A Triple Correspondence in Canonical Calculi: Strong Cut-Elimination, Coherence, and Non-deterministic Semantics

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Abstract. An (n, k)-ary quantifier is a generalized logical connective, binding k variables and connecting n formulas. Canonical systems with (n, k)-ary quantifiers form a natural class of Gentzen-type systems which in addition to the standard axioms and structural rules have only logical rules in which exactly one occurrence of a quantifier is introduced. The semantics for these systems is provided using two-valued non-deterministic matrices, a generalization of the classical matrix. In this paper we use a constructive syntactic criterion of coherence to characterize strong cut-elimination in such systems. We show that the following properties of a canonical system G with arbitrary (n, k)-ary quantifiers are equivalent: (i) G is coherent, (ii) G admits strong cut-elimination, and (iii) G has a strongly characteristic two-valued generalized non-deterministic matrix.

1 Introduction

The possibility to eliminate cuts is a crucial property of useful sequent calculi. This property was first established by Gentzen (in his classical [10]) for sequent calculi for classical and intuitionistic first-order logic. Since then many other cut-elimination theorems, for many systems, have been proved by various methods [1]. Now showing that a given sequent calculus admits cut-elimination is a difficult task, often carried out using heavy syntactic arguments and based on many case-distinctions. It is thus important to have some useful criteria that *characterize* cut-elimination (i.e., conditions which are both necessary and sufficient for having an appropriate cut-elimination theorem).

In this paper we give a constructive characterization of a general strong form of cut-elimination for a very natural class of Gentzen-type systems called *canonical systems*. These are systems which in addition to the standard axioms and structural rules have only logical introduction rules of the ideal type, in which exactly one occurrence of a connective or quantifier is introduced, and no other connective or quantifier is mentioned in the formulation of the rule. For the propositional case canonical systems were first introduced and investigated in [2]. There semantics for such systems was provided using two-valued

[1] We note that by 'cut-elimination' we mean here just the *existence* of proofs without (certain forms of) cuts, rather than an algorithm to transform a given proof to a cut-free one (for the assumptions-free case the term "cut-admissibility" is sometimes used, but this notion is too weak for our purposes).

non-deterministic matrices (called 2Nmatrices) [2]. It was shown in [2] that for propositional canonical systems there is an exact *triple correspondence* between cut-elimination, the existence of a characteristic 2Nmatrix for them, and a constructive syntactic property called *coherence*. In [17] the same *triple correspondence* was shown to hold also for canonical systems with unary quantifiers. The next natural stage was to consider languages and systems with arbitrary (n, k)-ary quantifiers. By an (n, k)-ary quantifier ([13, 15]) we mean a generalized logical connective, which binds k variables and connects n formulas. In particular, any n-ary propositional connective is an $(n, 0)$-ary quantifier, the unary quantifiers considered in [17] (including the standard first-order quantifiers $\exists$ and $\forall$) are $(1, 1)$-quantifiers, bounded universal and existential quantifiers used in syllogistic reasoning ($\forall x(p(x) \rightarrow q(x))$ and $\exists x(p(x) \wedge q(x))$) are (2,1)-ary quantifiers, while the simplest Henkin quantifier[3] $\mathcal{Q}^H$ is a (1,4)-quantifier:

$$\mathcal{Q}^H x_1 x_2 y_1 y_2 \psi(x_1, x_2, y_1, y_2) := \begin{matrix} \forall x_1 \ \exists y_1 \\ \forall x_2 \ \exists y_2 \end{matrix} \psi(x_1, x_2, y_1, y_2)$$

The first steps in investigating canonical systems with (n, k)-ary quantifiers were taken in [18]. There the semantics of 2Nmatrices and the coherence criterion were extended to languages with $(n, 1)$-ary quantifiers. Then it was shown that coherence is equivalent in this case to the existence of a characteristic 2Nmatrix, and it implies (but is not equivalent to) cut-elimination. When canonical systems are generalized to languages with arbitrary (n, k)-ary quantifiers, two serious problems emerge. The first problem is that the semantics of 2Nmatrices employed in [17, 18] is no longer adequate in case $k > 1$. The second problem is that even in case of $k = 1$, coherence is not a necessary condition for standard cut-elimination, and so the *triple correspondence* seems to be lost. Now the first problem was solved in [3] by introducing *generalized* two-valued Nmatrices (2GNmatrices), where a more complex approach to quantification is used. However, the problem of finding an appropriate form of cut-elimination for which coherence *is* a necessary condition, and reestablishing the *triple correspondence* was explicitly left open in [18] and then also in [3].

In this paper we provide a full solution to the second problem [4]. This is achieved in two main steps. The first is to include *substitution* as one of the structural rules of a canonical system [5]. The second step is to extend cut-elimination

[2] Non-deteministic matrices form a natural generalization of ordinary matrices, in which the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options.

[3] It should be noted though that the canonical systems with (n, k)-ary quantifiers studied in this paper are still not sufficient for treating Henkin quantifiers, as the representation language is not expressive enough to capture dependencies between variables; one direction is extending the representation language with function symbols, which would lead to the inevitable loss of the decidability of coherence.

[4] A partial solution for the restricted case $k = 1$ is already provided in [4] (which is a revised journal version of [18]).

[5] See [1] for the general need for this step, e.g. for the foundations of the resolution calculus, and for reasoning from assumptions in general.

to *deduction from assumptions* (following [1, 16]). For historical reasons, reasoning with non-logical assumptions (in the form of sequents) is usually reduced to pure provability of sequents. However, this is not always possible (the resolution calculus, primitive recursive arithmetics, pure equality reasoning, and disjunctive databases are four cases in point). Even when such reduction is possible, it is not necessarily desirable, as the case of first-order logic with equality shows (see e.g. [16]). Thus it is in fact very natural to investigate and characterize cut-elimination in the context of deduction from assumptions.

With the aid of the above two steps we again establish an exact *triple correspondence* between coherence of canonical systems with arbitrary (n, k)-ary quantifiers, their 2GNmatrices-based semantics, and a strong form of cut-elimination for them. More specifically, we show that the following properties of a canonical system G are equivalent: (i) G is coherent, (ii) G admits strong cut-elimination, and (iii) G has a strongly[6] characteristic 2GNmatrix.

2 Preliminaries

In what follows, L is a language with (n, k)-ary quantifiers, that is with quantifiers $\mathcal{Q}_1, ..., \mathcal{Q}_m$ with arities (n_1, k_1), ..., (n_m, k_m) respectively. For any $n > 0$ and $k \geq 0$, if a quantifier $\mathcal{Q}$ in a language L is of arity (n, k), then $\mathcal{Q}x_1...x_k(\psi_1, ..., \psi_n)$ is an L-formula whenever $x_1, ..., x_k$ are distinct variables and $\psi_1, ..., \psi_n$ are formulas of L. Denote by Frm_L (Frm_L^{cl}) the set of L-formulas (closed L-formulas). Denote by Trm_L (Trm_L^{cl}) the set of L-terms (closed L-terms). We write $\mathcal{Q}\overrightarrow{x}A$ instead of $\mathcal{Q}x_1...x_kA$, and $\psi\{\overrightarrow{\mathbf{t}}/\overrightarrow{z}\}$ instead of $\psi\{\mathbf{t}_1/z_1, ..., \mathbf{t}_k/z_k\}$.
A set of sequents $\mathcal{S}$ satisfies the *free-variable condition* if the set of variables occurring bound in $\mathcal{S}$ is disjoint from the set of variables occurring free in $\mathcal{S}$.
In the following two subsections, we briefly reproduce the relevant definitions from [18, 3] of canonical rules with (n, k)-ary quantifiers and of the framework of non-deterministic matrices. Note the important addition of the definition of *full canonical systems*, which include the rule of substitution.

2.1 Full canonical systems with (n, k)-ary quantifiers

We use the simplified representation language from [18, 3] for a schematic representation of canonical rules.

Definition 2.1 *For $k \geq 0$, $n \geq 1$, L_k^n is the language with n k-ary predicate symbols $p_1, ..., p_n$, the set of constants $Con = \{c_1, c_2, ..., \}$ and the set of variables $Var = \{v_1, v_2, ..., \}$.*

In this paper we assume for simplicity that L_k^n and L share their sets of variables and constants.

[6] See Defn. 2.15 below.

Definition 2.2 *A* canonical rule of arity (n, k) *has the form* $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/C$, *where* $m \geq 0$, C *is either* $\Rightarrow \mathcal{Q}v_1...v_k(p_1(v_1,...,v_k),...,p_n(v_1,...,v_k))$ *or* $\mathcal{Q}v_1...v_k(p_1(v_1,...,v_k),...,p_n(v_1,...,v_k)) \Rightarrow$ *for some* (n, k)*-ary quantifier* $\mathcal{Q}$ *of* L *and for every* $1 \leq i \leq m$: $\Pi_i \Rightarrow \Sigma_i$ *is a clause*[7] *over* L_k^n.

For a specific application of a canonical rule we need to instantiate the schematic variables by the terms and formulas of L. This is done using a mapping function:

Definition 2.3 *Let* $R = \Theta/C$ *be an* (n, k)*-ary canonical rule, where* C *is of one of the forms* $(\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v})) \Rightarrow)$ *or* $(\Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v})))$. *Let* Γ *be a set of* L*-formulas and* $z_1,...,z_k$ *- distinct variables of* L. *An* $\langle R, \Gamma, z_1,...,z_k\rangle$*-* mapping *is any function* χ *from the predicate symbols, terms and formulas of* L_k^n *to formulas and terms of* L, *satisfying the following conditions:*

- *For every* $1 \leq i \leq n$, $\chi[p_i]$ *is an* L*-formula.* $\chi[y]$ *is a variable of* L, *and* $\chi[x] \neq \chi[y]$ *for every two variables* $x \neq y$. $\chi[c]$ *is an* L*-term, such that* $\chi[x]$ *does not occur in* $\chi[c]$ *for any variable* x *occurring in* Θ.
- *For every* $1 \leq i \leq n$, *whenever* $p_i(\boldsymbol{t}_1,...,\boldsymbol{t}_k)$ *occurs in* Θ, *for every* $1 \leq j \leq k$: $\chi[\boldsymbol{t}_j]$ *is a term free for* z_j *in* $\chi[p_i]$, *and if* $\boldsymbol{t}_j$ *is a variable, then* $\chi[\boldsymbol{t}_j]$ *does not occur free in* $\Gamma \cup \{\mathcal{Q}z_1...z_k(\chi[p_1],...,\chi[p_n])\}$.
- $\chi[p_i(\boldsymbol{t}_1,...,\boldsymbol{t}_k)] = \chi[p_i]\{\chi[\boldsymbol{t}_1]/z_1,...,\chi[\boldsymbol{t}_k]/z_k\}$.

χ *is extended to sets of* L_k^n*-formulas as follows:* $\chi[\Delta] = \{\chi[\psi] \mid \psi \in \Delta\}$.

Definition 2.4 *Let* $R = \Theta/C$ *be an* (n, k)*-ary canonical rule, where* $\Theta = \{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}$ *and* C *has the form* $\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v})) \Rightarrow$. *An* application *of* R *is any inference step of the form:*

$$\frac{\{\Gamma, \chi[\Pi_i] \Rightarrow \Delta, \chi[\Sigma_i]\}_{1\leq i\leq m}}{\Gamma, \mathcal{Q}z_1...z_k\ (\chi[p_1],...,\chi[p_n]) \Rightarrow \Delta}$$

where $z_1,...,z_k$ *are variables,* Γ, Δ *are any sets of* L*-formulas and* χ *is some* $\langle R, \Gamma \cup \Delta, z_1,...,z_k\rangle$*-mapping.*
An application of a canonical rule of the form Θ/C' *there* C' *has the form* $\Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v}))$ *is defined similarly.*

Example 2.5 *The standard introduction rules for the* $(1, 1)$*-ary quantifier* $\forall$ *can be formulated as follows:* $\{p(c) \Rightarrow\}/\forall v_1\, p(v_1) \Rightarrow$ *and* $\{\Rightarrow p(v_1)\}/ \Rightarrow \forall v_1\, p(v_1)$. *Applications of these rules have the forms:*

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta}\ (\forall \Rightarrow) \quad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}\ (\Rightarrow \forall)$$

where z *is free for* w *in* A, z *is not free in* $\Gamma \cup \Delta \cup \{\forall w A\}$, *and* $\boldsymbol{t}$ *is any term free for* w *in* A.

[7] By a clause we mean a sequent containing only atomic formulas.

Notation 2.6 (Following [2, 18]). Let $-t = f, -f = t$. Let $ite(t, A, B) = A$ and $ite(f, A, B) = B$. Let Φ, A^s (where Φ may be empty) denote $ite(s, \Phi \cup \{A\}, \Phi)$. For instance, the sequents $A \Rightarrow$ and $\Rightarrow A$ are denoted by $A^{-a} \Rightarrow A^a$ for $a = f$ and $a = t$ respectively. With this notation, an (n, k)-ary canonical rule has the form $\{\Sigma_j \Rightarrow \Pi_j\}_{1 \le j \le m} / \mathcal{Q}\overrightarrow{z}(p_1(\overrightarrow{z}), ..., p_n(\overrightarrow{z}))^{-s} \Rightarrow \mathcal{Q}\overrightarrow{z}(p_1(\overrightarrow{z}), ..., p_n(\overrightarrow{z}))^s$ for some $s \in \{t, f\}$. For further abbreviation, we denote such rule by $\{\Sigma_j \Rightarrow \Pi_j\}_{1 \le j \le m} / \mathcal{Q}(s)$.

Definition 2.7 *A* full canonical calculus *G is a Gentzen-type system, which consists of (i) The α-axiom $\psi \Rightarrow \psi'$ for $\psi \equiv_\alpha \psi'$, (ii) The standard structural rules with the addition of the substitution rule, and (iii) Canonical inference rules.*

The *coherence* criterion used in [3, 18] can be straightforwardly extended to full canonical calculi:

Definition 2.8 *For two sets of clauses Θ_1, Θ_2 over L_k^n, $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is a set $\Theta_1 \cup \Theta_2'$, where Θ_2' is obtained from Θ_2 by a fresh renaming of constants and variables which occur in Θ_1.*

Definition 2.9 (Coherence[8]) *A full canonical calculus G is* coherent *if for every two canonical rules of the form $\Theta_1/ \Rightarrow A$ and $\Theta_2/A \Rightarrow$, the set of clauses $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically inconsistent.*

Example 2.10 *Consider the calculus G_1 consisting of the rules $\Theta_1/\forall v_1\, p(v_1) \Rightarrow$ and $\Theta_2/ \Rightarrow \forall v_1\, p(v_1)$ where $\Theta_1 = \{p(c) \Rightarrow\}$ and $\Theta_2 = \{\Rightarrow p(v_1)\}$ (these rules are from Example 2.5). $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \{p(c) \Rightarrow, \Rightarrow p(v_1)\}$ (note that no renaming is needed here) is clearly classically inconsistent and so G_1 is coherent.*

Proposition 2.11 (Decidability of coherence) *The coherence of a full canonical calculus is decidable.*

2.2 Generalized non-deterministic matrices

Our main semantic tool in this paper are *generalized non-deterministic matrices* introduced in [3], which are a generalization of non-deterministic structures used in [2, 17, 18].

Definition 2.12 *A* generalized non-deterministic matrix *(henceforth GNmatrix) for L is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where: (i) $\mathcal{V}$ is a non-empty set of truth values, (ii) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and (iii) For every (n, k)-ary quantifier $\mathcal{Q}$ of L, $\mathcal{O}$[9] includes a corresponding operation $\tilde{\mathcal{Q}}_S : (D^k \to \mathcal{V}^n) \to P^+(\mathcal{V})$*

[8] The coherence criterion was first introduced in [2]. A related criterion also called coherence was later used in [14], where linear logic is used to specify and reason about a number of sequent systems.

[9] Strictly speaking, the tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is not well-defined, since $\mathcal{O}$ is a proper class. Since all our results remain intact if we concentrate only on countable models, this technical problem can be overcome by assuming that the domains of all the structures are prefixes of the set of natural numbers.

for every L-structure $S = \langle D, I\rangle$.
A 2GNmatrix is any GNmatrix with $\mathcal{V} = \{t, f\}$ and $\mathcal{D} = \{t\}$.

The notion of an L-structure is defined standardly (see, e.g. [18]). In order to interpret quantifiers, the substitutional approach is used, which assumes that every element of the domain has a term referring to it. Thus given an L-structure $S = \langle D, I\rangle$, the language L is extended with *individual constants*: $\{\overline{a} \mid a \in D\}$. Call the extended language $L(D)$. The interpretation function I is extended to $L(D)$ as follows: $I[\overline{a}] = a$. An S-substitution σ is any function from variables to $Trm^{cl}_{L(D)}$. For an S-substitution σ and a term $\mathbf{t}$ (a formula ψ), the closed term $\sigma[\mathbf{t}]$ (the sentence $\sigma[\psi]$) is obtained from $\mathbf{t}$ (ψ) by substituting every variable x for $\sigma[x]$. We write[10] $\psi \sim^S \psi'$ if ψ' can be obtained from ψ by renamings of bound variables and by any number of substitutions of a closed term $\mathbf{t}$ for another closed term $\mathbf{s}$, so that $I[\mathbf{t}] = I[\mathbf{s}]$.

Definition 2.13 *Let $S = \langle D, I\rangle$ be an L-structure for a GNmatrix $\mathcal{M}$. An S-valuation $v : Frm^{\mathsf{cl}}_{L(D)} \to \mathcal{V}$ is* legal *in $\mathcal{M}$ if it satisfies the following conditions: $v[\psi] = v[\psi']$ for every two sentences ψ, ψ' of $L(D)$, such that $\psi \sim^S \psi'$, $v[p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)] = I[p][I[\boldsymbol{t}_1], ..., I[\boldsymbol{t}_n]]$, and $v[\mathcal{Q}x_1, ..., x_k(\psi_1, ..., \psi_n)$ is in the set $\tilde{\mathcal{Q}}_S[\lambda a_1, ..., a_k \in D.\langle v[\psi_1\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}], ..., v[\psi_n\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}]\rangle]$ for every (n, k)-ary quantifier $\mathcal{Q}$ of L.*

Definition 2.14 *Let $S = \langle D, I\rangle$ be an L-structure for an GNmatrix $\mathcal{M}$. An $\mathcal{M}$-legal S-valuation v is* a model of a sentence ψ in $\mathcal{M}$, *denoted by $S, v \models_{\mathcal{M}} \psi$, if $v[\psi] \in \mathcal{D}$. For an $\mathcal{M}$-legal S-valuation v, a sequent $\Gamma \Rightarrow \Delta$ is* $\mathcal{M}$-valid in $\langle S, v\rangle$ *if for every S-substitution σ: whenever $S, v \models_{\mathcal{M}} \sigma[\psi]$ for every $\psi \in \Gamma$, there is some $\varphi \in \Delta$, such that $S, v \models_{\mathcal{M}} \sigma[\varphi]$. A sequent $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid, denoted by $\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$, if for every L-structure S and every $\mathcal{M}$-legal S-valuation v, $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid in $\langle S, v\rangle$. For a set of sequents $\mathcal{S}$, $\mathcal{S} \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$ if for every L-structure S and every $\mathcal{M}$-legal S-valuation v: whenever the sequents of $\mathcal{S}$ are $\mathcal{M}$-valid in $\langle S, v\rangle$, $\Gamma \Rightarrow \Delta$ is also $\mathcal{M}$-valid in $\langle S, v\rangle$.*

Definition 2.15 *A system G is* strongly sound *for a GNmatrix $\mathcal{M}$ if for every set of sequents $\mathcal{S}$: $\mathcal{S} \vdash_G \Gamma \Rightarrow \Delta$ entails $\mathcal{S} \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$. A system G is* strongly complete *for a GNmatrix $\mathcal{M}$ if for every set of sequents $\mathcal{S}$: $\mathcal{S} \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$ entails $\mathcal{S} \vdash_G \Gamma \Rightarrow \Delta$. A GNmatrix $\mathcal{M}$ is* strongly characteristic *for G if G is strongly sound and strongly complete for $\mathcal{M}$.*

Note that strong soundness implies (weak) soundness[11]. A similar remark applies to completeness and a characteristic GNmatrix.

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[10] The motivation for this definition, in addition to capturing α-equivalence, is purely technical and is related to extending the language with the set of individual constants $\{\overline{a} \mid a \in D\}$. Suppose we have a closed term $\mathbf{t}$, such that $I[\mathbf{t}] = a \in D$. But a also has an individual constant $\overline{a}$ referring to it. We would like to be able to substitute $\mathbf{t}$ for $\overline{a}$ in every context.

[11] A system G is (weakly) sound for a GNmatrix $\mathcal{M}$ if $\vdash_G \Gamma \Rightarrow \Delta$ entails $\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$.

In addition to L-structures for languages with (n,k)-ary quantifiers, we will also use L_k^n-structures for the simplified languages L_k^n, using which the canonical rules are formulated. To make the distinction clearer, we shall use the metavariable S for the former and $\mathcal{N}$ for the latter. Since the formulas of L_k^n are always atomic, the specific 2GNmatrix for which $\mathcal{N}$ is defined is immaterial, and can be omitted. Henceforth we may speak simply of validity of sets of sequents over L_k^n.

Definition 2.16 *Let* $\mathcal{N} = \langle D, I\rangle$ *be a structure for* L_k^n. *The* functional distribution *of* $\mathcal{N}$ *is a function* $FDist_{\mathcal{N}} \in D^k \to \{t,f\}^n$, *such that:* $FDist_{\mathcal{N}} = \lambda a_1,...,a_k \in D.\langle I[p_1][a_1,...,a_k],...,I[p_n][a_1,...,a_k]\rangle$.

3 The Triple Correspondence

In this section we establish an exact *triple correspondence* between the coherence of full canonical systems, their 2GNmatrices-based semantics and *strong cut-elimination*, a version of cut-elimination for deduction with assumptions taken from [1]:

Definition 3.1 *Let* G *be a full canonical calculus and let* $\mathcal{S}$ *be some set of sequents. A proof* P *of* $\Gamma \Rightarrow \Delta$ *from* $\mathcal{S}$ *in* G *is* $\mathcal{S}$-cut-free *if all cuts in* P *are on substitution instances of formulas from* $\mathcal{S}$.

Definition 3.2 *A Gentzen-type calculus* G *admits* strong cut-elimination *if for every set of sequents* $\mathcal{S}$ *and every sequent* $\Gamma \Rightarrow \Delta$, *such that* $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ *satisfies the free-variable condition it holds that if* $\mathcal{S} \vdash_G \Gamma \Rightarrow \Delta$, *then* $\Gamma \Rightarrow \Delta$ *has an* $\mathcal{S}$*-cut-free proof in* G.

Note that strong cut-elimination implies standard cut-elimination (which corresponds to the case of an empty set $\mathcal{S}$).
Remark: At this point the importance of the substitution rule should be stressed. Consider for instance the canonical calculus with two standard $(1,1)$-ary rules for $\forall$ from Example 2.5. Consider the following deduction:

$$\dfrac{\dfrac{\Rightarrow p(x)}{\Rightarrow \forall x p(x)}\ (\Rightarrow \forall) \qquad \dfrac{p(c) \Rightarrow}{\forall x p(x) \Rightarrow}\ (\forall \Rightarrow)}{\Rightarrow}\ (Cut)$$

The above application of Cut can only be eliminated using an explicit application of the substitution rule. An alternative, less satisfactory solution (instead of adding the substitution rule to canonical calculi explicitly,) would be considering only sets of non-logical assumptions, which are closed under substitution[12].

In [3] a strongly sound and (weakly) complete 2GNmatrix $\mathcal{M}_G$ is defined for every coherent canonical calculus G. This can be straightforwardly extended to full canonical calculi. We strengthen this result in the sequel for full canonical calculi by showing that $\mathcal{M}_G$ is also *strongly* complete for G.

[12] This was done in [4] for the restricted class of canonical systems with $k = 1$.

Definition 3.3 *Let G be a coherent full canonical calculus. For every L-structure $S = \langle D, I \rangle$, the GNmatrix $\mathcal{M}_G$ contains the operation $\tilde{\mathcal{Q}}_S$ defined as follows. For every (n,k)-ary quantifier $\mathcal{Q}$ of L, every $r \in \{t, f\}$ and every $g \in D^k \to \{t, f\}^n$:*

$$\tilde{\mathcal{Q}}_S[g] = \begin{cases} \{r\} & \Theta/\mathcal{Q}(r) \in G \text{ and there is an } L_k^n - \text{structure } \mathcal{N} = \langle D_\mathcal{N}, I_\mathcal{N} \rangle \\ & \text{such that } D_\mathcal{N} = D,\ FDist_\mathcal{N} = g \text{ and } \Theta \text{ is valid in } \mathcal{N}. \\ \{t, f\} & \text{otherwise} \end{cases}$$

Proposition 3.4 *If a full canonical calculus G is coherent, then it is strongly sound for $\mathcal{M}_G$.*

Proof: The proof is similar to the proof of Theorem 23 in [3], with the addition of checking that the substitution rule is strongly sound for $\mathcal{M}_G$. In fact, it is easy to see that the substitution rule is strongly sound for any 2GNmatrix $\mathcal{M}$.

Now we come to the main result of this paper - establishing the correspondence:

Theorem 3.5 (The Triple Correspondence) *Let G be a full canonical calculus. Then the following statements concerning G are equivalent:*

1. *G is coherent.*
2. *G has a strongly characteristic 2GNmatrix.*
3. *G admits strong cut-elimination.*

Proof: We first prove that (1) $\Rightarrow$ (2). Suppose that G is coherent. By proposition 3.4, G is strongly sound for $\mathcal{M}_G$. For strong completeness, let $\mathcal{S}$ be some set of sequents. Suppose that a sequent $\Gamma \Rightarrow \Delta$ has no proof from $\mathcal{S}$ in G. Then it also has no $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G. If $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ does not satisfy the free-variable condition, obtain $\mathcal{S}' \cup \{\Gamma' \Rightarrow \Delta'\}$ by renaming the bound variables, so that $\mathcal{S}' \cup \{\Gamma' \Rightarrow \Delta'\}$ satisfies the condition (otherwise, take $\Gamma' \Rightarrow \Delta'$ and $\mathcal{S}'$ to be $\Gamma \Rightarrow \Delta$ and $\mathcal{S}$ respectively). Then $\Gamma' \Rightarrow \Delta'$ has no proof from $\mathcal{S}'$ in G (otherwise we could obtain a proof of $\Gamma \Rightarrow \Delta$ from $\mathcal{S}$ by using cuts on logical axioms), and so it also has no $\mathcal{S}'$-cut-free proof from $\mathcal{S}'$ in G. By proposition 3.6, $\mathcal{S}' \not\vdash_{\mathcal{M}_G} \Gamma' \Rightarrow \Delta'$. That is, there is an L-structure S and an $\mathcal{M}_G$-legal valuation v, such that the sequents in $\mathcal{S}'$ are $\mathcal{M}_G$-valid in $\langle S, v \rangle$, while $\Gamma' \Rightarrow \Delta'$ is not. Since v respects the $\equiv_\alpha$-relation, the sequents of $\mathcal{S}$ are also $\mathcal{M}_G$-valid in $\langle S, v \rangle$, while $\Gamma \Rightarrow \Delta$ is not. And so $\mathcal{S} \not\vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$. We have shown that G is strongly complete (and strongly sound) for $\mathcal{M}_G$. Thus $\mathcal{M}_G$ is a strongly characteristic 2GNmatrix for G.
Now we prove that (2) $\Rightarrow$ (1). Suppose that G has a strongly characteristic 2GN-matrix $\mathcal{M}$. Assume by contradiction that G is not coherent. Then there exist two (n,k)-ary rules of the forms $R_1 = \Theta_1/\Rightarrow A$ and $R_2 = \Theta_2/A \Rightarrow$ in G, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent and $A = \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))$. Recall that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \Theta_1 \cup \Theta_2'$, where Θ_2' is obtained from Θ_2 by renaming constants and variables that occur also in Θ_1 (see defn. 2.8). For simplicity [13] we assume that the fresh constants used for renaming are all in L. Let

[13] This assumption is not necessary and is used only for simplification of presentation, since we can instantiate the constants by any L-terms.

$\Theta_1 = \{\Sigma_j^1 \Rightarrow \Pi_j^1\}_{1\leq j\leq m}$ and $\Theta_2' = \{\Sigma_j^2 \Rightarrow \Pi_j^2\}_{1\leq j\leq r}$. Since $\Theta_1 \cup \Theta_2'$ is classically consistent, there exists an L_k^n-structure $\mathcal{N} = \langle D, I\rangle$, in which both Θ_1 and Θ_2' are valid. Recall that we also assume that L_k^n is a subset of L[14] and so the following are applications of R_1 and R_2 respectively:

$$\frac{\{\Sigma_j^1 \Rightarrow \Pi_j^1\}_{1\leq j\leq m}}{\Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))} \qquad \frac{\{\Sigma_j^2 \Rightarrow \Pi_j^2\}_{1\leq j\leq m}}{\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v})) \Rightarrow}$$

Let S be any extension of $\mathcal{N}$ to L and v - any $\mathcal{M}$-legal S-valuation. It is easy to see that the premises of the applications above are $\mathcal{M}$-valid in $\langle S, v\rangle$ (since the premises contain atomic formulas). Since G is strongly sound for $\mathcal{M}$, both $\Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))$ and $\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v})) \Rightarrow$ should also be $\mathcal{M}$-valid in $\langle S, v\rangle$, which is of course impossible.

Next, we prove that (1) $\Rightarrow$ (3). Let G be a coherent full canonical calculus. Let $\mathcal{S}$ be a set of sequents, and let $\Gamma \Rightarrow \Delta$ be a sequent, such that $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ satisfies the free-variable condition. Suppose that $\mathcal{S} \vdash_G \Gamma \Rightarrow \Delta$. We have already shown above that $\mathcal{M}_G$ is a strongly characteristic 2GNmatrix for G. Thus $\mathcal{S} \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$. Now we need the following proposition, the proof of which is given in Appendix A:

Proposition 3.6 *Let G be a coherent full canonical calculus. Let $\mathcal{S}$ be a set of sequents and $\Gamma \Rightarrow \Delta$ - a sequent such that $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ satisfies the free-variable condition. If $\Gamma \Rightarrow \Delta$ has no $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G, then $\mathcal{S} \not\vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$.*

By this proposition, $\Gamma \Rightarrow \Delta$ has an $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G. Thus G admits strong cut-elimination.

Finally, we prove that (3) $\Rightarrow$ (1). Suppose that G admits strong cut-elimination. Suppose by contradiction that G is not coherent. Then there are two canonical rules of G of the forms: $\Theta_1/ \Rightarrow A$ and $\Theta_2/A \Rightarrow$, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent. Let $\Theta = \mathsf{Rnm}(\Theta_1 \cup \Theta_2)$. Then $\Theta \cup \{\Rightarrow\}$ satisfy the free-variable condition, since only atomic formulas are involved and no variables are bound there. Since $\Theta \vdash_G \Rightarrow A$ and $\Theta \vdash_G A \Rightarrow$, by using cut we get: $\Theta \vdash_G \Rightarrow$. But $\Rightarrow$ has no Θ-cut-free proof in G from Θ since Θ is consistent, in contradiction to the fact that G admits strong cut-elimination.

Corollary 3.7 *For every full canonical calculus, the question whether it admits strong cut-elimination is decidable.*

Remark: The results presented above are related to [8], where a general class of sequent calculi with (n, k)-ary quantifiers and a (not necessarily standard) set of structural rules, are defined. Canonical calculi are a particular instance of such calculi which includes all of the standard structural rules. While handling a a wider class of calculi than canonical systems (different combinations of structural rules are allowed), [8] provides no semantics for them. Syntactic conditions are given, which are sufficient and under certain additional limitations also necessary for modular cut-elimination, a particular version of cut-elimination for deduction

[14] This assumption is again not essential for the proof, but it simplifies the presentation.

with non-logical assumptions containing only atomic formulas. In the context of canonical systems, these conditions can be shown to be equivalent to our coherence criterion, but (unlike coherence) they are not constructive.

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A Appendix: proof of Proposition 3.6

Let $\Gamma \Rightarrow \Delta$ be a sequent which satisfies the free-variable condition. Suppose that $\Gamma \Rightarrow \Delta$ has no $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G. To show that $\mathcal{S} \not\vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$, we will construct an L-structure S and an $\mathcal{M}_G$-legal valuation v, such that $\mathcal{S}$ is $\mathcal{M}_G$-valid in $\langle S, v\rangle$, while $\Gamma \Rightarrow \Delta$ is not.
It is easy to see that we can limit ourselves to the language L^*, which is a subset of L, consisting of all the constants and predicate and function symbols, occurring in $\Gamma \Rightarrow \Delta$. Let $\mathbf{T}$ be the set of all the terms in L^* which do not contain variables occurring bound in $\Gamma \Rightarrow \Delta$. It is a standard matter to show that Γ, Δ can be extended to two (possibly infinite) sets Γ', Δ' (where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$), satisfying the following properties:

1. For every finite $\Gamma_1 \subseteq \Gamma'$ and $\Delta_1 \subseteq \Delta'$, $\Gamma_1 \Rightarrow \Delta_1$ has no cut-free proof in G.
2. There are no $\psi \in \Gamma'$ and $\varphi \in \Delta'$, such that $\psi \equiv_\alpha \varphi$.
3. If $\{\Pi_j \Rightarrow \Sigma_j\}_{1\le j\le m}/\mathcal{Q}(r)$ is an (n,k)-ary rule of G and $\mathcal{Q}z_1...z_k\ (A_1,...,A_n) \in ite(r, \Delta', \Gamma')$ (recall Notation 2.6), then there is some $1 \le j \le m$ satisfying the following condition. Let $\mathbf{t}_1, ..., \mathbf{t}_m$ be all the L^n_k-terms occurring in $\Pi_j \cup \Sigma_j$, where $\mathbf{t}_{j_1}, ..., \mathbf{t}_{j_l}$ are all the constants and $\mathbf{t}_{j_{l+1}}, ..., \mathbf{t}_{j_m}$ are all the variables. Then for every $\mathbf{s}_1, ..., \mathbf{s}_l \in \mathbf{T}$ there are some[15] $\mathbf{s}_{l+1}, ..., \mathbf{s}_m \in \mathbf{T}$, such that whenever $p_i(\mathbf{t}_{n_1}, ..., \mathbf{t}_{n_k}) \in ite(r, \Pi_j, \Sigma_j)$, then $A_i\{\mathbf{s}_{n_1}/z_1, ..., \mathbf{s}_{n_k}/z_k\} \in ite(r, \Gamma', \Delta')$.
4. For every formula ψ occurring in $\mathcal{S}$ and every substitution instance ψ' of ψ: $\psi' \in \Gamma' \cup \Delta'$.

Note that the last condition can be satisfied because cuts on substitution instances of formulas from $\mathcal{S}$ are allowed in an $\mathcal{S}$-cut-free proof.
Let $S = \langle D, I\rangle$ be the L^*-structure defined as follows: $D = \mathbf{T}$, $I[c] = c$ for every constant c of L^*; $I[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$ for every n-ary function symbol f; $I[p][\mathbf{t}_1, ..., \mathbf{t}_n] = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$ for every n-ary predicate symbol p.
It is easy to show by induction on $\mathbf{t}$ that: **(lem1)** for every $\mathbf{t} \in \mathbf{T}$: $I[\sigma^*[\mathbf{t}]] = \mathbf{t}$.
Let σ^* be any S-substitution satisfying $\sigma^*[x] = \overline{x}$ for every $x \in \mathbf{T}$. (Note that every $x \in \mathbf{T}$ is also a member of the domain and thus has an individual constant referring to it in $L^*(D)$).
For an $L(D)$-formula ψ (an $L(D)$-term $\mathbf{t}$), we will denote by $\widehat{\psi}$ ($\widehat{\mathbf{t}}$) the L-formula (L-term) obtained from ψ ($\mathbf{t}$) by replacing every individual constant of the form $\overline{\mathbf{s}}$ for some $\mathbf{s} \in \mathbf{T}$ by the term $\mathbf{s}$.
Define the S-valuation v as follows: (i) $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$, (ii) If there is some $C \in \Gamma' \cup \Delta'$, s.t. $C \equiv_\alpha \widehat{\mathcal{Q}\overrightarrow{z}(\psi_1, ..., \psi_n)}$, then $v[\mathcal{Q}\overrightarrow{z}(\psi_1, ..., \psi_n)] = t$ iff $C \in \Gamma'$. Otherwise $v[\mathcal{Q}\overrightarrow{z}(\psi_1, ..., \psi_n)] = t$ iff
$\tilde{\mathcal{Q}}_S[\lambda a_1...a_k \in D.\{\langle v[\psi_1\{\overrightarrow{a}/\overrightarrow{z}\}], ..., v[\psi_n\{\overrightarrow{a}/\overrightarrow{z}\}]\rangle\}] = \{t\}$.
The proof of the following lemmas is not hard and is left to the reader:

(lem2): For every $L(D)$-formula ψ: $\psi \sim^S \sigma^*[\widehat{\psi}]$.
(lem3) For every $\psi \in \Gamma' \cup \Delta'$: $\widehat{\sigma^*[\psi]} = \psi$.

[15] Note that in contrast to $\mathbf{t}_1, ..., \mathbf{t}_m$, $\mathbf{s}_1, ..., \mathbf{s}_m$ are L-terms and not L^n_k-terms.

(lem4) v is legal in $\mathcal{M}_G$.

Next we prove: **(lem5)** For every $\psi \in \Gamma' \cup \Delta'$: $v[\sigma^*[\psi]] = t$ iff $\psi \in \Gamma'$. If $\psi = p(\mathbf{t}_1, ..., \mathbf{t}_n)$, then $v[\sigma^*[\psi]] = I[p][I[\sigma^*[\mathbf{t}_1]], ..., I[\sigma^*[\mathbf{t}_n]]]$. Note[16] that for every $1 \leq i \leq n$, $\mathbf{t}_i \in \mathbf{T}$. By **(lem1)**, $I[\sigma^*[\mathbf{t}_i]] = \mathbf{t}_i$, and by the definition of I, $v[\sigma^*[\psi]] = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$. Otherwise $\psi = Q\overrightarrow{z}(\psi_1, ..., \psi_n)$. If $\psi \in \Gamma'$, then by **(lem3)**: $\widehat{\sigma^*[\psi]} = \psi \in \Gamma'$ and so $v[\sigma^*[\psi]] = t$. If $\psi \in \Delta'$ then by property 2 of $\Gamma' \cup \Delta'$ it cannot be the case that there is some $C \in \Gamma'$, such that $C \equiv_\alpha \widehat{\sigma^*[\psi]} = \psi$ and so $v[\sigma^*[\psi]] = f$.
Finally, we prove that for every sequent $\Sigma \Rightarrow \Pi \in \mathcal{S}$, $\Sigma \Rightarrow \Pi$ is $\mathcal{M}_G$-valid in $\langle S, v \rangle$. Suppose by contradiction that there is some $\Sigma \Rightarrow \Pi \in \mathcal{S}$, which is not $\mathcal{M}_G$-valid in $\langle S, v \rangle$. Then there exists some S-substitution μ, such that for every $\psi \in \Sigma$: $S, v \models_{\mathcal{M}_G} \mu[\psi]$, and for every $\varphi \in \Pi$: $S, v \not\models_{\mathcal{M}_G} \mu[\varphi]$. Note that for every $\phi \in \Sigma \cup \Pi$, $\widehat{\mu[\phi]}$ is a substitution instance of ϕ. By property 5 of $\Gamma' \cup \Delta'$: $\widehat{\mu[\phi]} \in \Gamma' \cup \Delta'$. By **(lem5)**, if $\widehat{\mu[\phi]} \in \Gamma'$ then $v[\sigma^*[\widehat{\mu[\phi]}]] = t$, and if $\widehat{\mu[\phi]} \in \Delta'$ then $v[\sigma^*[\widehat{\mu[\phi]}]] = f$. By **(lem2)**, $\mu[\phi] \sim^S \sigma^*[\widehat{\mu[\phi]}]$. Since v is $\mathcal{M}_G$-legal, it respects the $\sim^S$-relation and so for every $\phi \in \Sigma \cup \Pi$: $v[\mu[\phi]] = v[\sigma^*[\widehat{\mu[\phi]}]]$. Thus $\widehat{\mu[\Sigma]} \subseteq \Gamma'$ and $\widehat{\mu[\Pi]} \subseteq \Delta'$. But $\widehat{\mu[\Sigma]} \Rightarrow \widehat{\mu[\Pi]}$ has an $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G (note that $\widehat{\mu[\Sigma]} \Rightarrow \widehat{\mu[\Pi]}$ is obtained from $\Sigma \Rightarrow \Pi$ by applying the substitution rule), in contradiction to property 1 of $\Gamma' \cup \Delta'$.
Thus, all sequents of $\mathcal{S}$ are $\mathcal{M}_G$-valid in $\langle S, v \rangle$. However, by **(lem5)**: $\Gamma \Rightarrow \Delta$ is not $\mathcal{M}_G$-valid in $\langle S, v \rangle$ (recall that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$). Thus $\mathcal{S} \not\vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$.

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[16] This is obvious if $\mathbf{t}_i$ does not occur in $\Gamma \Rightarrow \Delta$. If it occurs in $\Gamma \Rightarrow \Delta$, then since $\Gamma \Rightarrow \Delta$ satisfies the free-variable condition, $\mathbf{t}_i$ does not contain variables bound in this set and so $\mathbf{t}_i \in \mathbf{T}$ by definition of $\mathbf{T}$.