{n \to \infty} \frac{\log p_A(n)}{\log p(\alpha n)} = \lim_{n \to \infty} \frac{\log p_A(n)}{\pi \sqrt{2\alpha n/3}} = 1 \ . \tag{2}$$

More surprisingly, using the Hardy–Littlewood Tauberian Theorem [9], Erdős proved[5] an "inverse theorem", stating that if A satisfies (2) then A has density α. Together, these two theorems imply that A has density α if and only if (2) holds. Other inverse theorems of this type were obtained in [8, 12, 22].

Given Erdős's theorem [6], it is natural to ask if the lower and upper densities of A also uniquely determine the lower and upper limits of $p_A(n)$. That is, whether A has lower density α (respectively, upper density β) if and only if $\liminf_{n\to\infty} \frac{\log p_A(n)}{\log p(\alpha n)} = 1$ (respectively, $\limsup_{n\to\infty} \frac{\log p_A(n)}{\log p(\beta n)} = 1$). This question was first raised by Nathanson [15], who further proved the following theorem, which is a strengthening[6] of the first theorem of Erdős mentioned above.

Theorem 1.1 (Nathanson [15])**.** *Suppose A is a set of integers with $\gcd(A) = 1$ of lower density α and upper density β. Then*

$$\liminf_{n \to \infty} \frac{\log p_A(n)}{\log p(\alpha n)} \geq 1 \qquad and \qquad \limsup_{n \to \infty} \frac{\log p_A(n)}{\log p(\beta n)} \leq 1 \ . \tag{3}$$

Nathanson [15] asked if the above inequalities are in fact equalities, namely, whether one can prove inverse theorems (in the sense of Erdős's inverse theorem mentioned above) with respect to the lower and upper densities of A. Our main qualitative results in this paper are that (perhaps unexpectedly) the answers to both questions are negative. As we explain below, we moreover give optimal quantitative results relating the lower/upper densities of A and the lower/upper limits of $p_A(n)$.

Our first result deals with the upper density of A. It shows that for all small enough β there is a set A of upper density β so that $\limsup_{n\to\infty} \frac{\log p_A(n)}{\log p(\beta n)} < 1$. We in fact determine precisely how small

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[2]We use $[a, b]$ to denote the integers $\{a, \ldots, b\}$. We also use $[a]$ to denote the integers $\{1, \ldots, a\}$.

[3]Note that if $\gcd(A) = d > 1$ then trivially $p_A(n) = 0$ whenever n is not divisible by d. In our proofs it will be very easy to guarantee that $\gcd(A) = 1$ since all the sets A we construct contain two consecutive integers.

[4]For simplicity, we frequently remove floor/ceiling notation when they make no real difference. For example, in (2) the αn should really be $\lfloor \alpha n \rfloor$. Also, throughout the paper, logarithms are natural unless stated otherwise.

[5]A remark for the history buff: this result was actually stated as an open problem in the preliminary version of [6] and then sketched in the published version. A full proof was given by Nathanson [15], see also [17].

[6]Note that when $\alpha = \beta$ (i.e. when A has density α) this theorem is equivalent to Erdős's first theorem.

can this upper limit be. This, in particular, implies a negative answer to Nathanson's question for all small enough β.

Theorem 1.2. *For every $0 < \beta < 1$ there is a set of integers A with $\gcd(A) = 1$ of upper density β satisfying*

$$\limsup_{n \to \infty} \frac{\log p_A(n)}{\log p(\beta n)} \leq \frac{\sqrt{6} \log 2}{\pi} + o_\beta(1) \ . \tag{4}$$

Furthermore, the constant above is best possible. Namely, any A of upper density β satisfies

$$\limsup_{n \to \infty} \frac{\log p_A(n)}{\log p(\beta n)} \geq \frac{\sqrt{6} \log 2}{\pi} + o_\beta(1) \ . \tag{5}$$

Our second and main result deals with the lower density of A. Contrary to the case of the upper density, if A has lower density α then $\liminf_{n \to \infty} \frac{\log p_A(n)}{\log p(\alpha n)}$ cannot even be bounded from above by an absolute constant. Again, this implies a negative answer to Nathanson's question for all small enough α.

Theorem 1.3. *For every $0 < \alpha < 1$ there is a set of integers A of lower density α with $\gcd(A) = 1$ satisfying*

$$\liminf_{n \to \infty} \frac{\log p_A(n)}{\log p(\alpha n)} \geq (1 - o_\alpha(1)) \frac{\sqrt{6}}{\pi} \log(1/\alpha) \ . \tag{6}$$

Furthermore, the above lower bound is best possible. Namely, any A of lower density α satisfies

$$\liminf_{n \to \infty} \frac{\log p_A(n)}{\log p(\alpha n)} \leq (1 + o_\alpha(1)) \frac{\sqrt{6}}{\pi} \log(1/\alpha) \ . \tag{7}$$

Proof and paper overview: The proof of Theorem 1.3 appears in Section 2 and the proof of Theorem 1.2 appears in Section 3. All the proofs in this paper are elementary in the number theoretic sense [17], that is, they rely on combinatorial/counting arguments and do not use complex analysis which is frequently used when studying partition functions. We find it quite surprising that such elementary methods can yield the precise results stated in Theorems 1.2 and 1.3. The results that are the most challenging to prove are those stated in (4) and (6). In both cases, the constructions of the sets A are quite simple and rely on the following finitary intuition: if one has to choose a subset $A \subseteq [n]$ of size αn so as to maximize $p_A(m)$, one would choose $S = \{1, \ldots, \alpha n\}$, since small integers give more "freedom"[7]. Similarly, taking $S = \{n - \alpha n, \ldots, n\}$ would minimize $p_A(m)$. The constructions of A in both proofs are then an infinite variant of this finitary intuition. While the constructions of the sets A are simple, their analysis is quite involved, relying among other things, on special cases of the results of Szekeres [19, 20] and Diximier–Nicolas [5] mentioned above. While the original proofs of these two results were highly non-elementary, we will provide short and self-contained proofs of the special cases we need in this paper, see Lemmas 2.6 and 2.7. The latter proof might be of independent interest.

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[7]See Lemma 2.5 where this intuition is formalized.

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2 Proof of Theorem 1.3

We start with the proof of Theorem 1.3 equation (6). To this end, we will first consider the "easy" cases, handled by Lemma 2.1 below, and then move on to consider the harder cases, which will be dealt with in the proof itself later on. This proof will require a certain amount of preparation which will be given after the proof of Lemma 2.1.

In the next proof, as well as in the rest of this section, we will frequently use the basic inequalities $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ and $(n/e)^n \leq n! \leq en(n/e)^n$. Furthermore, throughout the paper we will frequently use the fact that there are $\binom{n+k-1}{k-1}$ solutions in nonnegative integers to the equation $\sum_{i=1}^{k} x_i = n$.

Lemma 2.1. *For every $0 < \alpha < 1$ there exists $n_1 = n_1(\alpha)$ such that the following holds for every integer $n > n_1$. Letting $A = \{1\} \cup [n, \alpha n^2]$ we have the following for all $m \in [16\alpha n^2, \alpha n^4/16]$*

$$\log p_A(m) \geq (2\log(1/\alpha) - 8)\sqrt{\alpha m} . \tag{8}$$

Proof. Let $0 < \alpha < 1$ and let n be a positive integer with $n > n_1 = n_1(\alpha)$ which will be specified later. Let $m \in [16\alpha n^2, \alpha n^4]$. The proof splits into two cases depending on the value of m.

Case 1: Assume $16\alpha n^2 \leq m \leq 4\alpha^3 n^4$. Our assumption in this case implies the following two inequalities:

$$2\sqrt{\alpha m} \left(n + \sqrt{m/16\alpha}\right) \leq m , \tag{9}$$

$$\sqrt{m/16\alpha} \leq \alpha n^2/2 . \tag{10}$$

Therefore, provided $n_1 \geq 2/\alpha$ we can deduce from (10) that

$$n + \sqrt{m/16\alpha} \leq \alpha n^2 .$$

Setting $B = \{1\} \cup [n, n + \sqrt{m/16\alpha}]$ we infer that,

$$B \subseteq \{1\} \cup [n, \alpha n^2] \subseteq A ,$$

implying that it is enough to prove a lower bound with respect to $p_B(m)$ in (8). We map each solution in nonnegative integers of the equation

$$\sum_{k=0}^{\sqrt{m/16\alpha}} x_k = 2\sqrt{\alpha m} \tag{11}$$

to a partition of m with parts in B as follows: if $(x_k)_{k=0}^{\sqrt{m/16\alpha}}$ is a solution in nonnegative integers of (11) then for every k we take the integer $n + k$ exactly x_k times and finally take 1 exactly $m - \sum_{k=0}^{\sqrt{m/16\alpha}} x_k = 2\sqrt{\alpha m}$ times. This map is well defined as by (9) and (11) we have,

$$\sum_{k=0}^{\sqrt{m/16\alpha}} x_k(n + k) \leq (n + \sqrt{m/16\alpha}) \sum_{k=0}^{\sqrt{m/16\alpha}} x_k = 2\sqrt{\alpha m}(n + \sqrt{m/16\alpha}) \leq m .$$

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Moreover, provided $n_1 \geq 2$ the above map is an injection as $1 \notin [n, n + \sqrt{m/16\alpha}]$. Combining the above observations we have

$$p_A(m) \geq p_B(m) \geq \binom{2\sqrt{\alpha m} + \sqrt{m/16\alpha}}{2\sqrt{\alpha m} - 1} \geq \binom{\sqrt{m/16\alpha}}{2\sqrt{\alpha m} - 1}$$

$$\geq \left(\frac{\sqrt{m/16\alpha}}{2\sqrt{\alpha m} - 1}\right)^{2\sqrt{\alpha m} - 1} \geq \left(\frac{1}{8\alpha}\right)^{2\sqrt{\alpha m} - 1}$$

$$\geq \exp\left((2\log(1/\alpha) - 8)\sqrt{\alpha m}\right) ,$$

where that last inequity holds provided $n \geq 1/4\sqrt{\alpha}$.

Case 2: Assume $n^2/4\alpha \leq m \leq \alpha n^4/16$. Note that provided $n_1 \geq 4/\alpha^2$ we have $n^2/4\alpha \leq 4\alpha^3 n^4$ for all $n > n_1$, hence Case 1 and Case 2 cover all $m \in [16\alpha n^2, \alpha n^4/16]$. Our assumption in this case implies the following two inequalities:

$$\sqrt{m/16\alpha}\left(n + 2\sqrt{\alpha m}\right) \leq m , \tag{12}$$
$$2\sqrt{\alpha m} \leq \alpha n^2/2 . \tag{13}$$

Therefore, provided $n_1 \geq 2/\alpha$ we can deduce from (13) that

$$n + 2\sqrt{\alpha m} \leq \alpha n^2 .$$

Setting $B = \{1\} \cup [n, n + 2\sqrt{\alpha m}]$ we deduce that,

$$B \subseteq \{1\} \cup \left[n + 1, \alpha n^2\right] \subseteq A .$$

Similar to the first case, we may thus prove a lower bound for $p_B(m)$ in (8). We map each solution in nonnegative integers of the equation

$$\sum_{k=0}^{2\sqrt{\alpha m}} x_k = \sqrt{m/16\alpha} \tag{14}$$

to a partition of m with parts in B as follows: if $(x_k)_{k=0}^{\sqrt{2\alpha m}}$ is a solution in nonnegative integers of (14) then for every k we take the integer $n + k$ exactly x_k times and 1 exactly $m - \sum_{k=0}^{2\sqrt{\alpha m}} x_k(n+k)$ times. This map is well defined as by (12) and (14) we have,

$$\sum_{k=0}^{2\sqrt{\alpha m}} x_k(n+k) \leq (n + 2\sqrt{\alpha m}) \sum_{k=0}^{2\sqrt{\alpha m}} x_k = \sqrt{m/16\alpha}(n + 2\sqrt{\alpha m}) \leq m .$$

Moreover, provided $n_1 \geq 2$ this map is an injection as $1 \notin [n, n + \sqrt{2\alpha m}]$. Combining the above observations we have,

$$p_A(m) \geq p_B(m) \geq \binom{2\sqrt{\alpha m} + \sqrt{m/16\alpha}}{2\sqrt{\alpha m}} \geq \binom{\sqrt{m/16\alpha}}{2\sqrt{\alpha m}}$$

$$\geq \left(\frac{\sqrt{m/16\alpha}}{2\sqrt{\alpha m}}\right)^{2\sqrt{\alpha m}} = \left(\frac{1}{8\alpha}\right)^{2\sqrt{\alpha m}} \geq \exp\left((2\log(1/\alpha) - 8)\sqrt{\alpha m}\right) . \qquad \square$$

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We will now prove several claims and lemmas which will be used in the proof of Theorem 1.3 equation (6). We start with the following very crude bound which will suffice for our purposes.

Claim 2.2. *Suppose n is positive integer and A is a set of positive integers with $|A| = k$. Then*

$$p_A(n) \leq (n+1)^k .$$

Proof. Since each partition of n using integers from A can contain each of these integers at most n times, the number of such partitions is clearly at most $(n+1)^k$. $\square$

For positive integers k, n we define $p_k(n)$ to be the number of ways to write n as a sum of *exactly* k nonnegative integers (without consideration of the ordering of the summands). The following three lemmas are folklore, and are proved here for the sake of completeness.

Lemma 2.3. *Suppose k, n are positive integers. Then,*

$$\binom{n-1}{k-1} \leq k! \cdot p_k(n) \leq \binom{n + \binom{k}{2} - 1}{k-1} .$$

Proof. There are exactly $\binom{n-1}{k-1}$ *ordered* partitions of n with k positive parts. This implies the first inequality. To see the second, suppose $y_1 \leq y_2 \leq \ldots \leq y_k$ satisfy $\sum_{i=1}^{k} y_i = n$. Defining $x_i = y_i + i - 1$ for all i, we have $\sum_{i=1}^{k} x_i = n + \binom{k}{2}$. As all x_i are distinct, each permutation of x_is give rise to a different *ordered* solution to the equation $\sum_{i=1}^{k} z_i = n + \binom{k}{2}$ with nonnegative integers. This implies the second inequality. $\square$

In the following lemma, as well as in the rest of the section, we use several times the notation $p_{[k]}(n)$. For clarity, we wish to emphasize that $p_{[k]}(n)$ stands for $p_A(n)$ where $A = [k]$.

Lemma 2.4. *Suppose k, n are positive integers. Then,*

$$p_{[k]}(n) = p_k(n + k) .$$

Proof. As it is well known, $p_{[k]}(n)$ is also the number of ways to write n as a sum of at most k integers. Let $(y_i)_{i=1}^{k}$ be a partition of $n + k$. Setting $x_i = y_i - 1$ we obtain a partition of n with at most k parts. This process is invertible and therefore we obtain the assertion of the lemma. $\square$

Lemma 2.5. *Suppose A is a set of positive integers with $|A| = k$. Then,*

$$p_A(n) \leq p_{[k]}(n) .$$

Proof. Let f be the bijection between A and $[k]$ defined by sending the ith largest integer of A to i. We now define an injection g between the partitions of n with parts in A and partitions of n with parts in $[k]$. Given a partition $x = (x_i)_{i=0}^{\ell}$ of n with parts in A, we define $g(x)$ to be the partition $y = (y_i)_{i=0}^{\ell + \ell'}$ where $\ell' = m - \sum_{i=0}^{\ell} f(x_i)$ and y_i is defined to be $f(x_i)$ for all $0 \leq i \leq \ell$ and 1 for all $\ell < i \leq \ell'$. To see that this is an injection let $x = (x_i)_{i=1}^{\ell_1}, x' = (x')_{i=1}^{\ell_2}$ be two partitions of n with all parts in A and assume that $g(x) = g(x')$. Let a the minimal integer in A. Since f is a bijection each of the integers in A besides a must appear in x and x' the same number of times. Since $\sum_{i=1}^{\ell_1} a_i = \sum_{i=1}^{\ell_2} b_i = n$, the integer a appears the same number of times in x and x'. This completes the proof. $\square$

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We now turn to prove the two key lemmas that will be used in the proof of (6). The first is Lemma 2.6 below. We remark that this result can be derived (with some effort) from the more precise bound due to Szekeres [19, 20] (see also [4, 18]), but the self contained elementary proof below is significantly simpler.

Lemma 2.6. *For every $0 < \gamma < 1$ there exists $n_2 = n_2(\gamma)$ such that for all $n > n_2$ we have*

$$\log p_{[\gamma\sqrt{n}]}(n) = (2\gamma \log(1/\gamma) + \Theta(\gamma))\sqrt{n} .$$

Proof. Let $0 < \gamma < 1$ be a real number and let n be an integer with $n > n_2 = n_2(\gamma)$ where n_2 will be specified later. Setting $k = \gamma\sqrt{n}$, Lemmas 2.3 and 2.4 implies that

$$\binom{n + \gamma\sqrt{n} - 1}{\gamma\sqrt{n} - 1} \leq (\gamma\sqrt{n})! \cdot p_{[\gamma\sqrt{n}]}(n) \leq \binom{n + \gamma\sqrt{n} + \binom{\gamma\sqrt{n}}{2} - 1}{\gamma\sqrt{n} - 1} . \tag{15}$$

Therefore we have,

$$p_{[\gamma\sqrt{n}]}(n) \leq \frac{\binom{n + \gamma\sqrt{n} + \binom{\gamma\sqrt{n}}{2} - 1}{\gamma\sqrt{n} - 1}}{(\gamma\sqrt{n})!} \leq \frac{\binom{2n}{\gamma\sqrt{n}}}{(\gamma\sqrt{n})!}$$

$$\leq \left(\frac{2e^2}{\gamma^2}\right)^{\gamma\sqrt{n}} \leq e^{(2\gamma\log(1/\gamma) + 4\gamma)\sqrt{n}} ,$$

where the second inequality holds provided $\gamma\sqrt{n_2} + \binom{\gamma\sqrt{n_2}}{2} - 1 \leq n_2$. As to the lower bound we have,

$$p_{[\gamma\sqrt{n}]}(n) \geq \frac{\binom{n + \gamma\sqrt{n} - 1}{\gamma\sqrt{n} - 1}}{(\gamma\sqrt{n})!} \geq \left(\frac{\sqrt{n}}{\gamma}\right)^{\gamma\sqrt{n} - 1} \left(\frac{e}{\gamma\sqrt{n}}\right)^{\gamma\sqrt{n}} \frac{1}{e\gamma\sqrt{n}}$$

$$= \left(\frac{e}{\gamma^2}\right)^{\gamma\sqrt{n}} \frac{1}{en} = e^{(2\gamma\log(1/\gamma) + \gamma)\sqrt{n} - \log(en)} \geq e^{(2\gamma\log(1/\gamma) + \gamma/2)\sqrt{n}} ,$$

where the second inequality holds provided $n_2 > 1/\gamma^2$, and the third inequality holds provided $\log(en_2) \leq \gamma\sqrt{n_2}/2$. $\square$

The second key lemma we will need is Lemma 2.7 below. We remark that (16) below is Theorem 2.6 in [5], see also [13] for a refined version of this result. We give a self contained elementary proof of (16) which is significantly simpler and also allows us to derive the stronger statement stated after (16).

Lemma 2.7. *There exists a positive real λ_0 such that for every $\lambda \geq \lambda_0$ there exists $n_3 = n_3(\lambda)$ such that for every integer $n > n_3$ we have*

$$\log p_{[\lambda\sqrt{n},n]}(n) = \left(\frac{2\log(\lambda) \pm \Theta(\log\log(\lambda))}{\lambda}\right)\sqrt{n} . \tag{16}$$

Furthermore, for every positive real $\varepsilon \leq 1$ the lower bound holds also for $\log p_{[\lambda\sqrt{n},\varepsilon n]}(n)$ provided $\lambda \geq \lambda_0(\varepsilon)$ and $n \geq n_3(\lambda, \varepsilon)$.

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Proof. Let $\lambda \geq \lambda_0$ be a real number where the value of λ_0 will be specified later and let n be an integer with $n > n_3 = n_3(\lambda)$ which will be specified later. We first claim that

$$p_{[\lambda\sqrt{n},n]}(n) \leq p_{[\sqrt{n}/\lambda]}(n) = p_{\sqrt{n}/\lambda}(n + \sqrt{n}/\lambda) \ .$$

To justify the inequality we observe that each partition of n with all integers from $[\lambda\sqrt{n}, n]$ uses at most $\sqrt{n}/\lambda$ integers. As noted earlier it is well know that the number of partitions of n with at most $\sqrt{n}/\lambda$ parts is precisely $p_{[\sqrt{n}/\lambda]}(n)$. Finally, the equality holds by Lemma 2.4. Applying Lemma 2.3 invoked with n replaced by $n + \sqrt{n}/\lambda$ and $k = \sqrt{n}/\lambda$ we obtain,

$$p_{[\lambda\sqrt{n},n]}(n) \leq \frac{\binom{n+\sqrt{n}/\lambda+\binom{\sqrt{n}/\lambda}{2}-1}{\sqrt{n}/\lambda-1}}{(\sqrt{n}/\lambda)!} \leq \frac{\binom{3n}{\sqrt{n}/\lambda}}{(\sqrt{n}/\lambda)!}$$

$$\leq \left(3e\lambda\sqrt{n}\right)^{\sqrt{n}/\lambda} \left(\frac{e\lambda}{\sqrt{n}}\right)^{\sqrt{n}/\lambda} = \left(3e^2\lambda^2\right)^{\sqrt{n}/\lambda} \leq e^{\left(\frac{2\log(\lambda)+4}{\lambda}\right)\sqrt{n}},$$

where the second inequality holds provided $\sqrt{n_3}/\lambda \leq n_3$, $\sqrt{n_3}/\lambda - 1 \leq 3n_3/2$ and provided $\lambda_0 > 1$ and as $\binom{\sqrt{n}/\lambda}{2} - 1 \leq n/\lambda^2$. This concludes the proof of the upper bound of (16).

For the lower bound of (16) we claim that for any positive integer k we have

$$p_{[\lambda\sqrt{n},n]}(n) \geq p_k(n - k \cdot \lfloor\lambda\sqrt{n}\rfloor) \ . \tag{17}$$

To see this we define the following one to one correspondence. For every partition of $n - k \cdot \lfloor\lambda\sqrt{n}\rfloor$ with positive integers $(x_i)_{i=1}^k$ we define $(y_i)_{i=1}^k$ with $y_i = x_i + \lfloor\lambda\sqrt{n}\rfloor$. This is clearly a one to one correspondence and furthermore $(y_i)_{i=1}^k$ is partition of n with all parts taken from $[\lambda\sqrt{n}, n]$. Setting

$$k = \left\lfloor \frac{\sqrt{n}}{\lambda + \lambda/\log(\lambda)} \right\rfloor$$

in (17) and applying Lemma 2.3 we obtain,

$$p_{[\lambda\sqrt{n},n]}(n) \geq \frac{\binom{n-k\lfloor\lambda n\rfloor-1}{k-1}}{k!}$$

$$\geq \left(\frac{\frac{n}{\log(\lambda)+1} - 1}{\frac{\sqrt{n}}{\lambda + \lambda/\log(\lambda)} - 1}\right) \bigg/ \left(\frac{\sqrt{n}}{\lambda + \lambda/\log(\lambda)}\right)!$$

$$\geq \left(\frac{\frac{n}{2(\log(\lambda)+1)}}{\frac{\sqrt{n}}{\lambda + \lambda/\log(\lambda)} - 1}\right) \bigg/ \left(\frac{\sqrt{n}}{\lambda + \lambda/\log(\lambda)}\right)!$$

$$\geq \left(\frac{\lambda\sqrt{n}}{2\log(\lambda)}\right)^{\frac{\sqrt{n}}{\lambda + \lambda/\log(\lambda)}} \left(\frac{e(\lambda + \lambda/\log(\lambda))}{\sqrt{n}}\right)^{\frac{\sqrt{n}}{\lambda + \lambda/\log(\lambda)}} \frac{2(\log(\lambda) + 1)}{en}$$

$$= \left(e\lambda^2\left(\frac{1}{2\log(\lambda)} + \frac{1}{2\log^2(\lambda)}\right)\right)^{\frac{\sqrt{n}}{\lambda} - \frac{\sqrt{n}}{\lambda(\log(\lambda)+1)}} \frac{2(\log(\lambda) + 1)}{en}$$

$$\geq \left(\frac{\lambda}{\log(\lambda)}\right)^{\frac{2\sqrt{n}}{\lambda} - \frac{4\sqrt{n}}{\lambda\log(\lambda)}} = \exp\left((\log(\lambda) - \log\log(\lambda))\left(\frac{2\sqrt{n}}{\lambda} - \frac{4\sqrt{n}}{\lambda\log(\lambda)}\right)\right)$$

$$\geq \exp\left(\frac{(2\log(\lambda) - 3\log\log(\lambda))\sqrt{n}}{\lambda}\right),$$

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where the third inequality holds provided $\frac{n_3}{2(\log(\lambda)+1)} \geq 1$ and provided $\lambda_0 \geq 1$, the fifth inequality holds provided $\log\left(\frac{en_3}{2(\log(\lambda)+1)}\right) \leq (\log(\lambda) - \log\log(\lambda))\frac{\sqrt{n_3}}{\lambda\log(\lambda)}$, and the sixth inequality holds provided $\log\log(\lambda_0) \geq 4$.

As for the furthermore part of the theorem, fix $0 < \varepsilon \leq 1$ and note the correspondence we used above when proving the lower bound of (16), took partitions of $n - k \cdot \lfloor \lambda\sqrt{n} \rfloor$ that use k integers and mapped them to partitions of n using integers in $[\lambda\sqrt{n}, \left(2\lambda + \frac{\log(\lambda)}{\lambda+\lambda\log(\lambda)}\right)\sqrt{n} + \frac{n}{\log(\lambda)+1}]$. Therefore, if we assume that $\frac{2}{\varepsilon} \leq \log(\lambda_0) + 1$ and $\sqrt{n_3} \geq \frac{2}{\varepsilon}\left(2\lambda + \frac{\log(\lambda)}{\lambda+\lambda\log(\lambda)}\right)$ then we in fact obtain the same lower bound stated in (16) even if using only integers from the interval $[\lambda\sqrt{n}, \varepsilon n]$. $\qquad\square$

We now use Lemmas 2.1, 2.6 and 2.7 in order to prove Theorem 1.3 equation (6).

Proof of Theorem 1.3 equation (6). By (1) it is sufficient to prove that there exists a set of positive integers A with lower density α and $\gcd(A) = 1$ satisfying

$$\log p_A(m) \geq (2\log(1/\alpha) - \Theta(\log\log(\alpha)))\sqrt{\alpha m} . \tag{18}$$

To this end suppose $\alpha < \alpha_0$ where α_0 is a small positive real which will be specified later. Let n be an integer with $n > n_0$ where $n_0 = n_0(\alpha)$ is some positive integer which will be specified later and will also be greater than $1/\alpha$. We claim that the set A which we introduce next satisfies (18). Define a sequence of sets A_i recursively as follows. Set $f(0) = 1$, and $f(1) = n_0$, and for any positive integer i let $f(i+1) = f(i)^2$, $A_{i+1} = [f(i), \alpha f(i+1)]$. Finally take $A = \bigcup_{n\geq 1} A_n$. It is easy to see that the lower density of A is α, and since $n_0 > 1/\alpha$ we have $1 \in A_1 \subseteq A$ which implies that $\gcd(A) = 1$.

Provided

$$n_0 > n_1(\alpha) \tag{19}$$

we may use Lemma 2.1 invoked with n replaced by $f(i) \geq n_0$ for $i \geq 1$, which asserts the following for all $m \in [16\alpha f(i+1), \alpha f(i+2)/16]$,

$$p_{\{1\}\cup[f(i),\alpha f(i+1)]}(m) \geq e^{(2\log(1/\alpha)-8)\sqrt{\alpha m}} .$$

For all $i \geq 2$ the set $\{1\}\cup[f(i), \alpha f(i+1)]$ is a subset of A and thus we obtain (18) for all $m \geq \alpha f(3)/16$ except for $m \in [\alpha f(i+1)/16, 16\alpha f(i+1)]$ where $i \geq 2$. Therefore, to complete the proof of (18) it remains to consider only $m \in [\alpha f(i+1)/16, 16\alpha f(i+1)]$ where $i \geq 2$ is some integer.

Therefore for the rest of the proof let us fix $i \geq 2$. For simplicity of presentation denote $f(i+1)$ by n^2 and then $f(i) = n$ and $f(i-1) = \sqrt{n}$. Let c be a real number with $1/16 \leq c \leq 16$ and set

$$m = c \cdot \alpha \cdot n^2 .$$

Since A contains both $\{1\}\cup A_i = \{1\}\cup[\sqrt{n}, \alpha n]$ and $A_{i+1} = [n, \alpha n^2]$ we deduce the following for all $0 \leq \delta \leq 1$,

$$p_A(m) \geq \sum_{k=0}^{m} p_{\{1\}\cup[\sqrt{n},\alpha n]}(k) \cdot p_{[n,\alpha n^2]}(m-k)$$
$$\geq p_{\{1\}\cup[\sqrt{n},\alpha n]}(\delta m) \cdot p_{[n,\alpha n^2]}((1-\delta)m) .$$

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Hence to establish (18) it is enough to show that there exists $0 < \delta < 1$ such that for all $1/16 \le c \le 16$ we have

$$\log(p_{\{1\}\cup[\sqrt{n},\alpha n]}(\delta m)) + \log(p_{[n,\alpha n^2]}((1-\delta)m)) \ge (2\log(1/\alpha) - \Theta(\log\log(1/\alpha)))\sqrt{\alpha m} \,. \tag{20}$$

To simplify (20) we observe that

$$p_{[\alpha n]}(\delta m) = \sum_{k=0}^{\delta m} p_{[2,\sqrt{n}-1]}(k) \cdot p_{\{1\}\cup[\sqrt{n},\alpha n]}(\delta m - k) \le (\delta m + 1)(\delta m + 1)^{\sqrt{n}} \cdot p_{\{1\}\cup[\sqrt{n},\alpha n]}(\delta m) \,, \tag{21}$$

where the inequality holds by Claim 2.2 and by the monotonicity of $p_{\{1\}\cup[\sqrt{n},\alpha n]}$. Hence, provided

$$\alpha \log\log(1/\alpha)\sqrt{n_0}/16 \ge 2\log(8n_0) \,, \tag{22}$$

to obtain (20) it is enough to prove that there exists $0 < \delta < 1$ such that the following holds for all $1/16 \le c \le 16$

$$\log(p_{[\alpha n]}(\delta m)) + \log(p_{[n,\alpha n^2]}((1-\delta)m)) \ge (2\log(1/\alpha) - \Theta(\log\log(1/\alpha)))\sqrt{\alpha m} \,. \tag{23}$$

Provided δ and α satisfy the right-hand inequality (the left-hand inequality holds as $c \ge 1/16$)

$$\frac{\alpha}{\delta \cdot c} \le \frac{\alpha}{\delta/16} \le 1 \,, \tag{24}$$

and n_0 satisfies

$$\delta m \ge \delta \alpha \cdot n_0/16 > n_2\left(\sqrt{\frac{\alpha}{16\delta}}\right) \ge n_2\left(\sqrt{\frac{\alpha}{\delta \cdot c}}\right) \,, \tag{25}$$

we may apply Lemma 2.6 invoked with γ replaced by $\sqrt{\frac{\alpha}{\delta \cdot c}}$ and with n replaced by δm and obtain,

$$\begin{aligned}
\log\left(p_{[\alpha n]}(\delta m)\right) &\ge \left(2\sqrt{\frac{\alpha}{\delta \cdot c}}\log\left(\sqrt{\frac{c \cdot \delta}{\alpha}}\right) + \Theta\left(\sqrt{\frac{\alpha}{\delta \cdot c}}\right)\right)\sqrt{\delta m} \\
&= \left(\alpha\log\left(\frac{c \cdot \delta}{\alpha}\right) + \Theta\left(\alpha\right)\right)n \\
&\ge \left(\alpha\log\left(\frac{\delta}{\alpha}\right) + \Theta\left(\alpha\right)\right)n \,, \tag{26}
\end{aligned}$$

where the second inequality holds as $c \ge 1/16$.

From now on let us assume (with foresight) that the δ which establishes (23) is less than $1/2$. Provided α_0 is small enough so that

$$\frac{1}{\sqrt{(1-\delta)c \cdot \alpha}} \ge \frac{1}{\sqrt{16\alpha}} \ge \lambda_0\left(\frac{1}{16}\right) \ge \lambda_0\left(\frac{1}{16(1-\delta)}\right) \,, \tag{27}$$

and n_0 satisfies

$$(1-\delta)m \ge \frac{\alpha \cdot n_0^2}{16} \ge n_3\left(\frac{1}{\sqrt{\alpha/32}}, \frac{1}{16}\right) \ge n_3\left(\frac{1}{\sqrt{(1-\delta)c \cdot \alpha}}, \frac{1}{16(1-\delta)}\right) \,, \tag{28}$$

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we may apply Lemma 2.7 invoked with $\varepsilon = \frac{1}{16(1-\delta)}$, $\lambda = \frac{1}{\sqrt{(1-\delta)c \cdot \alpha}}$, and with n replaced by $(1-\delta)m$. As $1/16 \le c \le 16$ and $0 < \delta < 1/2$ we obtain,

$$\log\left(p_{[n,m/16]}((1-\delta)m)\right) \ge \left(\frac{2\log\left(1/\sqrt{(1-\delta)c \cdot \alpha}\right) - \Theta\left(\log\log\left(1/\sqrt{(1-\delta)c \cdot \alpha}\right)\right)}{1/\sqrt{(1-\delta)c \cdot \alpha}}\right)\sqrt{(1-\delta)m}$$

$$\ge (1-\delta)c \cdot \alpha\left(\log\left(\frac{1}{\alpha}\right) - \Theta\left(\log\log\left(\frac{1}{\alpha}\right)\right)\right)n . \tag{29}$$

Since $m/16 \le \alpha n^2$ (29) implies

$$\log\left(p_{[n,\alpha n^2]}((1-\delta)m)\right) \ge (1-\delta)c \cdot \alpha\log\left(\frac{1}{\alpha}\right)n - \Theta\left(\alpha\log\log\left(\frac{1}{\alpha}\right)\right)n . \tag{30}$$

All that is left is to choose the optimal δ that will maximize the sum of (26) and (30). It is not hard to see that (up to lower order terms) the optimal choice is $\delta = 1/\log(1/\alpha)$, and that with this choice of δ, we can choose α_0 small enough so that (24) and (27) will hold, and then choose n_0 large enough so that $n_0 > 1/\alpha$ and (19), (22), (25) and (28) will hold. Plugging $\delta = 1/\log(1/\alpha)$ in (26) we obtain,

$$\log\left(p_{[\alpha n]}(\delta m)\right) \ge \left(\alpha\log\left(\frac{1}{\alpha\log(1/\alpha)}\right) + \Theta(\alpha)\right)n = \alpha\log(1/\alpha)n - \Theta(\alpha\log\log(1/\alpha))n . \tag{31}$$

Similarly plugging $\delta = 1/\log(1/\alpha)$ in (30) we obtain,

$$\log\left(p_{[n,\alpha n^2]}((1-\delta)m)\right) \ge \left(1 - \frac{1}{\log(\alpha)}\right)c \cdot \alpha\log\left(\frac{1}{\alpha}\right)n - \Theta\left(\alpha\log\log\left(\frac{1}{\alpha}\right)\right)n$$

$$\ge c \cdot \alpha\log(1/\alpha)n - \Theta(\alpha\log\log(1/\alpha))n \tag{32}$$

Combining (31) and (32) we obtain

$$\log(p_{[\alpha n]}(\delta m)) + \log(p_{[n,\alpha n^2]}((1-\delta)m)) \ge ((1+c)\log(1/\alpha) - \Theta(\log\log(1/\alpha)))\alpha n$$

$$= ((1/\sqrt{c} + \sqrt{c})\log(1/\alpha) - \Theta(\log\log(1/\alpha)))\sqrt{\alpha m}$$

$$\ge (2\log(1/\alpha) - \Theta(\log\log(1/\alpha)))\sqrt{\alpha m} ,$$

where the last inequality holds as $1/x + x \ge 2$ for all $x > 0$. This is (23), and the proof is completed. $\qquad\square$

We now turn to the proof of Theorem 1.3 equation (7). We will need the following lemma.

Lemma 2.8. *There exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$ there exists an integer $n_0 = n_0(\alpha)$ such that the following holds for every integer $n > n_0$. If A is a set of positive integers with $|A \cap [n]| = \alpha n$ then we have the following where $m = \alpha n^2$,*

$$p_A(m) \le e^{(2\log(1/\alpha) + \Theta(\log\log(1/\alpha)))\sqrt{\alpha m}} .$$

Proof. Let $A_1 = A \cap [n]$ and $A_2 = A \cap [n+1, m]$. We have,

$$p_A(m) = \sum_{k=0}^{m} p_{A_1}(k) \cdot p_{A_2}(m-k) \leq (m+1) \cdot \max_{k \in [m]} p_{A_1}(k) \cdot \max_{k \in [m]} p_{A_2}(k) .$$

Hence it is enough to show that the following holds for all $k \in [m]$ and $j = 1, 2$,

$$p_{A_j}(k) \leq e^{(\log(1/\alpha) + \Theta(\log\log(1/\alpha)))\sqrt{\alpha m}} . \tag{33}$$

Using Lemma 2.5 and the monotonicity of $p_{[\alpha n]}$ we deduce that,

$$p_{A_1}(k) \leq p_{[\alpha n]}(k) \leq p_{[\alpha n]}(m) = p_{[\sqrt{\alpha m}]}(m) .$$

Provided n_0 is large enough we may use Lemma 2.6 invoked with n replaced by m and γ replaced by $\sqrt{\alpha}$ and obtain

$$\log p_{[\sqrt{\alpha m}]}(m) \leq (2\sqrt{\alpha}\log(1/\sqrt{\alpha}) + \Theta(\sqrt{\alpha}))\sqrt{m} = (\log(1/\alpha) + \Theta(1))\sqrt{\alpha m} .$$

Combining the above we obtain (33) for $j = 1$.

We now prove (33) for $j = 2$. We first note that if $k \in [n]$ then (33) trivially holds since in this case $p_{A_2}(k) = 0$. Assume now $k \in [n+1, m]$. We may use Lemma 2.7 with $\lambda = 1/\sqrt{\alpha}$ and n replaced by k provided α_0 is small enough so that $1/\sqrt{\alpha_0} > \lambda_0$ and $k > n_0 > n_3(1/\sqrt{\alpha})$. We obtain

$$\log p_{[\sqrt{k/\alpha}, k]}(k) \leq \left(\frac{2\log(1/\sqrt{\alpha}) + \Theta(\log\log(1/\sqrt{\alpha}))}{1/\sqrt{\alpha}} \right) \sqrt{k}$$
$$= (\log(1/\alpha) + \Theta(\log\log(1/\alpha))) \sqrt{\alpha k}$$
$$\leq (\log(1/\alpha) + \Theta(\log\log(1/\alpha))) \sqrt{\alpha m}.$$

Since $k \leq m = \alpha n^2$ then $\sqrt{k/\alpha} \leq n$ and therefore $A_2 \cap [k] \subseteq [n+1, k] \subseteq [\sqrt{k/\alpha}, k]$ and thus

$$p_{A_2}(k) = p_{A_2 \cap [k]}(k) \leq p_{[\sqrt{k/\alpha}, k]}(k).$$

Combining the above we obtain (33) for $j = 2$. $\square$

Now using the above lemma we prove Theorem 1.3 equation (7).

Proof of Theorem 1.3 equation (7). Suppose $\alpha < \alpha_0$ where α_0 is given by Lemma 2.8 and A is a set of positive integers with lower density α. Similar to the proof of Theorem 1.3 equation (6) it is sufficient to prove that there exist infinitely many pairs of integer and real number (m_i, α_i) satisfying

$$\log p_A(m_i) \leq (2\log(1/\alpha_i) + \Theta(\log\log(1/\alpha_i)))\sqrt{\alpha_i m_i}, \ \lim_{i\to\infty} \alpha_i = \alpha, \ \text{and} \ \lim_{i\to\infty} m_i = \infty . \tag{34}$$

Since A has lower density α there exists an increasing sequence $\{n_i\}_{i=0}^{\infty}$ of positive integers such that setting $A_i = A \cap [n_i]$ and $\alpha_i = |A_i|/n_i$ we have $\lim_{i\to\infty} \alpha_i = \alpha$. Fix i_0 large enough so that for all $i > i_0$ we have $\alpha/2 < \alpha_i < \alpha_0$ and $n_i > n_0(\alpha/2)$ where $n_0(\alpha/2)$ is given by Lemma 2.8. Now we may use Lemma 2.8 invoked with α replaced by α_i and n replaced by n_i and obtain

$$p_A(m_i) \leq e^{2(\log(1/\alpha_i) + \Theta(\log\log(1/\alpha_i)))\sqrt{\alpha_i m_i}} ,$$

where $m_i = \alpha_i n_i^2$. Since $\lim_{i\to\infty} \alpha_i = \alpha$ and $\lim_{i\to\infty} m_i = \infty$ we obtain (34), thus completing the proof. $\square$

3 Proof of Theorem 1.2

We start with a proof of Theorem 1.2 equation (5) and then move on to proving Theorem 1.2 equation (4).

Lemma 3.1. *Suppose m is a positive integer and $A \subseteq [m]$ with $\beta = |A|/m$. Then, there exists $n \in [\beta^2 m^2/4, \beta(1 - \beta/4)m^2]$ satisfying*

$$\log p_A(n) \geq (2\log(2) - o_n(1))\sqrt{\frac{\beta n}{1 - \beta/4}} \ .$$

Proof. Let A_1 be the set of $\beta m/2$ smallest integers in A and A_2 the set of $\beta m/2$ largest integers in A. Note that any $\beta m/2$ integers in A_1 sum up to an integer in $[\beta(1 - \beta/2)m^2/2]$. Furthermore, the number of options to choose $\beta m/2$ integers from A_1 (not necessarily distinct) is exactly the same as the number of nonnegative solutions to $\sum_{i=1}^{\beta m/2} x_i = \beta m/2$. That can be seen by taking x_i to be the number of times the ith smallest element in A_1 is taken. Hence there are $\binom{\beta m - 1}{\beta m/2}$ such choices. Similarly, any $\beta m/2$ integers in A_2 (not necessarily distinct) sum up to an integer in $[\beta^2 m^2/4, \beta m^2/2]$. Similar to before, the number of ways to choose these integers is $\binom{\beta m - 1}{\beta m/2}$. Therefore, the number of ways one can choose βm integers from A, such that half of them are taken from A_1 and the other half from A_2 is $\binom{\beta m - 1}{\beta m/2}^2$. Since the sum of these integers is an integer in $[\beta^2 m^2/4, \beta(1 - \beta/4)m^2]$ we infer by the pigeonhole principle that there exists $n \in [\beta^2 m^2/4, \beta(1 - \beta/4)m^2]$ satisfying

$$p_A(n) \geq \frac{\binom{\beta m - 1}{\beta m/2}^2}{m^2} = \frac{\binom{\beta m}{\beta m/2}^2}{4m^2} \geq \frac{2^{2\beta m}}{4(\beta m + 1)^2 m^2} = 2^{(2 - o_n(1))\beta m} \geq 2^{(2 - o_n(1))\sqrt{\frac{\beta n}{1 - \beta/4}}} \ . \qquad \square$$

Proof of Theorem 1.2 equation (5). Let A be a set of positive integers with upper density $\beta < 1/2$. By (1) it is sufficient to prove that there exist infinitely many pairs of integer and real numbers (n_i, β_i) satisfying

$$\log p_A(n_i) \geq (2\log(2) - o_{n_i}(1))(1 + \beta_i/10)\sqrt{\beta_i n_i} \ , \tag{35}$$

$\lim_{i \to \infty} n_i = \infty$ and $\lim_{i \to \infty} \beta_i = \beta$. Since A has upper density β there exists an increasing sequence of integers $\{m_i\}_{i=0}^{\infty}$ such that setting $A_i = A \cap [m_i]$ and $\beta_i = |A_i|/m_i$ we have $\lim_{i \to \infty} \beta_i = \beta$. Lemma 3.1 implies that for every i there exists $n_i \in [\beta_i^2 m_i^2/4, \beta_i(1 - \beta_i/4)m_i^2]$ satisfying

$$\log p_A(n_i) \geq (2\log(2) - o_{n_i}(1))\sqrt{\frac{\beta_i n_i}{1 - \beta_i/4}} \geq (2\log(2) - o_{n_i}(1))(1 + \beta_i/10)\sqrt{\beta_i n_i} \ .$$

Since $\lim_{i \to \infty} n_i = \infty$ and $\lim_{i \to \infty} \beta_i = \beta$ we obtain (35). $\qquad \square$

To prove Theorem 1.2 equation (4) require the following lemma.

Lemma 3.2. *Suppose $0 < \beta < 1$. Then for all n, m positive integers we have,*

$$p_{[(1-\beta)n+1, n]}(m) \leq e^{2\log(2)\sqrt{\frac{\beta}{1-\beta}m}} \ .$$

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Proof of Lemma 3.2. Assume $n > 1/\beta$[8] and set $\gamma = m/n^2$. For simplicity of presentation we let A denote the set $[(1 - \beta)n + 1, n]$. Our goal is then to prove that for any $\gamma > 0$ we have,

$$p_A(\gamma n^2) \leq 2^{2\sqrt{\frac{\beta\gamma}{1-\beta}}n} . \tag{36}$$

Observe that a partition of γn^2 using integers from A can use at most $\gamma n/(1 - \beta)$ numbers. Hence, we can encode each such partition as a solution in nonnegative integers to the inequality $\sum_{i=0}^{\beta n-1} x_i \leq \gamma n/(1 - \beta)$. This is done by taking x_i to be the number of times $n - i$ appears in the partition. Therefore we obtain,

$$p_A(\gamma n^2) \leq \binom{\gamma n/(1 - \beta) + \beta n}{\gamma n/(1 - \beta)} . \tag{37}$$

We now use the well known inequality,

$$\binom{n}{k} \leq 2^{H_2(k/n)n} , \tag{38}$$

where $H_2(x)$ is the binary entropy function defined by

$$H_2(x) = -x \log_2(x) - (1 - x) \log_2(1 - x) .$$

From (37) and (38) we obtain,

$$p_A(\gamma n^2) \leq 2^{H_2\left(\frac{\gamma}{\gamma+\beta(1-\beta)}\right)(\gamma/(1-\beta)+\beta)n} . \tag{39}$$

Let

$$f_\beta(\gamma) = H_2\left(\frac{\gamma}{\gamma + \beta(1 - \beta)}\right) \frac{\gamma/(1 - \beta) + \beta}{\sqrt{\gamma}} ,$$

and observe that to prove (36) it is enough to prove that $f_\beta(\gamma) \leq 2\sqrt{\frac{\beta}{1-\beta}}$. Noting that $f_\beta(\beta(1-\beta)) = 2\sqrt{\frac{\beta}{1-\beta}}$ it is enough to prove that $f'_\beta(\gamma) > 0$ for $0 < \gamma < \beta(1 - \beta)$ and $f'_\beta(\gamma) < 0$ for $\gamma > \beta(1 - \beta)$.

We first note that

$$f'_\beta(\gamma) = \frac{\beta(1 - \beta)\log_2\left(\frac{\beta(1-\beta)}{\beta(1-\beta)+\gamma}\right) - \gamma \log_2\left(\frac{\gamma}{\beta(1-\beta)+\gamma}\right)}{2(1 - \beta)\gamma^{3/2}} .$$

Since the denominator above is always positive, we focus on the nominator. For every $a > 0$ define

$$g_a(x) = a \log_2\left(\frac{a}{a + x}\right) - x \log_2\left(\frac{x}{a + x}\right) .$$

We will now show that if $0 < x < a$ then $g_a(x) > 0$ and if $x > a$ then $g_a(x) < 0$, noting that this implies the required assertion regarding $f'_\beta(\gamma)$ upon taking $a = \beta(1 - \beta)$. Differentiating $g_a(x)$ we obtain

$$g'_a(x) = \frac{-2a - (a + x)\log\left(\frac{x}{a+x}\right)}{(a + x)\log(2)} .$$

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[8]This can be assumed as otherwise $[(1 - \beta)n + 1, n]$ is empty and the lemma holds trivially.

To see that $g_a(x) < 0$ for $x > a$ observe that $-(a+x)\log\left(\frac{x}{a+x}\right)$ is a decreasing function as its derivative is $-\frac{a}{x}+\log\left(1+\frac{a}{x}\right)$ which is negative by the well known inequality $\log(1+x) < x$. Therefore $-(a+x)\log\left(\frac{x}{a+x}\right) \leq -2a\log(1/2)$ implying

$$g'_a(x) \leq \frac{-2a(1+\log(1/2))}{(a+x)\log(2)} < 0 .$$

Thus, $g_a(x)$ is strictly decreasing for all $x > a$, implying that $g_a(x) < 0$ for all $x > a$ as $g_a(a) = 0$.

We now prove that $g_a(x) > 0$ for all $0 < x < a$. To this end note that

$$g''_a(x) = \frac{-a(a-x)}{x(a+x)^2\log(2)} ,$$

which is negative for all $0 < x < a$. Therefore $g_a(x)$ is concave in $(0,a)$, implying that $g_a(x) > 0$ for all $x \in (0,a)$ as $g_a(a) = 0$ and $\lim_{x\to 0} g_a(x) = 0$. Thus we have completed the proof that if $0 < x < a$ then $g_a(x) > 0$ and if $x > a$ then $g_a(x) < 0$. $\qquad\square$

Proof of Theorem 1.2 equation (4). Similar to the proof of Theorem 1.2 equation (5) it is enough to prove that there exists a set of positive integers A with upper density β and $\gcd(A) = 1$ satisfying,

$$\log p_A(m) \leq (2\log(2) + o_m(1))\sqrt{\frac{\beta m}{1-\beta}} \leq (2\log(2) + o_m(1))(1+\beta)\sqrt{\beta m} , \qquad (40)$$

where the second inequality holds provided $\beta \leq 1/2$. To this end fix $n_0 = \lceil\frac{1}{\beta}\rceil$. We define A as follows. Define a sequence of sets A_i recursively. Set $A_0 = [(1-\beta)n_0 + 1, n_0]$ and $f(0) = n_0$, for any positive integer i let $f(i+1) = 2^{f(i)}$ and $A_{i+1} = [(1-\beta)f(i+1)+1, f(i+1)]$. Now let $A = \bigcup_{n\in\mathbb{N}} A_n$. It is easy to see that the upper density of A is β. Further, since A contains two consecutive integers we have $\gcd(A) = 1$. We now prove that this set satisfies the first inequality in (40). Let m be any positive integer greater than n_0 and let i be the unique integer such that $m \in [f(i) + 1, f(i+1)]$. For simplicity of presentation we set $n = f(i)$ and thus $f(i+1) = 2^n$. We consider two cases, the first is when $m \in [n+1, (1-\beta)2^n]$ and the second is when $m \in [(1-\beta)2^n + 1, 2^n]$.

Case 1: Assume $m \in [n+1, (1-\beta)2^n]$. Let $B = \bigcup_{j\leq i-1} A_j$ and note that we have,

$$p_A(m) = \sum_{0\leq k\leq m} p_B(k)\cdot p_{A_i}(m-k) .$$

Note further that our choice of f guarantees that $|B| \leq \log_2(n) \leq \log_2(m)$. This, Claim 2.2 and Lemma 3.2 give the following bound

$$p_A(m) \leq \sum_{0\leq k\leq m} (k+1)^{\log_2(m)}\cdot e^{2\log(2)\sqrt{\frac{\beta(m-k)}{1-\beta}}}$$

$$\leq (m+1)(m+1)^{\log_2(m)}e^{2\log(2)\sqrt{\frac{\beta m}{1-\beta}}}$$

$$\leq e^{(2\log(2)+o_m(1))\sqrt{\frac{\beta m}{1-\beta}}} .$$

Taking logarithm from both sides of the inequality we obtain the first inequality in (40).

Case 2: Assume $m \in [(1 - \beta)2^n + 1, 2^n]$. Let $B = \bigcup_{j \leq i} A_j$ and note that we have,

$$p_A(m) = \sum_{0 \leq k \leq m} p_B(m - k) \cdot p_{A_{i+1}}(k)$$

$$= \sum_{0 \leq k \leq (1-\beta)2^n} p_B(m - k) \cdot p_{A_{i+1}}(k) + \sum_{(1-\beta)2^n+1 \leq k \leq m} p_B(m - k) \cdot p_{A_{i+1}}(k)$$

$$= p_B(m) + \sum_{(1-\beta)2^n+1 \leq k \leq m} p_B(m - k) \cdot p_{A_{i+1}}(k) .$$

For all $k \leq m \leq 2^n$ any partition of k with all parts in A_{i+1} uses at most $1/(1 - \beta)$ integers from $A_{i+1} \cap [m]$. Therefore for all $k \in [(1 - \beta)2^n + 1, m]$ we have

$$p_{A_{i+1}}(k) \leq m^{1/(1-\beta)} .$$

Next, if $k > (1 - \beta)2^n$ then

$$\log_2(k) + \log_2(1/(1 - \beta)) \geq \log_2(2^n) = n \geq |B| .$$

Therefore by Claim 2.2 we obtain that for all $k \in [(1 - \beta)2^n + 1, m]$ we have

$$p_B(k) \leq (k + 1)^{|B|} \leq (m + 1)^{\log_2(k)+\log_2(1/(1-\beta))} .$$

Combining the above two observations we obtain,

$$p_A(m) \leq \sum_{0 \leq k \leq m} (m + 1)^{\log_2(k)+\log_2(1/(1-\beta))+1/(1-\beta)} \leq (m + 1)(m + 1)^{\log_2(m)+4} \leq e^{o(\sqrt{m})} . \qquad \square$$

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