On Erdős's Method for Bounding the Partition Function

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Abstract

For fixed m and $R \subseteq \{0, 1, \ldots, m-1\}$, take A to be the set of positive integers congruent modulo m to one of the elements of R, and let $p_A(n)$ be the number of ways to write n as a sum of elements of A. Nathanson proved that $\log p_A(n) \le (1 + o(1))\pi\sqrt{2n|R|/3m}$ using a variant of a remarkably simple method devised by Erdős in order to bound the partition function. In this short note we describe a simpler and shorter proof of Nathanson's bound.

1 Introduction.

A partition of an integer n is a sequence of positive integers $a_1 \le a_2 \le \cdots$ whose sum is n. Let $p(n)$ denote the classical partition function of n, namely, the number of ways to write n as a sum of positive integers. The celebrated Hardy–Ramanujan formula [2] (discovered independently by Uspensky [6]) states that $p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$. Erdős [1] later devised a remarkably simple proof of the slightly weaker upper bound

$$\log p(n) \le \pi\sqrt{2n/3}\,. \tag{1}$$

Let $\mathbb{N}$ denote the set of positive integers, and suppose $S \subseteq \mathbb{N}$. We define $p_S(n)$ to be the number of partitions of n with all summands in S. For a fixed positive integer m and $R \subseteq \{0, 1, \ldots, m-1\}$, we take $A = A(m, R)$ to be the set of all positive integers a with $a \pmod m \in R$. Nathanson [4] used Erdős's method for proving (1) to obtain[1]

$$\log p_A(n) \le (1 + o(1))\pi\sqrt{2n|R|/3m}\,. \tag{2}$$

The argument in [4] was more complicated than Erdős's due to the need to control various error parameters (but was still simpler than the original proof of this result [3]); see the remark at the end of the proof.

Our goal in this short note is to give a proof of (2) which is as simple as Erdős's proof of (1). The main trick is that, instead of directly bounding $p_A(n)$, we will instead bound $p_{A^+}(n)$, where given m and R as above, we take $A^+ = A \setminus R$, that is, the set of all integers $a \ge m$ with $a \pmod m \in R$. Our main result here is the following generalization[2] of (1).

Theorem 1. *For every A^+ as above,* $\log p_{A^+}(n) \le \pi\sqrt{2n|R|/3m}$.

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[1] Nathanson [4] also proves that $\log p_A(n) \ge (1 - o(1))\pi\sqrt{2n|R|/3m}$.

[2] Indeed, when $m = 1$ and $R = \{0\}$, we have $p_{A^+}(n) = p(n)$.

It is easy to obtain (2) from the upper bound given by Theorem 1. Indeed, we first note that for every n' we have $p_{R^+}(n') \le (n'+1)^{|R|}$, where $R^+ = R \setminus \{0\}$. This follows immediately from the fact that in every partition of n', each of the integers of R^+ is used at most n' times. We thus infer that

$$p_A(n) = \sum_{0 \le n' \le n} p_{R^+}(n') \cdot p_{A^+}(n-n') \le (n+1)^{|R|} \sum_{0 \le n' \le n} e^{c\sqrt{n-n'}} \le (n+1)^{|R|+1} e^{c\sqrt{n}} ,$$

where $c = \pi\sqrt{2|R|/3m}$. Taking logs from both sides, we obtain (2).

The proof of Theorem 1 appears in the next section. At the end of that section we briefly explain why our proof is simpler than that of [4].

2 Proof of Theorem 1.

For a given fixed integer $m \ge 1$ and $R \subseteq \{0, 1, \ldots, m-1\}$, let A^+ denote the set of all integers $a \ge m$ with $a \pmod m \in R$. We start with a few observations that extend those used in [1]. We first note that, for every $0 < t < 1$, we have

$$\sum_{a \in A^+} a t^a = \sum_{r \in R} \frac{(r+m)t^{r+m} - r t^{2m+r}}{(1-t^m)^2} . \tag{3}$$

Indeed, $\sum_{a \in A^+} a t^a = \sum_{r \in R} \sum_{a \in A_r^+} a t^a$ where A_r^+ is the set of all integers $a \ge m$ with $a = r \pmod m$ (i.e., $A_r^+ = \{r+m, r+2m, r+3m, \ldots\}$). Hence, without loss of generality we may assume $|R| = 1$. Letting $r \in R$, we have

$$\sum_{a \in A_r^+} a t^a = t \sum_{a \in A_r^+} \frac{d}{dt} t^a = t \cdot \frac{d}{dt} \sum_{a \in A_r^+} t^a = t \cdot \frac{d}{dt} \frac{t^{r+m}}{1-t^m} = \frac{(r+m)t^{r+m} - r t^{2m+r}}{(1-t^m)^2} .$$

This proves (3). We next claim that, if $0 \le r \le m-1$ is an integer, then for all $x > 0$, we have

$$\frac{(r+m)e^{-(r+m)x} - r e^{-(2m+r)x}}{(1-e^{-mx})^2} \le \frac{1}{mx^2} . \tag{4}$$

Indeed, since $x > 0$, the power series expansion of e^x gives

$$e^{x/2} - e^{-x/2} = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{x}{2}\right)^{2k+1} = x + x^3 \sum_{k=1}^{\infty} \frac{x^{2k-2}}{(2k+1)! \cdot 2^{2k}} > x ,$$

implying that

$$\frac{e^{-x}}{(1-e^{-x})^2} = \frac{1}{(e^{x/2} - e^{-x/2})^2} < 1/x^2 .$$

We can thus infer that

$$\begin{aligned} \frac{(r+m)e^{-(r+m)x} - r e^{-(2m+r)x}}{(1-e^{-mx})^2} &= ((r+m)e^{-rx} - r e^{-(m+r)x}) \frac{e^{-mx}}{(1-e^{-mx})^2} \\ &\le ((r+m)e^{-rx} - r e^{-(m+r)x}) \frac{1}{m^2 x^2} . \end{aligned}$$

It remains to check that the expression in parentheses is bounded by m. Since the derivative of $(r+m)e^{-rx} - re^{-(m+r)x}$ (which is $r(r+m)(e^{-(m+r)x} - e^{-rx})$) is always nonpositive for $x \geq 0$, it is enough to check its value at $x = 0$ where it attains the value m. This proves (4).

We now note that (3) and (4) imply that, for every $x > 0$,

$$\sum_{a \in A^+} ae^{-ax} \leq \frac{|R|}{mx^2} \; . \tag{5}$$

The final observation we will need is the well-known fact that, for every set of positive integers S, we have

$$n \cdot p_S(n) = \sum_{s \in S \cap [n]} s \sum_{1 \leq k \leq n/s} p_S(n - sk) \; , \tag{6}$$

where we use $[n]$ for the integers $\{1, \ldots, n\}$. To see this, let $p_S(n, s, t)$ and $p'_S(n, s, t)$ be the number of partitions of n with summands in S where s appears exactly t times, and at least t times, respectively. Then by double counting,[3] we have

$$\begin{aligned} n \cdot p_S(n) &= \sum_{s \in S, t \in \mathbb{N}} s \cdot t \cdot p_S(n, s, t) = \sum_{s \in S \cap [n]} s \sum_{t \in \mathbb{N}} t \cdot p_S(n, s, t) \\ &= \sum_{s \in S \cap [n]} s \sum_{t \in \mathbb{N}} p'_S(n, s, t) = \sum_{s \in S \cap [n]} s \sum_{1 \leq k \leq n/s} p_S(n - sk) \; . \end{aligned}$$

This proves (6).

We are now ready to complete the proof of Theorem 1. In what follows we set $c = \pi\sqrt{2|R|/3m}$. We use induction on n, with the base case trivially holding. We have

$$\begin{aligned} n \cdot p_{A^+}(n) &= \sum_{a \in A^+ \cap [n]} a \sum_{1 \leq k \leq n/a} p_{A^+}(n - ak) \leq \sum_{a \in A^+ \cap [n]} a \sum_{1 \leq k \leq n/a} e^{c\sqrt{n - ak}} \\ &\leq e^{c\sqrt{n}} \sum_{a \in A^+ \cap [n]} a \sum_{1 \leq k \leq n/a} e^{-\frac{cak}{2\sqrt{n}}} \leq e^{c\sqrt{n}} \sum_{k=1}^{\infty} \sum_{a \in A^+} ae^{-\frac{cak}{2\sqrt{n}}} \\ &\leq e^{c\sqrt{n}} \sum_{k=1}^{\infty} \frac{4|R|n}{mc^2k^2} = ne^{c\sqrt{n}} \frac{4|R|}{mc^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = n \cdot e^{c\sqrt{n}} \; , \end{aligned}$$

where the first equality is (6), the first inequality is by the induction hypothesis, the second inequality uses the elementary fact $\sqrt{n - rk} \leq \sqrt{n} - \frac{rk}{2\sqrt{n}}$, and in the last inequality we applied (5) with $x = \frac{ck}{2\sqrt{n}}$. Dividing both sides by n we obtain the theorem.

Bounding $p_{A^+}(n)$ vs. bounding $p_A(n)$. The reader might be wondering why bounding $p_{A^+}(n)$ is so much easier than bounding $p_A(n)$. The answer is that the former gives us inequality (4) from which we obtain the clean inequality (5). To illustrate the complication that arises when working with $p_A(n)$, let us take A to be the set of odd integers. Then, running the same argument, instead of (4), one would have liked to use the inequality $\frac{e^{-x}+e^{-3x}}{(1-e^{-2x})^2} \leq \frac{1}{2x^2}$, which is false. To overcome this, one then needs to use the fact that this inequality is approximately correct for small x, which significantly complicates the proof.

[3] The two sides of the first equality count the sum of all integers that appear in all partitions of n using integers from S (there are $p_S(n)$ such partitions). As to the third equality, it follows by observing that each partition of n with exactly t occurrences of s contributes 1 to t of the summands $p'_S(n, s, t)$, namely $p'_S(n, s, 1), p'_S(n, s, 2), \ldots, p'_S(n, s, t)$. See Theorem 15.1 in [5] for a full detailed proof.

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