Every orientation of a 4-chromatic graph has a non-bipartite acyclic subgraph

Asaf Shapira*

Abstract

Let $f(n)$ denote the smallest integer such that every directed graph with chromatic number larger than $f(n)$ contains an acyclic subgraph with chromatic number larger than n. The problem of bounding this function was introduced by Addario-Berry et al., who noted that $f(n) \leq n^2$. The only improvement over this bound was obtained by Nassar and Yuster, who proved that $f(2) = 3$ using a deep theorem of Thomassen. Yuster asked if this result can be proved using elementary methods. In this short note we provide such a proof.

1 Introduction

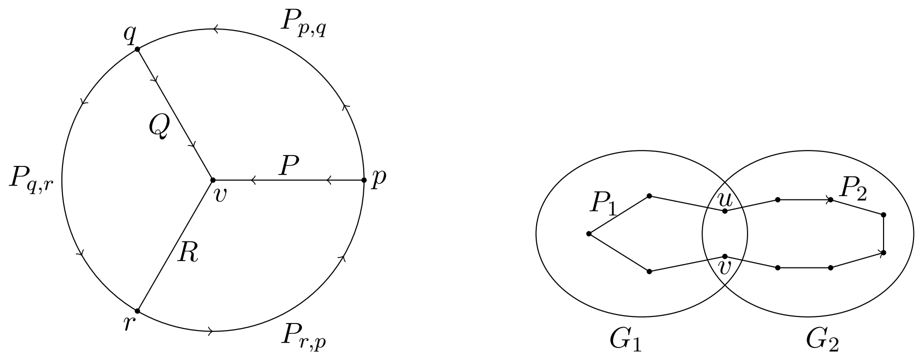
The relation between the chromatic number of a graph and properties of orientations of its edges have long been investigated. For the sake of brevity, we refer the reader to [4] for a general survey on this topic, and to the discussions in [3, 6], which are more closely related to our investigation here.

We consider the following problem introduced by Addario-Berry, Havet, Sales, Reed and Thomassé; given an integer n, what is the smallest integer $f(n)$ so that if G has chromatic number more than $f(n)$ then in every orientation of G's edges, one can find an acyclic subgraph of chromatic number more than n. The best known general upper bound for this function is $f(n) \leq n^2$. This follows from taking any oriented version of G, splitting it into two acyclic subgraphs, denoted G_1, G_2, and applying the well known fact that the chromatic number of G is at most the product of the chromatic numbers of G_1, G_2. The only known improvement over this general bound was obtained by Nassar and Yuster [6] who proved that $f(2) = 3$, by establishing the following.

Theorem 1 (Nassar–Yuster [6]). *Suppose G is a graph of chromatic number 4. Then every orientation of its edges contains an acyclic odd cycle.*

The proof in [6] relied on a deep theorem of Thomassen [7], which confirmed a conjecture of Toft [8]. Yuster [9] asked if one can prove Theorem 1 using elementary methods. In this short paper we give such a proof. The main idea is to take advantage of properties of 4-critical graphs.

*School of Mathematics, Tel Aviv University, Tel Aviv 69978, Israel. Email: asafico@tau.ac.il. Supported in part by ISF Grant 1028/16, ERC Consolidator Grant 863438 and NSF-BSF Grant 20196.



(a) G is 3-connected

(b) G is not 3-connected

Figure 1: The two cases considered in the proof

2 An elementary proof of Theorem 1

We may and will assume that G is 4-critical, that is, that the removal of every edge of G reduces its chromatic number. This will allow us to use important properties of 4-critical graphs. We proceed by induction on $|V(G)|$, with the base case being K_4. It is easy to see that every orientation of K_4 contains an acyclic K_3 (in fact, two) so the base case holds. We now proceed with the induction step. We consider separately the case where G is 3-connected (in which case we will not need induction) and the case where it has a separating pair of vertices.

Assume first that G is 3-connected, and let C be a shortest odd cycle in G. Since C must be induced and G has chromatic number 4, there must be a vertex $v \notin C$. Since G is assumed to be 3-connected, there are 3 vertex disjoint paths connecting v to C. Let P, Q, R denote these paths, and p, q, r denote their meeting points with C, see Figure 1a. If C is acyclic we are done, so suppose wlog that C is oriented as in Figure 1a. Clearly not all three paths $P \cup Q$, $P \cup R$ and $Q \cup R$ can be directed paths, as they all intersect internally in the vertex v. Assume wlog that $P \cup Q$ is not directed. Then, since $|C|$ is odd, one of the cycles $P \cup Q \cup P_{pq}$ or $P \cup Q \cup P_{rp} \cup P_{qr}$ is an acyclic odd cycle.

Suppose now that G is not 3 connected, that is, it has a pair of vertices u, v whose removal breaks it into at least two (non-empty) connected components. In what follows, if $(u, v) \notin E(G)$ then we use $G + (u, v)$ to denote the graph obtained by adding the edge (u, v) to G. We use $G/\{u, v\}$ to denote the graph obtained from G by contracting u, v, that is, the graph obtained by replacing u, v with a new vertex and connecting it to all the vertices that were connected to either[1] v or u. We will need the following well known result of Dirac [2], see also Problem 9.22 in [5] for a short proof.

Lemma 2.1 (Dirac [2]). *Let $k \geq 4$ be an integer, let G be a k-critical graph, and let $u, v \in V(G)$ be such that $G \setminus \{u, v\}$ (the graph obtained from G by deleting the vertices u, v) is disconnected. Then*

1. *$u \neq v$, that is, G is 2-connected*
2. *$(u, v) \notin E(G)$*

[1] We will only apply this operation when u, v are not connected and have no common neighbor, so this operation will not create loops or parallel edges.

3. *$G \setminus \{u,v\}$ has exactly two components*

4. *There are unique proper induced subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{u,v\}$, and the graphs $G_1 \setminus \{u,v\}$ and $G_2 \setminus \{u,v\}$ are the two components of $G \setminus \{u,v\}$. Also, u,v have no common neighbor in G_2, and $G_1 + (u,v)$ and $G_2/\{u,v\}$ are k-critical.*

By induction and Lemma 2.1, the graph $G_2/\{u,v\}$ has an acyclic odd cycle C_2. If C_2 does not contain the vertex w that resulted from contracting $\{u,v\}$, it is also a cycle in G and we are done. Also, if the two neighbors of w on C_2 are both neighbors of v or both neighbors of u, then we can again conclude that C_2 is also an acyclic odd cycle in G. So assume one neighbor of w is a neighbor of v and one is a neighbor of u. Then we may infer that in G we have a path P_2 connecting u and v, so that $|P|$ is even and P is *not* directed from u to v or from v to u. See Figure 1b.

By induction and Lemma 2.1, the graph $G_1 + (u,v)$ has an acyclic odd cycle C_1 (no matter how we orient the edge (u,v)). If C_1 does not use the edge (u,v), it is also an acyclic odd cycle in G and we are done. Suppose then that it does, implying that G contains a path P_1 connecting u to v with $|P_1|$ odd. Then item (4) in Lemma 2.1 guarantees that $|P_1 \cup P_2| = |P_1| + |P_2| - 2$ so $P_1 \cup P_2$ is an odd cycle. The assertion at the end of the previous paragraph guarantees that it is acyclic. This completes the proof of Theorem 1.

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