Reasoning with Uncertainty by Nmatrix–Metric Semantics

Ofer Arieli[1] and Anna Zamansky[2*]

[1] Department of Computer Science, The Academic College of Tel-Aviv, Israel.
oarieli@mta.ac.il
[2] Department of Computer Science, Tel-Aviv University, Israel.
annaz@post.tau.ac.il

Abstract. Non-deterministic matrices, a natural generalization of many-valued matrices, are semantic structures in which the value assigned to a complex formula may be chosen non-deterministically from a given set of options. We show that by combining Nmatrices and preferential metrics-based considerations, one obtains a family of logics that are useful for reasoning with uncertainty. We investigate the basic properties of these logics and demonstrate their usefulness in handling incomplete and inconsistent information.

1 Introduction

One of the main challenges of commonsense reasoning is dealing with phenomena that are inherently non-deterministic. The causes of non-determinism may vary: partially unknown information, faulty behavior of devices and ambiguity of natural languages are just a few cases in point. It is clear that truth-functional semantics, in which the truth-value of a complex formula is completely determined by the truth-values of its subformulas, cannot capture non-deterministic behaviour, the very essence of which is, in some sense, contradictory to the principle of truth-functionality. One possible solution is to borrow the idea of non-deterministic computations from automata and computability theory and apply it to evaluations of formulas. This idea led to introducing *non-deterministic matrices* (Nmatrices) in [8]. These structures are a natural generalization of standard multi-valued matrices [13, 25], in which the truth-value of a complex formula can be chosen *non-deterministically* out of some non-empty set of options. The use of Nmatrices preserves many attractive properties of logics with ordinary finite-valued logics, such as decidability and compactness. Moreover, as in many-valued logics, the consequence relations induced by Nmatrices are monotonic (i.e., the set of conclusions monotonically grow in the size of the premises), and are trivialized in the presence of inconsistency (i.e., any inconsistent set of premises entails every formula). In real life, however, both of these properties are not always desirable as, e.g., it is often the case that new information requires a retraction of old assertions. To cope with this, Shoham [22] introduced the notion

* Supported by the Israel Science Foundation, grant No. 809–06.

of *preferential semantics* (see also [20]), according to which an order relation, reflecting some condition or preference criteria, is defined on a set of valuations, and only the valuations that are minimal with respect to this order are relevant for making inferences from a given theory. Following this idea, we use metric-like considerations as our primary preference criteria. Such a distance minimization consideration is a cornerstone behind many paradigms of handling incomplete or inconsistent information, such as belief revision [9, 14, 18, 23] database integration systems [1, 5, 10, 19], and formalisms for commonsense reasoning in the context of social choice theory [16, 21]. In [2, 3, 7] this approach is described in terms of entailment relations, based on a standard truth-functional semantics. As argued above, this cannot capture non-deterministic behavior, so instead, in this paper, we use logics based on Nmatrices as the underlying formalism for a preferential metrics-based approach. We also consider some of the properties of the entailment relations that are obtained, demonstrate their applicability for reasoning under uncertainty by some case studies, and show the relation between reasoning in these cases and some well-known SAT problems.

2 Distance-Based Non-Deterministic Semantics

2.1 Non-Deterministic Matrices

In what follows, $\mathcal{L}$ denotes a propositional language with a set Atoms of atomic formulas. A theory Γ is a *finite multiset* of $\mathcal{L}$-formulas, for which $\mathsf{Atoms}(\Gamma)$ and $\mathsf{SF}(\Gamma)$ denote, respectively, the atomic formulas of Γ and the subformulas of Γ. Below, we shortly reproduce the main definitions from [8].

Definition 1. A *non-deterministic matrix* (henceforth, *Nmatrix*) for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} - \{\emptyset\}$.

Definition 2. An $\mathcal{M}$-*valuation* is a function $\nu : \mathcal{L} \to \mathcal{V}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1 \ldots \psi_n \in \mathcal{L}$:

$$\nu(\diamond(\psi_1 \ldots \psi_n)) \in \widetilde{\diamond}(\nu(\psi_1) \ldots \nu(\psi_n)).$$

We denote by $\Lambda_{\mathcal{M}}$ the space of all the $\mathcal{M}$-valuations.

It is important to stress that in Nmatrices the truth-values assigned to $\psi_1, \ldots, \psi_n$ do not uniquely determine the truth-value assigned to $\diamond(\psi_1, \ldots, \psi_n)$, as ν makes a non-deterministic choice out of the set of options $\tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$. Thus, the non-deterministic semantics is non-truth-functional, as opposed to standard many-valued logics.

Example 1. Let $\mathcal{M} = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$, where $\mathcal{O}$ contains the following operators:

	$\neg$
t	$\{f\}$
f	$\{t\}$

$\to$	t	f
t	$\{t\}$	$\{f\}$
f	$\{t\}$	$\{t\}$

$\leftrightarrow$	t	f
t	$\{t\}$	$\{f\}$
f	$\{f\}$	$\{t\}$

$\vee$	t	f
t	$\{t\}$	$\{t\}$
f	$\{t\}$	$\{f\}$

$\curlywedge$	t	f
t	$\{t, f\}$	$\{f\}$
f	$\{f\}$	$\{f\}$

Let $p, q \in \mathsf{Atoms}$ and $\nu_1, \nu_2 \in \Lambda_{\mathcal{M}}$, such that $\nu_1(p) = \nu_2(p) = \nu_1(q) = \nu_2(q) = t$, $\nu_1(p \curlywedge q) = t$ and $\nu_2(p \curlywedge q) = f$. While ν_1 and ν_2 coincide on, e.g., $p \vee q$, and on the proper subformulas of $p \curlywedge q$, they make different non-deterministic choices for $p \curlywedge q$.

Definition 3. A valuation $\nu \in \Lambda_{\mathcal{M}}$ is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ if $\nu(\psi) \in \mathcal{D}$. ν is a *model* in $\mathcal{M}$ of a set Γ of formulas if it satisfies every formula in Γ. A formula ψ is $\mathcal{M}$-*satisfiable* if it is satisfied by a valuation in $\Lambda_{\mathcal{M}}$. ψ is an $\mathcal{M}$-*tautology* if it is satisfied by every valuation in $\Lambda_{\mathcal{M}}$.

Definition 4. For an Nmatrix $\mathcal{M}$, a formula ψ, and a theory Γ in $\mathcal{L}$, denote: $mod_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$ and $mod_{\mathcal{M}}(\Gamma) = \cap_{\psi \in \Gamma}\, mod_{\mathcal{M}}(\psi)$.

Definition 5. The consequence relation that is induced by an Nmatrix $\mathcal{M}$ is defined by: $\Gamma \models_{\mathcal{M}} \psi$ if $mod_{\mathcal{M}}(\Gamma) \subseteq mod_{\mathcal{M}}(\psi)$.

In this paper we concentrate on two-valued Nmatrices with $\mathcal{V} = \{t, f\}$ and $\mathcal{D} = \{t\}$, and denote by $\mathcal{M}$ such an Nmatrix.

2.2 Preferential Distance-Based Entailments

Next, we augment non-deterministic semantics with preferential considerations. The idea is simple: given a distance function d on a space of valuations, reasoning with a set of premises Γ is based on those valuations that are 'd-closest' to Γ (called the *most plausible valuations* of Γ). For instance, under the standard interpretation of negation, it is intuitively clear that valuations in which q is true should be closer to $\Gamma = \{p, \neg p, q\}$ than valuations in which q is false, and so q should follow from Γ while $\neg q$ should *not* follow from Γ, although Γ is not consistent. The formal details are given in [2, 3] and are adapted to the non-deterministic case in what follows.

Definition 6. A *pseudo-distance* on a set U is a total function $d : U \times U \to \mathbb{R}^+$, satisfying the following conditions:

- *symmetry:* for all $\nu, \mu \in U$ $d(\nu, \mu) = d(\mu, \nu)$,
- *identity preservation:* for all $\nu, \mu \in U$ $d(\nu, \mu) = 0$ iff $\nu = \mu$.

A pseudo-distance d is a *distance* (*metric*) on U if it has the following property:

- *triangular inequality:* for all $\nu, \mu, \sigma \in U$ $d(\nu, \sigma) \leq d(\nu, \mu) + d(\mu, \sigma)$.

Example 2. For every $\mathcal{M}$, the following two functions are distances on $\Lambda_{\mathcal{M}}$.

- *The drastic distance:* $d_U(\nu, \mu) = 0$ if $\nu = \mu$ and $d_U(\nu, \mu) = 1$ otherwise.
- *The Hamming distance:* $d_H(\nu, \mu) = |\{p \in \mathsf{Atoms} \mid \nu(p) \neq \mu(p)\}|$. [3] [4]

[3] Here, the set Atoms of the atomic formulas in the language is assumed to be *finite*.

[4] The drastic distance is also known as the discrete metric, and Hamming distance is sometimes called Dalal distance [11], or the symmetric difference. For other representations of distances between propositional valuations see, e.g., [16].

The non-deterministic character of our framework induces some further restrictions on the distances that we shall use. This is so, since two valuations for an Nmatrix can agree on all the atoms of a formula, but still assign two different values to that formula, thus for computing distances between valuations it is not enough to consider only atomic formulas.[5] It follows that even under the assumption that the set of atoms is finite, there are infinitely many complex formulas to consider. To handle this, the distance computations in the sequel are *context dependent*, that is: restricted to a certain set of relevant formulas.

Definition 7. A *context* C is a finite set of $\mathcal{L}$-formulas closed under subformulas. The *restriction to* C of a valuation $\nu \in \Lambda_{\mathcal{M}}$ is a valuation $\nu^{\downarrow \mathsf{C}}$ on C, such that $\nu^{\downarrow \mathsf{C}}(\psi) = \nu(\psi)$ for every ψ in C. The restriction to C of $\Lambda_{\mathcal{M}}$ is the set $\Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}} = \{\nu^{\downarrow \mathsf{C}} \mid \nu \in \Lambda_{\mathcal{M}}\}$, that is, $\Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}}$ consists of all the $\mathcal{M}$-valuations on C.

Example 3. Consider the following functions on $\Lambda_{\mathcal{M}}^{\downarrow \mathsf{SF}(\Gamma)} \times \Lambda_{\mathcal{M}}^{\downarrow \mathsf{SF}(\Gamma)}$:

- $d_U^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \mu) = \begin{cases} 0 & \text{if } \nu(\psi) = \mu(\psi) \text{ for every } \psi \in \mathsf{SF}(\Gamma), \\ 1 & \text{otherwise.} \end{cases}$
- $d_H^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \mu) = |\{\psi \in \mathsf{SF}(\Gamma) \mid \nu(\psi) \neq \mu(\psi)\}|$.

Proposition 1. *$d_U^{\downarrow \mathsf{SF}(\Gamma)}$ and $d_H^{\downarrow \mathsf{SF}(\Gamma)}$ are distance functions on $\Lambda_{\mathcal{M}}^{\downarrow \mathsf{SF}(\Gamma)}$.* [6]

Definition 8. Let d be a function on $\cup_{\mathcal{M}} = \bigcup_{\{\mathsf{C} = \mathsf{SF}(\Gamma) \mid \Gamma \in 2^{\mathcal{L}}\}} \Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}} \times \Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}}$

- The *restriction* of d to a context C is a function $d^{\downarrow \mathsf{C}}$ on $\Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}} \times \Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}}$, defined for every $\nu, \mu \in \Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}}$ by $d^{\downarrow \mathsf{C}}(\nu, \mu) = d(\nu, \mu)$.
- d is a *generic (pseudo) distance on* $\Lambda_{\mathcal{M}}$, if for every context C, $d^{\downarrow \mathsf{C}}$ is a (pseudo) distance on $\Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}}$.

Example 4. Given an Nmatrix $\mathcal{M}$ for $\mathcal{L}$, define the functions d_U and d_H on $\cup_{\mathcal{M}}$ as follows: for every context C and every $\nu, \nu \in \Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}}$,

- $d_U(\nu, \mu) = \begin{cases} 0 & \text{if } \nu = \mu, \\ 1 & \text{otherwise.} \end{cases}$
- $d_H(\nu, \mu) = |\{\psi \in \mathsf{C} \mid \nu(\psi) \neq \mu(\psi)\}|$.

The restrictions of the two functions to a context $\mathsf{C} = \mathsf{SF}(\Gamma)$ are given in Example 3. By Proposition 1, then, both of these functions are generic distances on $\Lambda_{\mathcal{M}}$ for every Nmatrix $\mathcal{M}$.

[5] Thus, e.g., the Hamming distance defined in the last example should be adjusted to the non-deterministic case, so that differences in the truth assignment of complex formulas will be taken into consideration as well.

[6] This proposition is easily verifiable. Proofs of some other propositions in this paper appear in the appendix.

Note 1. Denote by $\mathcal{M}_c$ the Nmatrix for the language $\{\neg, \wedge, \vee, \rightarrow\}$ with the classical interpretations of the connectives (i.e., $\mathcal{M}_c$ is similar to the classical deterministic matrix, except that its valuation functions return singletons of truth-values instead of truth-values). Under the assumption that the set of atoms is finite, the distance functions in Example 2 can be represented in the non-deterministic case as metrics on $\Lambda_{\mathcal{M}_c}^{\downarrow \mathsf{Atoms}}$; In the notations of Example 4, they are generic distances on $\Lambda_{\mathcal{M}_c}$, denoted by $d_U^{\downarrow \mathsf{Atoms}}$ and $d_H^{\downarrow \mathsf{Atoms}}$.

Definition 9. A *numeric aggregation function* is total function f whose argument is a multiset of real numbers and whose values are real numbers, such that: (i) f is non-decreasing in the value of its argument,[7] (ii) $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = x_2 = \ldots x_n = 0$, and (iii) $f(\{x\}) = x$ for every $x \in \mathbb{R}$.

Definition 10. A (distance-based, nondeterministic) *setting* for a language $\mathcal{L}$, is a triple $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$, where $\mathcal{M}$ is a non-deterministic matrix for $\mathcal{L}$, d is a generic distance on $\Lambda_{\mathcal{M}}$, and f is an aggregation function.

Definition 11. Given a setting $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$ for a language $\mathcal{L}$, a valuation $\nu \in \Lambda_{\mathcal{M}}$, and a set $\Gamma = \{\psi_1, \ldots, \psi_n\}$ of formulas in $\mathcal{L}$, define:

– $d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi_i) =$
$$\begin{cases} \min\{d^{\downarrow \mathsf{SF}(\Gamma)}(\nu^{\downarrow \mathsf{SF}(\Gamma)}, \mu^{\downarrow \mathsf{SF}(\Gamma)}) \mid \mu \in mod_{\mathcal{M}}(\psi_i)\} & \text{if } mod_{\mathcal{M}}(\psi_i) \neq \emptyset, \\ 1 + \max\{d^{\downarrow \mathsf{SF}(\Gamma)}(\mu_1^{\downarrow \mathsf{SF}(\Gamma)}, \mu_2^{\downarrow \mathsf{SF}(\Gamma)}) \mid \mu_1, \mu_2 \in \Lambda_{\mathcal{M}}\} & \text{otherwise.} \end{cases}$$

– $\delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \Gamma) = f(\{d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi_1), \ldots, d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi_n)\})$.

Note 2. In every setting $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$, the following properties hold:

1. In the two extreme degenerate cases, when ψ is either a tautology or a contradiction w.r.t. $\mathcal{M}$, all the valuations are equally distant from ψ. Otherwise, the valuations that are closest to ψ are its models and their distance to ψ is zero. This also implies that $\delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \Gamma) = 0$ iff $\nu \in mod_{\mathcal{M}}(\Gamma)$ (see also [3]).
2. A natural property of distances between valuations and formulas is that they are not affected (biased) by irrelevant formulas (those that are not part of the relevant context):

   Proposition 2 (unbiasedness). *For every $\nu_1, \nu_2 \in \Lambda_{\mathcal{M}}$, $\mathsf{C} = \mathsf{SF}(\Gamma)$, and $\psi \in \Gamma$, if $\nu_1^{\downarrow \mathsf{C}} = \nu_2^{\downarrow \mathsf{C}}$ then $d^{\downarrow \mathsf{C}}(\nu_1, \psi) = d^{\downarrow \mathsf{C}}(\nu_2, \psi)$ and $\delta_{d,f}^{\downarrow \mathsf{C}}(\nu_1, \Gamma) = \delta_{d,f}^{\downarrow \mathsf{C}}(\nu_2, \Gamma)$.*

Now we define entailment relations based on distance minimization.

Definition 12. Given a setting $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$, the *most plausible valuations* of a theory Γ are defined as follows:

$$\Delta_{\mathcal{S}}(\Gamma) = \begin{cases} \{\nu \in \Lambda_{\mathcal{M}} \mid \forall \mu \in \Lambda_{\mathcal{M}} \ \delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \Gamma) \leq \delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \Gamma)\} & \text{if } \Gamma \neq \emptyset, \\ \Lambda_{\mathcal{M}} & \text{otherwise.} \end{cases}$$

Definition 13. Let $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$. Define: $\Gamma \models_{\mathcal{S}} \psi$ if $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod_{\mathcal{M}}(\psi)$.

[7] That is, the function value is non-decreasing when an element in the multiset is replaced by a larger element.

2.3 Examples of Reasoning with $\models_{\mathcal{S}}$

Notation. Given a theory Γ with $\mathsf{SF}(\Gamma) = \{\psi_1, \psi_2, \ldots, \psi_n\}$, a valuation $\nu \in \Lambda_{\mathcal{M}}^{\downarrow \mathsf{SF}(\Gamma)}$ is represented by $\{\psi_1 : \nu(\psi_1), \psi_2 : \nu(\psi_2), \ldots, \psi_n : \nu(\psi_n)\}$.

Example 5. Let $\mathcal{S} = \langle \mathcal{M}, d_U, \Sigma \rangle$, where $\mathcal{M}$ is the Nmatrix considered in Example 1. Let $\Gamma = \{p, \neg p, q, \neg(p \curlywedge q)\}$. Then:

$$\Delta_{\mathcal{S}}(\Gamma) = \left\{ \begin{array}{l} \{p\!:\!t,\ \neg p\!:\!f,\ q\!:\!t,\ p \curlywedge q\!:\!f,\ \neg(p \curlywedge q)\!:\!t\}, \\ \{p\!:\!f,\ \neg p\!:\!t,\ q\!:\!t,\ p \curlywedge q\!:\!f,\ \neg(p \curlywedge q)\!:\!t\} \end{array} \right\}.$$

Thus, $\Gamma \models_{\mathcal{S}} q$ and $\Gamma \models_{\mathcal{S}} \neg(p \curlywedge q)$, while $\Gamma \not\models_{\mathcal{S}} p$ and $\Gamma \not\models_{\mathcal{S}} \neg p$.

Example 6. A reasoner wants to learn as much as possible about a (black-box) circuit, the structure of which is assumed to be the following:

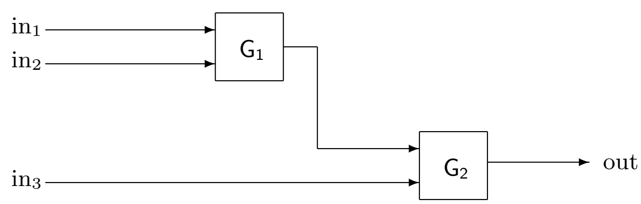

Fig. 1.

Here, G_1 and G_2 are two AND gates that are faulty or behave unpredictably when both of their input lines are 'on'.[8] After experimenting with the circuit, the reasoner concludes that if one of the input lines is 'on' then so is the output line. This situation may be represented by Nmatrix $\mathcal{M}$ of Example 1 as follows:

$$\Gamma = \left\{\, (in_1 \vee in_2 \vee in_3) \rightarrow out \,\right\},$$

where *out* denotes the formula $((in_1 \curlywedge in_2) \curlywedge in_3)$. Here, $\Lambda_{\mathcal{M}}^{\downarrow \mathsf{SF}(\Gamma)}$ has 11 elements (see the appendix), two of them are models of Γ. Thus, by Lemma 1 below, for *every* setting $\mathcal{S}$,

$$\Delta_{\mathcal{S}}(\Gamma) = mod_{\mathcal{M}}(\Gamma) = \left\{ \begin{array}{l} \left\{in_1\!:\!t,\ in_2\!:\!t,\ in_3\!:\!t,\ in_1 \curlywedge in_2\!:\!t,\ out\!:\!t\right\}, \\ \left\{in_1\!:\!f,\ in_2\!:\!f,\ in_3\!:\!f,\ in_1 \curlywedge in_2\!:\!f,\ out\!:\!f\right\} \end{array} \right\},$$

so the reasoner may conclude that when all the input lines have the same value, the output line of the circuit preserves this value.

Suppose now that the reasoner learns that the value of the output line is always different than the value of G_1. The new situation can be represented by

$$\Gamma' = \Gamma \cup \left\{(in_1 \curlywedge in_2) \leftrightarrow \neg out\right\}.$$

[8] This may happen due to noises on or off chip, variations in the manufacturing process, adversary operations, etc.

It is easy to verify that Γ' is not $\mathcal{M}$-satisfiable anymore, i.e. the new information is inconsistent with the reasoner's previous knowledge. In such cases the usual $\models_{\mathcal{M}}$ entailment is trivialized: everything can be inferred from Γ'. This, however, is not the case for $\models_{\mathcal{S}}$. For instance, when $\mathcal{S} = \langle \mathcal{M}, d_U, \Sigma \rangle$, we have that

$$\Delta_{\mathcal{S}}(\Gamma') = \left\{ \begin{array}{l} \{in_1\!:\!t,\ in_2\!:\!t,\ in_3\!:\!t,\ in_1 \curlywedge in_2\!:\!t,\ out\!:\!t\}, \\ \{in_1\!:\!t,\ in_2\!:\!t,\ in_3\!:\!t,\ in_1 \curlywedge in_2\!:\!t,\ out\!:\!f\}, \\ \{in_1\!:\!t,\ in_2\!:\!t,\ in_3\!:\!f,\ in_1 \curlywedge in_2\!:\!t,\ out\!:\!f\}, \\ \{in_1\!:\!f,\ in_2\!:\!f,\ in_3\!:\!f,\ in_1 \curlywedge in_2\!:\!f,\ out\!:\!f\} \end{array} \right\}.$$

Using $\models_{\mathcal{S}}$, the reasoner may still conclude from Γ' that if the value of all the input lines is 'off', this is also the value of the output line. This shows that $\models_{\mathcal{S}}$ is inconsistency-tolerant (see Proposition 4 below). On the other hand, a stronger assertion, that when the values of all input lines coincide the value of the output line is the same, is no longer a valid consequence of Γ'. This shows that $\models_{\mathcal{S}}$ is non-monotonic (see Proposition 6 below).

3 General Properties of $\models_{\mathcal{S}}$

In this section, we consider some basic properties of the entailments that are induced by a setting $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$. First, we consider the relation between basic and distance-based entailments.

Proposition 3. [6] *For every setting $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$, if $\Gamma \models_{\mathcal{S}} \psi$ then $\Gamma \models_{\mathcal{M}} \psi$. Moreover, if Γ is $\mathcal{M}$-satisfiable, then $\Gamma \models_{\mathcal{S}} \psi$ iff $\Gamma \models_{\mathcal{M}} \psi$.*

Proposition 3 follows from the fact that if Γ is not $\mathcal{M}$-satisfiable then $\Gamma \models_{\mathcal{M}} \psi$ for *every* ψ, and from the following lemma:

Lemma 1. [6] *Γ is $\mathcal{M}$-satisfiable iff $\Delta_{\mathcal{S}}(\Gamma) = mod_{\mathcal{M}}(\Gamma)$.*

Thus, $\models_{\mathcal{S}}$ coincides with $\models_{\mathcal{M}}$ w.r.t. $\mathcal{M}$-consistent premises. In contrast to $\models_{\mathcal{M}}$, however, $\models_{\mathcal{S}}$ tolerates inconsistent information in a non-trivial way, thus, as Proposition 4 shows, $\models_{\mathcal{S}}$ is paraconsistent.

Definition 14. Γ_1 and Γ_2 are called *independent* if $\mathsf{Atoms}(\Gamma_1) \cap \mathsf{Atoms}(\Gamma_2) = \emptyset$.

The next proposition is an improvement of a similar proposition in [6].

Proposition 4 (paraconsistency). *For every Γ and every ψ such that Γ and $\{\psi\}$ are independent, $\Gamma \models_{\mathcal{S}} \psi$ iff ψ is an $\mathcal{M}$-tautology.*

Corollary 1 (weak paraconsistency). *For every Γ there is a ψ s.t. $\Gamma \not\models_{\mathcal{S}} \psi$.*

A related property is that $\models_{\mathcal{S}}$ preserves the consistency of its conclusions:

Definition 15. An Nmartix $\mathcal{M} = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ is *with negation*, if there is a unary function $\tilde{\neg}$ in $\mathcal{O}$ such that $\tilde{\neg}(t) = \{f\}$ and $\tilde{\neg}(f) = \{t\}$. A setting $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$ is with negation if its Nmatrix $\mathcal{M}$ is with negation.

Proposition 5. [6] *Let $\mathcal{S}$ be a setting with negation. Then for every Γ and every ψ, if $\Gamma \models_{\mathcal{S}} \psi$ then $\Gamma \not\models_{\mathcal{S}} \neg\psi$.*

We now consider to what extent the entailment relations of our framework are non-monotonic (i.e., whether conclusions may be revised in light of new information).

Proposition 6 (non-monotonicity). *Let $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$ be a setting with negation. Then $\models_{\mathcal{S}}$ is non-monotonic.*

In spite of Proposition 6, even for settings with negation, one may specify conditions under which the entailment relations have some monotonic characteristics.

Definition 16. An aggregation function f is *hereditary*, if $f(\{x_1, \ldots, x_n\}) < f(\{y_1, \ldots, y_n\})$ entails $f(\{x_1, \ldots, x_n, z_1, \ldots, z_m\}) < f(\{y_1, \ldots, y_n, z_1, \ldots, z_m\})$.

Example 7. The aggregation function Σ is hereditary, while max is not.

The following proposition shows that in light of new information that is unrelated to the premises, previously drawn conclusions should not be retracted.[9]

Proposition 7 (rational monotonicity). *Let $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$ be a setting in which f is hereditary. If $\Gamma \models_{\mathcal{S}} \psi$, then $\Gamma, \phi \models_{\mathcal{S}} \psi$ for every formula ϕ such that $\Gamma \cup \{\psi\}$ and $\{\phi\}$ are independent.*

The discussion above, on the non-monotonicity of $\models_{\mathcal{S}}$, brings us to the question to what extent these entailments can be considered as consequence relations.

Definition 17. A Tarskian *consequence relation* [24] for a language $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ and formulas of $\mathcal{L}$ that satisfies the following conditions:

Reflexivity: if $\psi \in \Gamma$, then $\Gamma \vdash \psi$.
Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \varphi$, then $\Gamma, \Gamma' \vdash \varphi$.

As follows from Example 5 and Proposition 6, entailments of the form $\models_{\mathcal{S}}$ are, in general, neither reflexive nor monotonic. It is also not difficult to verify that in general $\models_{\mathcal{S}}$ is not transitive either. In the context of non-monotonic reasoning, however, it is usual to consider the following weaker conditions that guarantee a 'proper behaviour' of nonmonotonic entailments in the presence of inconsistency (see, e.g., [4, 15, 17, 20]):

Definition 18. A *cautious* consequence relation for $\mathcal{L}$ is a relation $\mid\!\sim$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, that satisfies the following conditions:

Cautious Reflexivity: if Γ is $\mathcal{M}$-satisfiable and $\psi \in \Gamma$, then $\Gamma \mid\!\sim \psi$.
Cautious Monotonicity [12]: if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \phi$, then $\Gamma, \psi \mid\!\sim \phi$.
Cautious Transitivity [15]: if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$.

[9] This type of monotonicity is a kind of *rational monotonicity*, considered in [17].

The next result is another improvement of a similar proposition in [6].

Proposition 8. *Let $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$ be a setting where f is hereditary. Then $\models_{\mathcal{S}}$ is a cautious consequence relation.*

Regarding the computability of our entailments, we show that in most practical cases entailment checking is decidable.

Definition 19. A setting $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$ is *computable*, if f is computable, and there is an algorithm that computes $d(\mu, \nu)$ for every context C and $\mu, \nu \in \Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}}$.

Note 3. Clearly, all the distance and aggregation functions considered in this paper are computable. Yet, as the following example shows, this is not always the case. Let $\mathcal{L} = \{\wedge\}$ be a propositional language and L a first-order language with a constant c, a unary function g and a binary relation R. Consider the following one-to-one mapping Θ from L-formulas to $\mathcal{L}$-formulas: every symbol s in L is associated with an atomic formula p_s in $\mathcal{L}$, and every L-formula ψ is mapped to the $\mathcal{L}$-formula $\Theta(\psi)$, obtained by taking the conjunction of all the atomic formulas to which the symbols of ψ are mapped. For instance, the formula $\forall x_1 \forall x_2 R(x_1, x_2)$ is mapped to $p_\forall \wedge p_{x_1} \wedge p_\forall \wedge p_{x_2} \wedge p_R \wedge p_(\wedge p_{x_1} \wedge p_, \wedge p_{x_2} \wedge p_)$. A formula ψ in $\mathcal{L}$ is called *proper* if there is an L-formula ψ' s.t. $\psi = \Theta(\psi')$. Now, consider the following pseudo distance:

$$d(\nu, \mu) = \begin{cases} 0 & \text{if } \nu = \mu, \\ 1 & \text{if } \nu \neq \mu \text{ and there is a proper } \psi \text{ s.t. } \nu, \mu \in \Lambda_{\mathcal{M}}^{\downarrow \mathsf{SF}(\psi)}, \\ & \text{and } \Theta^{-1}(\psi) \text{ is satisfiable,} \\ 2 & \text{otherwise.} \end{cases}$$

Since $\mathsf{SF}(\psi) \neq \mathsf{SF}(\phi)$ whenever $\psi \neq \phi$, the pseudo distance above is well defined. Now, as the satisfiability problem for L-formulas is undecidable, d is not computable.

Proposition 9. *For every computable setting $\mathcal{S}$, the question whether $\Gamma \models_{\mathcal{S}} \psi$ is decidable.*

4 Some Particular Cases of Reasoning with $\models_{\mathcal{S}}$

In this section we focus on drastic settings, i.e., settings with a drastic distance (see Examples 2 and 4). In this context we investigate the following aggregation functions:

Definition 20. An aggregation function f is *range restricted* if $f(\{x_1, \ldots, x_n\}) \in \{x_1, \ldots, x_n\}$; f is called *additive* if for any non-empty set S it can be represented as $f(S) = g(|S|) \cdot \Sigma_{x \in S}\, x$, for some function $g : \mathbb{N}^+ \to \mathbb{R}^+$.

Example 8. The maximum function is a range-restricted but not additive, while the summation (respectively, the average) is additive where g is uniformly 1 (respectively, $g(n) = \frac{1}{n}$), but it is not range-restricted.

The next proposition should be compared with Proposition 4.

Proposition 10. *Let $\mathcal{S} = \langle \mathcal{M}, d_U, f \rangle$ be a drastic setting in which f is range restricted. Let Γ be a set of formulas that is not $\mathcal{M}$-satisfiable. Then $\Gamma \models_{\mathcal{S}} \psi$ iff ψ is an $\mathcal{M}$-tautology.*

Corollary 2. *Let $\mathcal{S}$ be a drastic setting with a range-restricted aggregation function. If $\Gamma \models_{\mathcal{S}} \psi$ then either $\Gamma \models_{\mathcal{M}} \psi$ or ψ is an $\mathcal{M}$-tautology.*

The last corollary shows that reasoning with drastic distances and range-restricted functions has a somewhat 'crude nature': either the set of premises is $\mathcal{M}$-consistent, in which case the set of conclusions coincide with that of the basic entailment, or, in case of contradictory premises, only tautologies are entailed.

The behavior of drastic settings with additive functions is completely different: entailments in this case are closely related to the maximum satisfiability problem:

Definition 21. Let $\mathsf{SAT}_{\mathcal{M}}(\Gamma)$ be the set of all the $\mathcal{M}$-satisfiable subsets of Γ. The set $\mathsf{mSAT}_{\mathcal{M}}(\Gamma)$ of the *maximally $\mathcal{M}$-satisfiable* subsets of Γ consistent of all the elements $\Upsilon \in \mathsf{SAT}_{\mathcal{M}}(\Gamma)$ such that $|\Upsilon'| \leq |\Upsilon|$ for every $\Upsilon' \in \mathsf{SAT}_{\mathcal{M}}(\Gamma)$.

Note 4. Clearly, $\mathsf{mSAT}_{\mathcal{M}}(\Gamma)$ is nonempty whenever Γ contains an $\mathcal{M}$-satisfiable element.

Proposition 11. *Let $\mathcal{S} = \langle \mathcal{M}, d_U, f \rangle$ be a drastic setting with additive f and let Γ be a finite set of formulas. Then:*

$$\Delta_{\mathcal{S}}(\Gamma) = \begin{cases} \{\nu \in mod_{\mathcal{M}}(\Upsilon) \mid \Upsilon \in \mathsf{mSAT}_{\mathcal{M}}(\Gamma)\} & \textit{if } \mathsf{mSAT}_{\mathcal{M}}(\Gamma) \neq \emptyset, \\ \Lambda_{\mathcal{M}} & \textit{otherwise.} \end{cases}$$

Corollary 3. *Let $\mathcal{S}$ be a drastic setting with additive f. If $\mathsf{mSAT}_{\mathcal{M}}(\Gamma) \neq \emptyset$ and $\Gamma' \models_{\mathcal{M}} \psi$ for every $\Gamma' \in \mathsf{mSAT}_{\mathcal{M}}(\Gamma)$, then $\Gamma \models_{\mathcal{S}} \psi$.*

Example 9. By taking $\mathcal{S} = \langle \mathcal{M}_c, d_U, \Sigma \rangle$ in the last corollary, we get that reasoning with summation of drastic distances is equivalent to checking classical entailments from the maximally consistent subsets of the premises.

Note 5. It is easy to verify that all the results in this section still hold for settings $\mathcal{S} = \langle \mathcal{M}, d, f \rangle$, where for every context $\mathsf{C} = \mathsf{SF}(\Gamma)$ there is some constant $k_{\mathsf{C}} > 0$, such that for all $\psi \in \Gamma$ and $\nu \in \Lambda_{\mathcal{M}}$,

$$d^{\downarrow \mathsf{C}}(\nu, \psi) = \begin{cases} 0 & \text{if } \nu \in mod_{\mathcal{M}}(\psi), \\ k_{\mathsf{C}} & \text{otherwise.} \end{cases}$$

Note that the drastic setting $\mathcal{S} = \langle \mathcal{M}, d_U, f \rangle$ is a particular instance of this definition in which $k_{\mathsf{C}} = 1$ for every context C.

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A Supplementary Material

Elaboration on Example 6: Below, we use the following abbreviations:

$$\begin{aligned} \mathsf{G}_1 &= (in_1 \curlywedge in_2), & \psi_1 &= (in_1 \vee in_2 \vee in_3) \to out, \\ out &= ((in_1 \curlywedge in_2) \curlywedge in_3), & \psi_2 &= (in_1 \curlywedge in_2) \leftrightarrow \neg out. \end{aligned}$$

In these notations, $\Gamma = \{\psi_1\}$ and $\Gamma' = \{\psi_1, \psi_2\}$. Distances to elements of $\Lambda_{\mathcal{M}}^{\downarrow \mathsf{SF}(\Gamma)}$ are given below, where $\delta(\cdot)$ abbreviates $\delta_{d_U,\Sigma}(\nu, \cdot)$ for the relevant valuation ν.

	in_1	in_2	in_3	G_1	out	$\delta(\psi_1)$	$\delta(\psi_2)$	$\delta(\Gamma)$	$\delta(\Gamma')$
ν_1	t	t	t	t	t	0	1	**0**	**1**
ν_2	t	t	t	t	f	1	0	1	**1**
ν_3	t	t	t	f	f	1	1	1	2
ν_4	t	t	f	t	f	1	0	1	**1**
ν_5	t	t	f	f	f	1	1	1	2
ν_6	t	f	t	f	f	1	1	1	2
ν_7	t	f	f	f	f	1	1	1	2
ν_8	f	t	t	f	f	1	1	1	2
ν_9	f	t	f	f	f	1	1	1	2
ν_{10}	f	f	t	f	f	1	1	1	2
ν_{11}	f	f	f	f	f	0	1	**0**	**1**

Thus, $\Delta_{\mathcal{S}}(\Gamma) = \{\nu_1, \nu_{11}\}$ and $\Delta_{\mathcal{S}}(\Gamma') = \{\nu_1, \nu_2, \nu_4, \nu_{11}\}$.

We turn now to the proofs of the propositions in the paper: Proposition 1 and Proposition 2 are easy. The proofs of Propositions 3, 5, and 6 appear in [6]. The proof of Proposition 4 is a variation of the proof of Proposition 39 in [6]. Below, we show the other results:

Proof of Proposition 7: Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$ and $\mu \in \Lambda_{\mathcal{M}}$, s.t. $\mu(\psi) = f$. As $\Gamma \models_{\mathcal{S}} \psi$, $\mu \notin \Delta_{\mathcal{S}}(\Gamma)$, so there is $\nu \in \Delta_{\mathcal{S}}(\Gamma)$ with $\delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \Gamma) < \delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \Gamma)$, i.e., $f(\{d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi_1), \ldots, d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi_n)\}) < f(\{d^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \psi_1), \ldots, d^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \psi_n)\})$. As $\Gamma \models_{\mathcal{S}} \psi$, it follows that $\nu(\psi) = t$. Now, as $\mathsf{Atoms}(\Gamma \cup \{\psi\}) \cap \mathsf{Atoms}(\{\phi\}) = \emptyset$, one can easily define an $\mathcal{M}$-valuation σ such that $\sigma(\varphi) = \nu(\varphi)$ for every $\varphi \in \mathsf{SF}(\Gamma \cup \{\psi\})$ and $\sigma(\varphi) = \mu(\varphi)$ for every $\varphi \in \mathsf{SF}(\{\phi\})$. By Proposition 2, and since f is hereditary, we have:

$$\begin{aligned} \delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\sigma, \Gamma \cup \{\phi\}) &= f(\{d^{\downarrow \mathsf{SF}(\Gamma)}(\sigma, \psi_1), \ldots, d^{\downarrow \mathsf{SF}(\Gamma)}(\sigma, \psi_n), d^{\downarrow \mathsf{SF}(\Gamma)}(\sigma, \phi)\}) \\ &= f(\{d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi_1), \ldots, d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi_n), d^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \phi)\}) \\ &< f(\{d^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \psi_1), \ldots, \delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \psi_n), d^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \phi)\}) \\ &= \delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \Gamma \cup \{\phi\}) \end{aligned}$$

Thus, for every $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(\psi) = f$, there is some $\sigma \in \Lambda_{\mathcal{M}}$ such that $\sigma(\psi) = t$ and $\delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\sigma, \Gamma \cup \{\phi\}) < \delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \Gamma \cup \{\phi\})$. It follows that the elements of $\Delta_{\mathcal{S}}(\Gamma \cup \{\phi\})$ must satisfy ψ, and so $\Gamma, \phi \models_{\mathcal{S}} \psi$. □

Proof of Proposition 8: Cautious reflexivity follows from Proposition 3. The proofs for cautious monotonicity and cautious transitivity are an adaptation of the ones for the deterministic case (see [3]):

For cautious monotonicity, let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ and suppose that $\Gamma \models_{\mathcal{S}} \psi$, $\Gamma \models_{\mathcal{S}} \phi$, and $\nu \in \Delta_{\mathcal{S}}(\Gamma \cup \{\psi\})$. We show that $\nu \in \Delta_{\mathcal{S}}(\Gamma)$ and since $\Gamma \models_{\mathcal{S}} \phi$ this implies that $\nu \in mod_{\mathcal{M}}(\{\phi\})$. Indeed, if $\nu \notin \Delta_{\mathcal{S}}(\Gamma)$, there is a valuation $\mu \in \Delta_{\mathcal{S}}(\Gamma)$ so that $\delta_{d,f}(\mu, \Gamma) < \delta_{d,f}(\nu, \Gamma)$, i.e., $f(\{d(\mu, \gamma_1), \dots, d(\mu, \gamma_n)\}) < f(\{d(\nu, \gamma_1), \dots, d(\nu, \gamma_n)\})$. Also, as $\Gamma \models_{\mathcal{S}} \psi$, $\mu \in mod_{\mathcal{M}}(\{\psi\})$, thus $d(\mu, \psi) = 0$. By these facts, then,

$$\begin{aligned}\delta_{d,f}(\mu, \Gamma \cup \{\psi\}) &= f(\{d(\mu, \gamma_1), \dots, d(\mu, \gamma_n), 0\}) \\ &< f(\{d(\nu, \gamma_1), \dots, d(\nu, \gamma_n), 0\}) \\ &\leq f(\{d(\nu, \gamma_1), \dots, d(\nu, \gamma_n), d(\nu, \psi)\}) = \delta_{d,f}(\nu, \Gamma \cup \{\psi\}),\end{aligned}$$

a contradiction to $\nu \in \Delta_{\mathcal{S}}(\Gamma \cup \{\psi\})$.

For cautious transitivity, let again $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ and assume that $\Gamma \models_{\mathcal{S}} \psi$, $\Gamma, \psi \models_{\mathcal{S}} \phi$, and $\nu \in \Delta_{\mathcal{S}}(\Gamma)$. We have to show that $\nu \in mod_{\mathcal{M}}(\{\phi\})$. Indeed, since $\nu \in \Delta_{\mathcal{S}}(\Gamma)$, for all $\mu \in \Lambda_{\mathcal{M}}$, $f(\{d(\nu, \gamma_1), \dots, d(\nu, \gamma_n)\}) \leq f(\{d(\mu, \gamma_1), \dots, d(\mu, \gamma_n)\})$. Moreover, since $\Gamma \models_{\mathcal{S}} \psi$, $\nu \in mod_{\mathcal{M}}(\{\psi\})$, and so $d(\nu, \psi) = 0 \leq d(\mu, \psi)$. It follows, then, that for every $\mu \in \Lambda_{\mathcal{M}}$,

$$\begin{aligned}\delta_{d,f}(\nu, \Gamma \cup \{\psi\}) &= f(\{d(\nu, \gamma_1), \dots, d(\nu, \gamma_n), d(\nu, \psi)\}) \\ &\leq f(\{d(\mu, \gamma_1), \dots, d(\mu, \gamma_n), d(\nu, \psi)\}) \\ &\leq f(\{d(\mu, \gamma_1), \dots, d(\mu, \gamma_n), d(\mu, \psi)\}) = \delta_{d,f}(\mu, \Gamma \cup \{\psi\}).\end{aligned}$$

Thus, $\nu \in \Delta_{\mathcal{S}}(\Gamma \cup \{\psi\})$, and since $\Gamma, \psi \models_{\mathcal{S}} \phi$, necessarily $\nu \in mod_{\mathcal{M}}(\{\phi\})$. □

Proof outline of Proposition 9: Suppose that $\mathcal{S}$ is a computable setting. By Definition 17, in order to check whether $\Gamma \models_{\mathcal{S}} \psi$, one has to check whether $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod_{\mathcal{M}}(\psi)$. For decidability, we show that this condition, which involves infinite sets, can be reduced to an equivalent condition in terms of finite sets. For this, we denote by $mod_{\mathcal{M}}^{\downarrow \mathsf{C}}(\psi)$ the set $\{\mu^{\downarrow \mathsf{C}} \mid \mu \in mod_{\mathcal{M}}(\psi)\}$. Next, we extend the notions of distance between a valuation and a formula and distance between a valuation and a theory to *partial valuations* as follows: for every context C such that $\mathsf{SF}(\Gamma) \subseteq \mathsf{C}$, define, for every $\nu \in \Lambda_{\mathcal{M}}^{\downarrow \mathsf{SF}(\Gamma)}$ and every $\psi \in \Gamma$,

– $d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi) =$

$$\begin{cases} \min\{d^{\downarrow \mathsf{SF}(\Gamma)}(\nu^{\downarrow \mathsf{SF}(\Gamma)}, \mu^{\downarrow \mathsf{SF}(\Gamma)}) \mid \mu \in mod_{\mathcal{M}}^{\downarrow \mathsf{C}}(\psi)\} & \text{if } mod_{\mathcal{M}}^{\downarrow \mathsf{C}}(\psi) \neq \emptyset, \\ 1 + \max\{d^{\downarrow \mathsf{SF}(\Gamma)}(\mu_1^{\downarrow \mathsf{SF}(\Gamma)}, \mu_2^{\downarrow \mathsf{SF}(\Gamma)}) \mid \mu_1, \mu_2 \in \Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}}\} & \text{otherwise.} \end{cases}$$

– $\delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \Gamma) = f(\{d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi_1), \dots, d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi_n)\})$.

Note that since all the partial valuations involved in the definitions above are defined on finite contexts, there are finitely many such valuations to check, and so $d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \psi)$ and $\delta_{d,f}^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \Gamma)$ are computable for every $\nu \in \Lambda_{\mathcal{M}}^{\downarrow \mathsf{C}}$. Now, consider the following set of partial valuations on C:

$$\Delta_{\mathcal{S}}^{\downarrow\mathsf{C}}(\Gamma) = \begin{cases} \{\nu \in \Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}} \mid \forall \mu \in \Lambda_{\mathcal{M}} \ \delta_{d,f}^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\Gamma) \leq \delta_{d,f}^{\downarrow\mathsf{SF}(\Gamma)}(\mu,\Gamma)\} & \text{if } \Gamma \neq \emptyset, \\ \Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}} & \text{otherwise.} \end{cases}$$

Clearly, $\Delta_{\mathcal{S}}^{\downarrow\mathsf{C}}(\Gamma)$ and $mod_{\mathcal{M}}^{\downarrow\mathsf{C}}(\psi)$ are computable. Decidability now follows from the fact that $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod_{\mathcal{M}}(\psi)$ if and only if $\Delta_{\mathcal{S}}^{\downarrow\mathsf{C}}(\Gamma) \subseteq mod_{\mathcal{M}}^{\downarrow\mathsf{C}}(\psi)$. □

Proof of Proposition 10: Let $\mu \in \Lambda_{\mathcal{M}}$. As $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ is not $\mathcal{M}$-satisfiable, μ is not a model of Γ, and so there is some formula $\varphi_j \in \Gamma$ such that $d_U^{\downarrow\mathsf{SF}(\Gamma)}(\mu, \varphi_j) = 1$. Moreover, for every $\varphi_i \in \Gamma$ we have that $d_U^{\downarrow\mathsf{SF}(\Gamma)}(\mu, \varphi_i) \in \{0,1\}$ and so, since f is range-restricted,

$$\delta_{d_U,f}^{\downarrow\mathsf{SF}(\Gamma)}(\mu,\Gamma) = f(\{d_U^{\downarrow\mathsf{SF}(\Gamma)}(\mu,\varphi_1), \ldots, d_U^{\downarrow\mathsf{SF}(\Gamma)}(\mu,\varphi_n)\}) = 1.$$

This shows that all the valuations in $\Lambda_{\mathcal{M}}$ are equally distant from Γ and so $\Delta_{\mathcal{S}}(\Gamma) = \Lambda_{\mathcal{M}}$. Thus, $\Gamma \models_{\mathcal{S}} \psi$ iff $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod_{\mathcal{M}}(\psi)$, iff $mod_{\mathcal{M}}(\psi) = \Lambda_{\mathcal{M}}$, iff ψ is a tautology. □

Proof of Proposition 11: Consider a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$, and assume first that $\mathsf{mSAT}_{\mathcal{M}}(\Gamma) \neq \emptyset$. Since $\mathcal{S}$ is drastic, for every $\psi \in \Gamma$ and every $\nu \in \Lambda_{\mathcal{M}}$, $d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu, \psi) = 0$ if $\nu \in mod_{\mathcal{M}}(\psi)$, and otherwise $d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu, \psi) = 1$. Now, since f is additive, we have that

$$\begin{aligned} \delta_{d_U,f}^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\Gamma) &= f\{d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\psi_1), \ldots, d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\psi_n)\} \\ &= g(n) \cdot (d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\psi_1) + \ldots + d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\psi_n)) \\ &= g(n) \cdot |\{\psi \in \Gamma \mid \nu \notin mod_{\mathcal{M}}(\psi)\}|. \end{aligned}$$

Thus, $\nu \in \Delta_{\mathcal{S}}(\Gamma)$ iff the set $\{\psi \in \Gamma \mid \nu \notin mod_{\mathcal{M}}(\psi)\}$ is minimal in its size, iff $\{\psi \in \Gamma \mid \nu \in mod_{\mathcal{M}}(\psi)\}$ is maximal in its size, iff this set belongs to $\mathsf{mSAT}_{\mathcal{M}}(\Gamma)$.

Now assume that $\mathsf{mSAT}_{\mathcal{M}}(\Gamma) = \emptyset$. In this case none of the formulas in Γ is $\mathcal{M}$-satisfiable (see Note 4). Thus, as

$$\mathsf{M}_{d_U}(\Gamma) = \max\{d_U^{\downarrow\mathsf{SF}(\Gamma)}(\mu_1^{\downarrow\mathsf{SF}(\Gamma)}, \mu_2^{\downarrow\mathsf{SF}(\Gamma)}) \mid \mu_1, \mu_2 \in \Lambda_{\mathcal{M}}^{\downarrow\mathsf{SF}(\Gamma)}\} = 1,$$

we have that for every $\nu \in \Lambda_{\mathcal{M}}$,

$$\begin{aligned} \delta_{d_U,f}^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\Gamma) &= f\{d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\psi_1), \ldots, d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\psi_n)\} \\ &= g(n) \cdot (d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\psi_1) + \ldots + d_U^{\downarrow\mathsf{SF}(\Gamma)}(\nu,\psi_n)) \\ &= g(n) \cdot n \cdot (1 + \mathsf{M}_{d_U}(\Gamma)) \\ &= 2n \cdot g(n). \end{aligned}$$

Thus, all the elements in $\Lambda_{\mathcal{M}}$ are equally distant from Γ, and so $\Delta_{\mathcal{S}}(\Gamma) = \Lambda_{\mathcal{M}}$.
□