Tutorial on Non-Deterministic Semantics
Part III: More Advanced Topics

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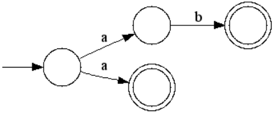

What this tutorial is about

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Non-deterministic Semantics (Matrices):

Incorporating the notion of *"non-deterministic computations"* from automata and computability theory into logical truth-tables.

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Incorporating the notion of *"non-deterministic computations"* from automata and computability theory into logical truth-tables.

We would like to show:

Non-deterministic semantics is a natural and useful paradigm.

Already covered topics

- Basic definitions and properties of Nmatrices.
- Application: canonical Gentzen-type systems.
- Application: semantics and sequent calculi for Logics of Formal (In)consistency

Overview of Part III - More Advanced Topics

Characterization of constructive connectives

- Extending the notion of canonical systems to the framework of single-conclusioned Gentzen-type calculi.
- Semantics: a combination of non-deterministic semantics with Kripke-style frames

Characterization of constructive connectives

- Extending the notion of canonical systems to the framework of single-conclusioned Gentzen-type calculi.
- Semantics: a combination of non-deterministic semantics with Kripke-style frames
- Application: constructive connectives can be characterized proof-theoretically by *a set of canonical rules in single-conclusion canonical systems*.

Reminder: what is a multiple-conclusioned canonical rule?

Stage 1.

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

Stage 2.

$$\frac{\psi, \varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi}$$

Stage 3.

$$\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \; ; \; \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

What is a single-conclusioned canonical rule?

Stage 1.

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta}{\Gamma, \psi \wedge \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \psi \quad \Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \wedge \varphi}$$

Stage 2.

$$\frac{\psi, \varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi}$$

Stage 3.

$$\{p_1, p_2 \Rightarrow\} / p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \;;\; \Rightarrow p_2\} / \Rightarrow p_1 \wedge p_2$$

Example 1

Implication rules:

$$\{p_1 \Rightarrow p_2\} \ / \ \Rightarrow p_1 \supset p_2 \qquad \{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \supset p_2 \Rightarrow$$

Their applications:

$$\frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \psi \supset \varphi} \qquad \frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \supset \varphi \Rightarrow \theta}$$

Example 2

Semi-implication rules (Gurevich):

$$\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \rightsquigarrow p_2 \Rightarrow \qquad \{\Rightarrow p_2\} \ / \ \Rightarrow p_1 \rightsquigarrow p_2$$

Their applications:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \rightsquigarrow \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \rightsquigarrow \varphi}$$

Coherence

- A canonical single-conclusioned calculus G is *coherent* if for every pair of rules $\Theta_1 / \Rightarrow \diamond(p_1, \ldots, p_n)$ and $\Theta_2 / \diamond(p_1, \ldots, p_n) \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically unsatisfiable (and so inconsistent, i.e., the empty sequent can be derived from it using only cuts)
- Examples of coherent calculi:

$$\{p_1 \Rightarrow p_2\} \;/\; \Rightarrow p_1 \supset p_2 \qquad \{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; p_1 \supset p_2 \Rightarrow$$

$$\{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; p_1 \rightsquigarrow p_2 \Rightarrow \qquad \{\Rightarrow p_2\} \;/\; \Rightarrow p_1 \rightsquigarrow p_2$$

- For a canonical calculus G, $\vdash_G$ is consistent iff G is coherent.

Characterization of constructiveness

Constructive connective

A connective is called constructive iff it can be defined by a coherent set of canonical rules.

Semantics

A generalized Kripke-frame

A triple $\mathcal{W} = \langle W, <, v \rangle$, where:

- $\langle W, < \rangle$ is a nonempty partially ordered set
- $v : W \times F \to \{\mathbf{t}, \mathbf{f}\}$ is a persistent function: if $v(w, \psi) = \mathbf{t}$, then for every $w' \geq w$, $v(w', \psi) = \mathbf{t}$.

- A sequent $\Gamma \Rightarrow \Delta$ is locally true in $w \in W$ if either $v(w, \psi) = \mathbf{f}$ for some $\psi \in \Gamma$, or $v(w, \psi) = \mathbf{t}$ for some $\psi \in \Delta$.
- A sequent is true in $w \in W$ if it is locally true in every $w' \geq w$.
- $\mathcal{W}$ is a model of a sequent if it is locally true in every $w \in W$.

G-legality of frames

Let G be a canonical coherent single-conclusioned system. A generalized frame is G-legal if it respects the introduction and elimination rules of G.

Respecting introduction rules

The conclusion is locally true in $w \in W$ whenever the premises are true in w.

Respecting elimination rules

The conclusion is locally true in $w \in W$ whenever the definite premises are true in w and the negative premises are locally true in w.

Example 1

Implication rules:

$$\{p_1 \Rightarrow p_2\} \;/\; \Rightarrow p_1 \supset p_2$$

$$v(w, \psi \supset \varphi) = \mathbf{t} \text{ if } v(w', \psi) = \mathbf{f} \text{ or } v(w', \varphi) = \mathbf{t} \text{ for every } w' \geq w$$

$$\{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; p_1 \supset p_2 \Rightarrow$$

$$v(w, \psi \supset \varphi) = \mathbf{f} \text{ if } v(w, \psi) = \mathbf{t} \text{ and } v(w, \varphi) = \mathbf{f}$$

The known semantics for intuitionistic implication!

Example 2

Semi-implication rules (Gurevich):

$$\{\Rightarrow p_2\} \;/\; \Rightarrow p_1 \rightsquigarrow p_2$$

$$v(w, \psi \rightsquigarrow \varphi) = \mathbf{t} \text{ if } v(w, \varphi) = \mathbf{t}$$

$$\{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; p_1 \rightsquigarrow p_2 \Rightarrow$$

$$v(w, \psi \rightsquigarrow \varphi) = \mathbf{f} \text{ if } v(w, \psi) = \mathbf{t} \text{ and } v(w, \varphi) = \mathbf{f}$$

Non-deterministic (e.g., for the case when $v(w', \psi) = v(w', \varphi) = \mathbf{f}$ for every $w' \geq w$)

Main results

Soundness and completeness:

A sequent s is provable from a set of sequents S in G iff every G-legal frame which is a model of S is also a model of s.

Decidability:

Every coherent canonical system is decidable.

Cut-elimination:

Every coherent canonical system admits strong cut-elimination.

Modularity:

The characterization of a constructive connective is independent of the system in which it is included.

Extension: basic systems

- Unlike in canonical systems, in *basic* sequent systems it is possible to control the *context* formulas.
- This allows one to have a larger variety of rules, and thus to handle more logics. For example,
  - Bi-intuitionistic logic:

$$\frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \psi \supset \varphi} \qquad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$$

$$\frac{\psi \Rightarrow \varphi, \Delta}{\psi \prec \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \psi \prec \varphi, \Delta}$$

  - The modal logic K:

$$\frac{\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi}$$

Basic systems - main results

- Every basic system has a (non-determinstic) Kripke-style semantics.
- In fact, there is a general method to obtain a (non-determinstic) Kripke-style semantics for a given basic systems.
- In addition, there are complete semantic characterizations of analyticity and (strong) cut-admissibility in basic systems.

Extension: canonical Gödel systems

- For some important logics, sequent systems do not suffice $\Rightarrow$ *hypersequent* systems.
- A single-conclusion hypersequent is a set of single-conclusion sequents denoted by:

$$\Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2 \mid \ldots \mid \Gamma_n \Rightarrow E_n$$

- The only known "ideal" system for Gödel logic is the single-conclusion hypersequent system **HG** based on the rule:

$$\frac{H \mid \Gamma, \Delta \Rightarrow E_1 \quad H \mid \Gamma, \Delta \Rightarrow E_2}{H \mid \Gamma \Rightarrow E_1 \mid \Delta \Rightarrow E_2} \ (com)$$

- Canonical Gödel systems: single-conclusion hypersequent systems with standard structural rules, (com), and any finite set of canonical single-conclusion logical rules.

Canonical Gödel systems - main results

- A general method to obtain (strongly) sound and complete Kripke semantics for canonical Gödel systems, based on linearly ordered frames.
- A general method to obtain (strongly) sound and complete many-valued semantics for canonical Gödel systems, based on the truth-values $[0, 1]$.
- The coherence criterion (from canonical single-conclusion sequent system) characterizes (strong) cut-admissibility in Canonical Gödel systems as well.

Summary

Non-deterministic semantics combined with Kripke-style frames are a powerful semantic formalism:

- Providing semantics for many natural classes of calculi (*canonical single-conclusioned, basic, canonical Gödel,...*)
- Semantic characterization of proof-theoretical properties of calculi.

Reminder: First-order languages

A first-order language L includes:

- *A set of variables* $x_1, x_2, \ldots,$
- *Parentheses, logical connectives (e.g.* $\wedge, \vee, \supset, \neg$*) and quantifiers (e.g.,* $\forall$ *and* $\exists$*)*
- *The signature of L:*
  - *a (non-empty) set of predicate symbols*
  - *a set of constants*
  - *a set of function symbols*

Matrices with unary quantifiers

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a (deterministic) matrix for a language L with unary quantifiers if:

1. $\mathcal{V}$ is a nonempty set of truth-values,
2. $\emptyset \neq \mathcal{D} \subset \mathcal{V}$ is a set of designated truth-values,
3. for every n-ary connective $\diamond$ of L, $\mathcal{O}$ includes an operation $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$,
4. for every unary quantifier Q of L, $\mathcal{O}$ includes an operation $\widetilde{Q} : P^+(\mathcal{V}) \to \mathcal{V}$.

Distribution quantifiers (coined by W.A. Carnielli)

Example

H	$\tilde{\forall}(H)$
{t}	t
{t,f}	f
{f}	f

H	$\tilde{\exists}(H)$
{t}	t
{t,f}	t
{f}	f

Matrices: objectual quantification

- *Variables range over objects from the domain and assignments map variables to elements of the domain.*

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- $S = \langle D, I \rangle$ - an L-structure.
  An *assignment* G in S maps the variables of L to D.
  Extend G to terms:

$$G(c) = I(c), \; G(f(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(f)(G(\mathbf{t}_1), ..., G(\mathbf{t}_n))$$

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The valuation $v_{S,G}$

- $v_{S,G}(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(G(\mathbf{t}_1), ..., G(\mathbf{t}_n))$.
- $v_{S,G}(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v_{S,G}(\psi_1), ..., v_{S,G}(\psi_n))$.
- $v_{S,G}(Qx\psi) = \tilde{Q}(\{v_{S,G\{x:=a\}}(\psi) \mid a \in D\})$.
  where $G\{x := a\}$ *coincides with* G *except for assigning* $a \in D$ *to* x.

Matrices: substitutional quantification

- *In classical first-order substitutional semantics, a universally quantified sentence is true iff each of its substitution instances is true.*
- Assumption: every element of the domain has a name.
  Given an L-structure $S = \langle D, I \rangle$, extend the language with the set of individual constants $\{\overline{a} \mid a \in D\}$ interpreted as the corresponding domain elements.

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  Given an L-structure $S = \langle D, I \rangle$, extend the language with the set of individual constants $\{\bar{a} \mid a \in D\}$ interpreted as the corresponding domain elements.

The valuation v_S

- $v_S(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$
- $v_S(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v_S(\psi_1), ..., v_S(\psi_n))$
- $v_S(Qx\psi) = \tilde{Q}(\{v_S(\psi\{\bar{a}/x\}) \mid a \in D\})$

Nmatrices with unary quantifiers

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a non-deterministic matrix (Nmatrix) for a language L with unary quantifiers if:

1. $\mathcal{V}$ is a nonempty set of truth-values,
2. $\emptyset \neq \mathcal{D} \subset \mathcal{V}$ is a set of designated truth-values,
3. for every n-ary connective $\diamond$ of L, $\mathcal{O}$ includes an operation $\widetilde{\diamond} : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$,
4. for every unary quantifier Q of L, $\mathcal{O}$ includes an operation $\widetilde{Q} : P^+(\mathcal{V}) \rightarrow P^+(\mathcal{V})$.

Example

Consider the two-valued Nmatrix $\mathcal{M}_1 = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ for a language L over $\{Q, \forall, \neg\}$, where $\mathcal{O}$ contains the following operations:

H	$\tilde{Q}$(H)
{t}	{t}
{t,f}	{t,f}
{f}	{f}

H	$\tilde{\forall}$(H)
{t}	{t}
{t,f}	{f}
{f}	{f}

a	$\neg a$
t	{t,f}
f	{t}

Nmatrices: objectual quantification

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Nmatrices: objectual quantification

- $v_{S,G}(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(G(\mathbf{t}_1), ..., G(\mathbf{t}_n))$.
- $v_{S,G}(\diamond(\psi_1, ..., \psi_n)) \in \tilde{\diamond}(v_{S,G})(\psi_1), ..., v_{S,G}(\psi_n))$.
- $v_{S,G}(Qx\psi) \in \tilde{Q}[\underbrace{\{v_{S,G[x:=a]}(\psi)}_{???} \mid a \in D\})$.

Substitutional quantification

Reminder: For $S = \langle D, I \rangle$, the language extended by individual constants is denoted by $L(D)$

Let $S = \langle D, I \rangle$ be an L-structure. A valuation in an Nmatrix $\mathcal{M}$ for L is a function v from sentences of $L(D)$ to $\mathcal{V}$, satisfying:

- $v((p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$
- $v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$
- $v(Qx\psi) \in \widetilde{Q}(\{v(\psi\{\overline{a}/x\}) \mid a \in D\})$

The problem of α-equivalence

- $\psi \equiv_\alpha \psi'$ if ψ can be obtained from ψ' by renaming bound variables.
- Problem: two α-equivalent sentences are not necessarily assigned the same truth-value.
- Example:

H	$\tilde{\forall}$[H]
{t}	{t}
{t,f}	{f}
{f}	{f}

a	$\neg a$
t	{t,f}
f	{t}

Consider: $\neg\forall x p(x)$ and $\neg\forall y p(y)$

Definition of a non-deterministic valuation - corrected

Let $S = \langle D, I \rangle$ be an L-structure. A valuation in an Nmatrix $\mathcal{M}$ for L is a function v from closed sentences of $L(D)$ to $\mathcal{V}$ satisfying:

- $v(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$.
- $v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$.
- $v(Qx\psi) \in \widetilde{Q}(\{v(\psi\{\overline{a}/x\}) \mid a \in D\})$.
- If $\psi_1 \equiv_\alpha \psi_2$, then $v(\psi_1) = v(\psi_2)$.

Other problems to handle

- Terms denoting the same objects cannot be used interchangeably.
- Void quantification for first-order quantifiers $\forall$ and $\exists$.
- Example:

H	$\tilde{\forall}$[H]
{t}	{t}
{t,f}	{f}
{f}	{f}

a	$\neg a$
t	{t,f}
f	{t}

Let $S = \langle\{1,2\}, I\rangle$, $I(p)(1) = I(p)(2) = t$ *and* $I(c) = I(d) = 1$.
Consider: (i) $\neg p(c)$ and $\neg p(d)$, (ii) $\neg\forall x p(c)$ and $\neg p(c)$.

- Solution: add appropriate congruence relations. For instance, $A \sim_{void} QxA$ if $x \notin Fv(A)$.

Analyticity

Analyticity of an Nmatrix M

for every L-structure S and every partial **M**-legal S-valuation v_p defined on a set of L-sentences closed under subformulas: v_p can be extended to a full **M**-legal valuation.

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- Analyticity is not guaranteed anymore when congruence relations are involved.
- Some good cases:
  - Analyticity for $\equiv_\alpha$ is always guaranteed.
  - Denote $\varphi_1 \sim^{dc} \varphi_2$ if φ_2 can be obtained from φ_1 by renaming bound variables and deleting/adding void quantifiers. Analyticity for $\sim^{dc}$ is guaranteed iff $a \in \tilde{Q}_{\mathbf{M}}(\{a\})$ for every quantifier Q of L and every $a \in \mathcal{V}$.

Using congruences in the propositional case

- Introducing congruences can be useful also in the propositional case *(e.g. equivalence in all contexts of $\psi \wedge \varphi$ and $\varphi \wedge \psi$)*.
- Analyticity should be handled with care *(question for further research)*

Application: first-order C-systems

Language: $\mathcal{L}_{\mathrm{QC}} = \{\wedge, \vee, \supset, \neg, \circ, \forall, \exists\}$.
Logic: **QBK** is obtained by adding the following axioms to some standard Hilbert-type system for classical positive FOL:

(N1) $\neg\varphi \vee \varphi$

(b) $(\circ\varphi \wedge \varphi \wedge \neg\varphi) \supset \psi$

(k) $\circ\psi \vee (\psi \wedge \neg\psi)$

(DC) $\varphi_1 \supset \varphi_2$ whenever $\varphi_1 \sim^{dc} \varphi_2$.

$\varphi_1 \sim^{dc} \varphi_2$ if φ_2 can be obtained from φ_1 by renaming bound variables and deleting/adding void quantifiers.

Extensions of QBK

(c) $\neg\neg\varphi \supset \varphi$

(e) $\varphi \supset \neg\neg\varphi$

$\ldots$

(a$_\forall$) $\forall x \circ\varphi \supset \circ(\forall x \varphi)$

(a$_\exists$) $\forall x \circ\varphi \supset \circ(\exists x \varphi)$

(o$_\forall$) $\exists x \circ\varphi \supset \circ(\forall x \varphi)$

(o$_\exists$) $\exists x \circ\varphi \supset \circ(\exists x \varphi)$

Example: da-Costa's original C_1^* is equivalent to **QBKcila**.

The idea of semantics

- Truth-value: $v(\varphi) = \langle x, y \rangle$, where x expresses truth/falsity of φ and y expresses truth/falsity of $\neg\varphi$.

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- Truth-value: $v(\varphi) = \langle x, y \rangle$, where x expresses truth/falsity of φ and y expresses truth/falsity of $\neg\varphi$.
- Possible values:
  - $v(\varphi) = \langle 1, 0 \rangle = \mathbf{t}$ - φ is true and $\neg\varphi$ is false
  - $v(\varphi) = \langle 0, 1 \rangle = \mathbf{f}$ - φ is false and $\neg\varphi$ is true
  - $v(\varphi) = \langle 1, 1 \rangle = \top$ - φ is true and $\neg\varphi$ is true
- **Addition:** Every **M**-legal valuation should also respect the congruences for α-equivalence and void quantification (but analyticity is preserved!).

3-valued Semantics for QBK

The Nmatrix $\mathcal{QM} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined by:

$\mathcal{V} = \{\mathbf{t}, \top, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}, \top\}$, and $\mathcal{F} = \{\mathbf{f}\}$:

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \end{cases} \qquad a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } \ a \in \{\top, \mathbf{f}\} \\ \mathcal{F} & \text{if } \ a = \mathbf{t} \end{cases} \qquad \widetilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \{\mathbf{t}, \mathbf{f}\} \\ \mathcal{F} & \text{if } a = \top \end{cases}$$

$$\widetilde{\forall}(H) = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases} \qquad \widetilde{\exists}(H) = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

Effects of $(\mathbf{a_Q})$ and $(\mathbf{o_Q})$ for $Q \in \{\forall, \exists\}$

$(\mathbf{a}_\forall)$ $\forall x \circ\varphi \supset \circ(\forall x \varphi)$ $\quad Cond(\mathbf{a}_\forall) : \tilde{\forall}(\{\mathbf{t}\}) = \{\mathbf{t}\}$

$(\mathbf{a}_\exists)$ $\forall x \circ\varphi \supset \circ(\exists x \varphi)$ $\quad Cond(\mathbf{a}_\forall) : \tilde{\exists}(\{\mathbf{t}\}) = \tilde{\exists}(\{\mathbf{t}, \mathbf{f}\}) = \{\mathbf{t}\}$

$(\mathbf{o}_\forall)$ $\exists x \circ\varphi \supset \circ(\forall x \varphi)$ $\quad Cond(\mathbf{o}_\forall) : \tilde{\forall}(\{\mathbf{t}\}) = \tilde{\exists}(\{\mathbf{t}, \top\}) = \{\mathbf{t}\}$

$(\mathbf{o}_\exists)$ $\exists x \circ\varphi \supset \circ(\exists x \varphi)$ $\quad Cond(\mathbf{o}_\exists) : \tilde{\exists}(\{\mathbf{t}\}) = \tilde{\exists}(\{\mathbf{t}, \top\}) = \tilde{\exists}(\{\mathbf{t}, \mathbf{f}\}) = \tilde{\exists}(\{\mathbf{t}, \top, \mathbf{f}\}) = \{\mathbf{t}\}$

QBK :

H	$\tilde{\forall}[H]$	$\tilde{\exists}[H]$
$\{\mathbf{t}\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\{\top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \top\}$
$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\{\mathbf{f}, \top\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \top\}$
$\{\mathbf{t}, \mathbf{f}, \top\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \top\}$

QBK + (**a**) :

H	$\tilde{\forall}[H]$	$\tilde{\exists}[H]$
$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\{\top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$
$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\{\mathbf{f}, \top\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \top\}$
$\{\mathbf{t}, \mathbf{f}, \top\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \top\}$

QBK + (**o**) :

H	$\tilde{\forall}[H]$	$\tilde{\exists}[H]$
$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\{\top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$
$\{\mathbf{t}, \top\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\mathbf{f}, \top\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$
$\{\mathbf{t}, \mathbf{f}, \top\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$

The Nmatrix $\mathbf{M}_{C_1^*}$

$\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$, $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{\top_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{F} = \{f\}$,
$\mathcal{D} = \mathcal{T} \cup \mathcal{I}$.

$$a \tilde{\supset} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \text{ and } b \notin \mathcal{I}, \text{ or} \\ & \text{if } b \in \mathcal{T} \text{ and } a \notin \mathcal{I} \\ \mathcal{D} & \text{otherwise} \end{cases} \qquad a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{T} \text{ and } b \in \mathcal{T}, \text{ or} \\ & \text{if } a = \top_i^j \text{ and } b \in \{\top_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \\ \{\top_i^{j+1}, t_i^{j+1}\} & \text{if } a = \top_i^j \end{cases}$$

$$\tilde{\forall}(H) = \begin{cases} \mathcal{T} & \text{if } H \subseteq \mathcal{T} \\ \mathcal{D} & \text{if } H \subseteq \mathcal{D} \text{ and } H \cap \mathcal{I} \neq \emptyset \\ \mathcal{F} & f \in H \end{cases} \qquad \tilde{\exists}(H) = \begin{cases} \mathcal{T} & \text{if } H \subseteq \mathcal{T} \cup \mathcal{F} \text{ and } H \cap \mathcal{T} \neq \emptyset \\ \mathcal{D} & \text{if } H \cap \mathcal{I} \neq \emptyset \\ \mathcal{F} & H = \{f\} \end{cases}$$

Application: $\neg\exists x \neg p(x) \not\vdash_{C_1^*} \forall x p(x)$

- A rather complex syntactic proof of da Costa (1974).
- A much easier semantic proof: refutation using $\mathbf{M}_{C_1^*}$.

$$S = \langle \{a, b\}, I \rangle$$

$$I(p)(a) = \top_0^0 \quad I(p)(b) = f$$

Next define a partial valuation v on the set of subformulas of $\{\neg\exists x \neg p(x), \forall x p(x)\}$ as follows:

$$v(p(\overline{a})) = \top_0^0 \quad v(p(\overline{b})) = f \quad v(\neg p(\overline{a})) = \top_0^1 \quad v(\neg p(\overline{b})) = t_0^0$$

$$v(\exists x \neg p(x)) = \top_0^1 \quad v(\neg\exists x \neg p(x)) = t_0^2 \quad v(\forall x p(x)) = f$$

v is $\mathbf{M}_{C_1^*}$-legal, and (by the analyticity of $\mathbf{M}_{C_1^*}$) it can be extended to a full $\mathbf{M}_{C_1^*}$-legal valuation.

Reminder: propositional canonical systems

Each logical rule satisfies:

1. Introduces exactly one formula in its conclusion.
2. The introduced formula: $\diamond(\psi_1, \ldots, \psi_n)$.
3. All active formulas in its premises are in $\{\psi_1, \ldots, \psi_n\}$.
4. No restrictions on the side formulas.

Direct correspondence: *A canonical system is coherent iff it admits cut-elimination iff it has a characteristic 2Nmatrix.*

Canonical quantifier rules

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}$$

where z is a variable free for w in A, z is not free in $\Gamma \cup \Delta \cup \{\forall w A\}$, and $\mathbf{t}$ is any term free for w in A.

$$\Downarrow$$

$$\frac{A\{\mathbf{t}/w\} \Rightarrow}{\forall w\, A \Rightarrow} \qquad \frac{\Rightarrow A\{z/w\}}{\Rightarrow \forall w\, A}$$

$$\Downarrow$$

$$\{p(c) \Rightarrow\}/\forall w\, p(w) \Rightarrow \qquad \{\Rightarrow p(y)\}/ \Rightarrow \forall w\, p(w)$$

An eigenvariable is marked by a variable, and a term is marked by a constant.

Canonical systems

A canonical system includes

1. Axioms: $\psi \Rightarrow \psi'$ for $\psi \equiv_\alpha \psi'$
2. Structural Weakening and Cut rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathit{Weakening}) \qquad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}\ (\mathit{Cut})$$

3. Substitution rule:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'}\ (S)$$

where Γ', Δ' *are substitution instances of* Γ, Δ *resp.*

4. Canonical introduction rules.

Coherence

- A canonical calculus G is coherent if for every two canonical rules of G of the form $\Theta_1 / \Rightarrow A$ and $\Theta_2 / A \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent.
- *The coherence of a canonical calculus G is decidable.*
- Examples:

  Coherent:

$$\{p(c) \Rightarrow\} \,/\, \forall x\, p(x) \Rightarrow \qquad \{\Rightarrow p(y)\} \,/\, \Rightarrow \forall x\, p(x)$$

  Non-coherent:

$$\{\Rightarrow p(c)\} \,/\, \Rightarrow Qxp(x) \qquad \{p(d) \Rightarrow\} \,/\, Qxp(x) \Rightarrow$$

Correspondence Theorem

The following statements concerning a canonical system G with unary quantifiers are equivalent:

1. G is coherent.
2. G has a characteristic 2Nmatrix.
3. G admits strong cut elimination.

Strong cut-elimination

G admits strong cut-elimination if whenever $S \vdash s$, then s has a proof from S in G, where cuts are applied only on substitution instances of formulas from S.

More General Quantifiers

- A natural step: *n*-ary quantifiers:

  If Q is an n-ary quantifier, then $Qx(\psi_1, ..., \psi_n)$ *is a formula.*

- Examples:
  1. *Unary quantifiers:* $\forall$, $\exists$.
  2. *Binary quantifiers: bounded universal and existential quantifiers* $\overline{\forall}$ *and* $\overline{\exists}$*, where:*
     - $\overline{\forall}(\psi_1, \psi_2)$ means $\forall x(\psi_1 \rightarrow \psi_2)$.
     - $\overline{\exists}(\psi_1, \psi_2)$ means $\exists x(\psi_1 \wedge \psi_2)$.

Nmatrices with *n*-ary quantifiers

- An *n*-ary quantifier Q in an Nmatrix $\mathbf{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is interpreted by a function $\tilde{Q} : P^{+}(\mathcal{V}^{n}) \rightarrow P^{+}(\mathcal{V})$.
- Example: for every $\mathcal{E} \in P^{+}(\{t, f\}^{2})$:

$$\tilde{\tilde{\forall}}(\mathcal{E}) = \begin{cases} \{f\} & \textit{if } \langle t, f \rangle \in \mathcal{E} \\ \{t\} & \textit{otherwise} \end{cases} \qquad \tilde{\tilde{\exists}}(\mathcal{E}) = \begin{cases} \{t\} & \textit{if } \langle t, t \rangle \in \mathcal{E} \\ \{f\} & \textit{otherwise} \end{cases}$$

The definition of an **M**-valuation v is now modified as follows:

$$v(Qx(\psi_1, ..., \psi_n)) \in \tilde{Q}_{\mathbf{M}}(\{\langle v(\psi_1\{\overline{a}/x\}), ..., v(\psi_n\{\overline{a}/x\})\rangle \mid a \in D\})$$

The framework of canonical systems can be extended to the case of *n*-ary quantifiers, the direct correspondence still holds.

Example

$\mathbf{H}$	$\tilde{\tilde{\forall}}(\mathbf{H})$	$\tilde{\tilde{\exists}}[\mathbf{H}]$	$\tilde{Q}_2[\mathbf{H}]$
$\{\langle \mathbf{t}, \mathbf{t} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{f}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{f}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{f}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{f}, \mathbf{f} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

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Summary

- Non-deterministic semantics is a useful paradigm:
  - Semantic tool for proof-theoretical investigations
  - Characterization of various non-classical logics
- Allows for a systematic and modular approach
- Insights into the syntax-semantics interface
- Provides important tools for Universal Logic.

Thank you for your attention!