Tutorial on Non-Deterministic Semantics Part II: Logics of Formal (In)consistency

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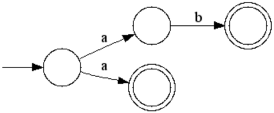
UNILOG 2013, Rio de Janeiro



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Non-deterministic Semantics (Matrices):

Incorporating the notion of *"non-deterministic computations"* from automata and computability theory into logical truth-tables.

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We would like to show:

Non-deterministic semantics is a natural and useful paradigm.

Already covered topics

- Basic definitions and properties of Nmatrices.
- Application of two-valued Nmatrices: canonical Gentzen-type systems.

Overview of Part II - Logics of Formal (In)consistency

What kind of logic is needed for reasoning with inconsistencies?

- Within classical logic, inconsistency leads to trivialization of knowledge bases, as everything becomes derivable:

$$A, \neg A \vdash B$$

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- Within classical logic, inconsistency leads to trivialization of knowledge bases, as everything becomes derivable:

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- Paraconsistent logic is a logic which allows contradictory but non-trivial theories.

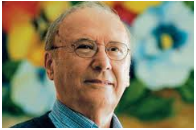
Definition

A propositional logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is *paraconsistent* (with respect to $\neg$) if there are $\mathcal{L}$-formulas A, B, such that $A, \neg A \nvdash B$.

The fathers of paraconsistent logic

S. Jaśkowski, 1948:

...PL should be rich enough to enable practical inferences.



N.C.A. da Costa, 1963:

...PL should contain as much as possible of classical logic.

The Brazilian school of paraconsistent logics

- Divide propositions into two sorts: consistent and inconsistent ones.
- Reflect this classification within the language.
- The class of C-systems:
  - Employ a special (primitive or defined) connective $\circ$.
  - Intuitive meaning of $\circ A$: "A is consistent".
  - Explosive character of contradictions is restricted:

$$\psi, \neg\psi \vdash \varphi \quad \Rightarrow \quad \psi, \neg\psi, \circ\psi \vdash \varphi$$

An example: da Costa's system C_1

Obtained by:

- Taking $\circ\varphi = \neg(\varphi \wedge \neg\varphi)$
- Adding to some Hilbert-style system for positive classical logic the following axioms concerning negation:

  (N1) $\neg\varphi \vee \varphi$
  (N2*) $\neg\neg\varphi \supset \varphi$
  ($\mathbf{a}_{\wedge}$) $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$
  ($\mathbf{a}_{\vee}$) $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi)$
  ($\mathbf{a}_{\supset}$) $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \supset \psi)$

  and *either* of the following two axioms:

  (N∘1) $\circ\varphi \supset (\psi \supset \varphi) \supset (\psi \supset \neg\varphi) \supset \neg\psi$
  (N∘2) $(\circ\varphi \wedge \varphi \wedge \neg\varphi) \supset \psi$

Logics of Formal (In)consistency

Logics of Formal (In)consistency

A paraconsistent logic **L** is an LFI if there is an atomic variable p and a set $X(p)$ of formulas, such that $A, \neg A, X\{A/p\} \vdash B$ for all A and B.

Studied by W.A. Carnielli, J. Marcos, M.E. Coniglio and others.

C-systems

A (bit modified) definition

L is a C-system if (i) **L** contains the positive fragment of classical logic, and (ii) **L** has a (primitive or defined) unary connective $\circ$, for which the following are valid:

(N1) $\neg\psi \vee \psi$

(b) $\circ\psi \supset ((\psi \wedge \neg\psi) \supset \varphi)$

(k) $\circ\psi \vee (\psi \wedge \neg\psi)$

The basic C-system BK

The system **BK** extends the positive fragment of classical logic with **(t)**, **(b)** and **(k)**.
The system **B** is **BK** without **(k)**.

Extensions of BK

For $\sharp \in \{\wedge, \vee, \supset\}$:

(c) $\neg\neg\varphi \supset \varphi$

(e) $\varphi \supset \neg\neg\varphi$

($\mathbf{i}_1$) $\neg\circ\varphi \supset \varphi$

($\mathbf{i}_2$) $\neg\circ\varphi \supset \neg\varphi$

($\mathbf{a}_\sharp$) $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi\sharp\psi)$

($\mathbf{o}^1_\sharp$) $\circ\varphi \supset \circ(\varphi\sharp\psi)$

($\mathbf{o}^2_\sharp$) $\circ\psi \supset \circ(\varphi\sharp\psi)$

(l) $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$

(d) $\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$

Example: C_1 is equivalent to **BKcila**$(=$ **Bcila**$)$

Semantics

- C-systems were mostly introduced in proof-theoretic terms.
- After some years several semantic approaches were proposed (da Costa, Carnielli and Marcos, Béziau,...):
  - Bivaluation semantics
  - Possible translations semantics

Semantics

- C-systems were mostly introduced in proof-theoretic terms.
- After some years several semantic approaches were proposed (da Costa, Carnielli and Marcos, Béziau,...):
  - Bivaluation semantics
  - Possible translations semantics
- These semantic frameworks are very general and as such lack some useful properties, such as a general **analyticity** theorem.

Non-deterministic semantics - the idea

- Truth-value: $v(\varphi) = \langle x, y\rangle$, where x expresses truth/falsity of φ and y expresses truth/falsity of $\neg\varphi$.
- Possible values:
  - $v(\varphi) = \langle 1, 0\rangle = \mathbf{t}$ - φ is true and $\neg\varphi$ is false
  - $v(\varphi) = \langle 0, 1\rangle = \mathbf{f}$ - φ is false and $\neg\varphi$ is true
  - $v(\varphi) = \langle 1, 1\rangle = \top$ - φ is true and $\neg\varphi$ is true
  - $v(\varphi) = \langle 0, 0\rangle = \bot$ - φ is false and $\neg\varphi$ is false

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  - $v(\varphi) = \langle 1, 1 \rangle = \top$ - φ is true and $\neg\varphi$ is true

(N1) $(\neg\varphi \vee \varphi)$ leads to the deletion of $\perp$

Semantics for BK - the Nmatrix $\mathbf{M}^3$

- Truth-values: $\mathbf{t} = \langle 1, 0\rangle$, $\top = \langle 1, 1\rangle$, $\mathbf{f} = \langle 0, 1\rangle$
- Designated truth-values: $\mathbf{t} = \langle 1, 0\rangle$, $\top = \langle 1, 1\rangle$

a	$\neg a$	$\circ a$
$\mathbf{t}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \top\}$
$\top$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$
f	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$

$\wedge$	$\mathbf{t}$	$\top$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$
$\top$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$

$\vee$	$\mathbf{t}$	$\top$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\top$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\mathbf{f}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$

$\supset$	$\mathbf{t}$	$\top$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$
$\top$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$

Soundness and completeness theorem

$T \vdash_{HBK} \psi$ iff $T \vdash_{\mathbf{M}^3} \psi$.

Semantic effects of the axioms

An addition of an axiom leads to a refinement of the basic Nmatrix.

Reminder:

$\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ is a refinement of $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ if:

1. $\mathcal{V}_1 \subseteq \mathcal{V}_2$
2. $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$
3. $\widetilde{\diamond}_{\mathcal{M}_1}(x_1 \dots x_n) \subseteq \widetilde{\diamond}_{\mathcal{M}_2}(x_1 \dots x_n)$ for every n-ary connective $\diamond$ and every $x_1 \dots x_n, y \in \mathcal{V}_1$.

Adding (c) $\neg\neg\varphi \supset \varphi$

- Possible refutations:

$$v(\varphi) = f,\ v(\neg\varphi) = \top \text{ and } v(\neg\neg\varphi) \in \{t, \top\}$$

- Imposed semantic condition:

$$\neg f = \{t\}$$

a	$\neg a$
$\mathbf{t}$	$\{\mathbf{f}\}$
$\top$	$\{\mathbf{t}, \top\}$
$\mathbf{f}$	$\{\mathbf{t}, \top\}$

$\Rightarrow$

a	$\neg a$
$\mathbf{t}$	$\{\mathbf{f}\}$
$\top$	$\{\mathbf{t}, \top\}$
$\mathbf{f}$	$\{\mathbf{t}\}$

Adding $(\mathbf{o}^{\mathbf{1}}_{\vee})$ $\circ\varphi \supset \circ(\varphi \vee \psi)$

- Possible refutations:

$$v(\circ\varphi) = t/\top$$
$$v(\varphi) = \mathbf{t}/f,\ v(\psi) = \dots$$
$$v(\varphi \vee \psi) = \top$$
$$v(\circ(\varphi \vee \psi)) = \mathbf{f}$$

- Imposed semantic conditions:

$$t \vee t = t \vee f = t \vee \top = \{t\}$$

$$f \vee t = f \vee \top = \{t\}$$

$\vee$	$\mathbf{t}$	$\top$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\top$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\mathbf{f}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$

$\Rightarrow$

$\vee$	$\mathbf{t}$	$\top$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\top$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\mathbf{f}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$

More semantic conditions

	ax	$C(ax)$
(c)	$\neg\neg\varphi \supset \varphi$	$\neg f = \{t\}$
(e)	$\varphi \supset \neg\neg\varphi$	$\neg\top = \{\top\}$
(i₁)	$\neg\circ\varphi \supset \varphi$	$\circ f = \{t\}$
(i₂)	$\neg\circ\varphi \supset \neg\varphi$	$\circ t = \{t\}$
(a$_\wedge$)	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$	$t \wedge t = \{t\}$
...	...	...

	ax	$C(ax)$
(a$_\vee$)	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi)$	$t \vee t = t \vee f = \{t\}$ $t \vee t = f \vee t = \{t\}$
(a$_\supset$)	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \supset \psi)$	$f \supset t = f \supset f = \{t\}$ $f \supset t = t \supset t = \{t\}$
(o$^1_\wedge$)	$\circ\varphi \supset \circ(\varphi \wedge \psi)$	$t \wedge t = t \wedge \top = \{t\}$
(o$^2_\wedge$)	$\circ\psi \supset \circ(\varphi \wedge \psi)$	$t \wedge t = \top \wedge t = \{t\}$
...	...	...

Soundness and completeness for $A \subseteq Ax' = Ax \setminus \{(\mathbf{l}), (\mathbf{d})\}$

$\mathbf{M}^3_{\mathbf{BK}}[A]$ - the simplest refinement of $\mathbf{M}^3_{\mathbf{BK}}$, for which all the semantic conditions induced by the axioms of A hold.

Theorem

$T \vdash_{\mathbf{M}^3_{\mathbf{BK}}[A]} \psi$ *iff* $T \vdash_{\mathbf{BK}[A]} \psi$.

The axioms **(l)** and (**d**) are a bit problematic, we will handle them later.

Some applications

- $(\mathbf{a}_\sharp)$ follows in **BK** from $(\mathbf{o}^1_\sharp)$ and $(\mathbf{o}^2_\sharp)$.
- 1. $\vdash_{BKia} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$
  2. $\nvdash_{BKcie} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$
- **BK[X]** is decidable for every $\mathbf{X} \subseteq Ax'$.
- Let **L** be a logic in a language which includes $\{\neg, \wedge, \vee, \supset\}$. If **BKcioe** is an extension of **L** then two formulas in $\{\neg, \wedge, \vee, \supset\}$ are logically indistinguishable in **L** iff they are identical.

Logical indistinguishability

A and B are *logically indistinguishable* in **L** if $\varphi(A) \vdash_{\mathbf{L}} \varphi(B)$ and $\varphi(B) \vdash_{\mathbf{L}} \varphi(A)$ for every formula $\varphi(p)$ in the language of **L**.

No improvements possible

No characteristic finite matrices

$\mathcal{L}$ - either $\{\neg, \wedge, \vee, \supset\}$ or $\mathcal{L}_C$.
L - a logic in $\mathcal{L}$, such that its set of theorems includes that of positive classical logic, and is included in that of **BKcioe**.
Then there is no finite (deterministic) matrix P, such that $T \vdash_{\mathbf{L}} \psi$ iff $T \vdash_P \psi$.

No improvements possible

No characteristic finite matrices

$\mathcal{L}$ - either $\{\neg, \wedge, \vee, \supset\}$ or $\mathcal{L}_C$.
L - a logic in $\mathcal{L}$, such that its set of theorems includes that of positive classical logic, and is included in that of **BKcioe**.
Then there is no finite (deterministic) matrix P, such that $T \vdash_{\mathbf{L}} \psi$ iff $T \vdash_P \psi$.

No weakly characteristic finite matrices

L - as above.
Then there is no finite (deterministic) matrix P, such that $\vdash_{\mathbf{L}} \psi$ iff $\vdash_P \psi$.

Analytic calculi for C-systems

- Sequent and tableaux systems were proposed:
  - da Costa's C_1: *Raggio, Béziau, Carnielli and Marcos*.
  - Other particular C-systems: *Carnielli and Marcos, Gentillini, Finger et al.*
- Methods tailored for specific systems, rules are not uniform.
- Is systematic approach possible?

Can Nmatrices help?

There is an algorithm for constructing cut-free sequent calculi for logics, which:

1. have a finite-valued characteristic Nmatrix **M**
2. have a language expressive enough with respect to **M**

Intuition: $\mathcal{L}$ is expressive enough for **M** if we can "characterize" each truth-value of **M** using a finite set of $\mathcal{L}$-sequents.

Expressivity in our case

- $v(A) = \mathbf{t}$ iff $\neg A \Rightarrow$ is true in v.
- $v(A) = \mathbf{f}$ iff $A \Rightarrow$ is true in v.
- $v(A) = \top$ iff $\Rightarrow A$ and $\Rightarrow \neg A$ are both true in v.
- $v(A) \in \{\mathbf{f}, \top\}$ iff $\Rightarrow \neg A$ is true in v.
- $v(A) \in \{\mathbf{t}, \top\}$ iff $\Rightarrow A$ is true in v.
- $v(A) \in \{\mathbf{t}, \mathbf{f}\}$ iff $A, \neg A \Rightarrow$ is true in v.

Reminder: A sequent $\Gamma \Rightarrow \Delta$ is true in v if $v(\psi) \notin \mathcal{D}$ for some $\psi \in \Gamma$ or $v(\psi) \in \mathcal{D}$ for some $\psi \in \Delta$.

Example: the truth-table for $\vee$ in $\mathbf{M}^3_{\mathbf{BK}}$

$\vee$	$\mathbf{t}$	$\top$	$\mathbf{f}$
t	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\top$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
f	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$

$$f \vee f = \{f\}$$

$$\Downarrow$$

if A is false and B is false, then $A \vee B$ is false

$$\Downarrow$$

if $A \Rightarrow$ is true and $B \Rightarrow$ is true then $A \vee B \Rightarrow$ is true.

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

Example: the truth-table for $\vee$ in $\mathbf{M}^3_{\mathbf{BK}}$

$\vee$	$\mathbf{t}$	$\top$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\top$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\mathbf{f}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$

$$\mathbf{f} \vee \top = \{\mathbf{t}, \top\}$$

$$\Downarrow$$

if $A \Rightarrow$ is true and $\Rightarrow B$ is true and $\Rightarrow \neg B$ is true, then $\Rightarrow A \vee B$ is true.

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, B \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, A \vee B}$$

Example: the truth-table for $\vee$ in $\mathbf{M}^3_{\mathbf{BK}}$

$\vee$	$\mathbf{t}$	$\top$	f
$\mathbf{t}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\top$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$
$\mathbf{f}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{t}, \top\}$	$\{\mathbf{f}\}$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

The system $\mathbf{G_K}$

$$(\wedge\Rightarrow)\quad \frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \wedge \phi \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\quad \frac{\Gamma \Rightarrow \Delta, \psi \qquad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \wedge \phi}$$

$$(\vee\Rightarrow)\quad \frac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \vee \phi \Rightarrow \Delta} \qquad (\Rightarrow\vee)\quad \frac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \vee \phi}$$

$$(\supset\Rightarrow)\quad \frac{\Gamma \Rightarrow \psi, \Delta \qquad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \supset \phi \Rightarrow \Delta} \qquad (\Rightarrow\supset)\quad \frac{\Gamma, \psi \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \psi \supset \phi, \Delta}$$

$$(\Rightarrow\neg)\quad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$(\circ\Rightarrow)\quad \frac{\Gamma \Rightarrow \psi, \Delta \qquad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma, \circ\psi \Rightarrow \Delta} \qquad (\Rightarrow\circ)\quad \frac{\Gamma, \psi, \neg\psi \Rightarrow \Delta}{\Gamma \Rightarrow \circ\psi, \Delta}$$

- $\mathbf{G_K}$ is equivalent to **BK** and enjoys cut-admissibility.
- The rules of $\mathbf{G_K}$ for $\wedge, \vee, \supset, \circ$ are invertible.

Example 1: (**c**) $\neg\neg A \supset A$

- Semantic condition:

$$\neg f = \{t\}$$

- Translation: if $A \Rightarrow$ is true, then $\neg\neg A \Rightarrow$ is true.

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta}$$

Example 2: $(\mathbf{o}^{\mathbf{1}}_{\vee})$ $\circ A \supset \circ(A \vee B)$

- The semantic conditions: $(i)\ t \vee t = t \vee f = t \vee \top = \{t\}$
  $(ii)\ f \vee t = f \vee \top = \{t\}$
- Translate (i): if $\neg A \Rightarrow$ is true, then $\neg(A \vee B) \Rightarrow$ is true

$$\frac{\Gamma, \neg A \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta}$$

- Translate (ii): if $A \Rightarrow$ is true and $\Rightarrow B$ is true, then $\neg(A \vee B) \Rightarrow$ is true

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta}$$

Rules for axioms from Ax'

	ax	C(ax)	R(ax)
(c)	$\neg\neg\varphi \supset \varphi$	$\neg f = \{t\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$
(e)	$\varphi \supset \neg\neg\varphi$	$\neg\top = \{\top\}$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$
($\mathbf{i_1}$)	$\neg\circ\varphi \supset \varphi$	$\circ f = \{t\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\circ\varphi \Rightarrow \Delta}$
($\mathbf{i_2}$)	$\neg\circ\varphi \supset \neg\varphi$	$\circ t = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg\circ\varphi \Rightarrow \Delta}$
($\mathbf{a}_\wedge$)	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$ ...	$t \wedge t = \{t\}$ ...	$\dfrac{\Gamma, \neg\varphi \Rightarrow, \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$ ...

Soundness, completeness and cut-elimination

Theorem

For all $A \subseteq Ax'$:

1. $\mathbf{G_K}[A]$ *is equivalent to* $\mathbf{BK}[A]$.
2. $\mathbf{G_K}[A]$ *enjoys cut-admissibility.*

Soundness, completeness and cut-elimination

Theorem

For all $A \subseteq Ax'$*:*

1. $\mathbf{G_K}[A]$ *is equivalent to* $\mathbf{BK}[A]$.
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A systematic way to construct cut-free systems:

- Modularity: each axiom corresponds to a set of Gentzen-type rules, which are easily computed from the semantic conditions induced by the axiom.
- Uniformity: The rules of the obtained calculi have a simple, intuitive and uniform form $\Rightarrow$ *Quasi-Canonical Systems!*

Reminder: canonical systems

In canonical Gentzen-type systems, each logical rule satisfies:

1. Introduces exactly one formula in its conclusion.
2. The introduced formula: $\diamond(\psi_1, \ldots, \psi_n)$.
3. All active formulas in its premises are in $\{\psi_1, \ldots, \psi_n\}$.
4. No restrictions on the side formulas.

Direct correspondence: *A canonical system is coherent iff it admits cut-elimination iff it has a characteristic 2Nmatrix.*

Quasi-canonical systems

A ¬-quasi-canonical logical rule:

1. Introduces exactly one formula in its conclusion.
2. The introduced formula: $\diamond(\psi_1, \ldots, \psi_n)$ or $\neg \diamond (\psi_1, \ldots, \psi_n)$.
3. All active formulas in its premises are in $\{\psi_1, \ldots, \psi_n, \neg\psi_1, \ldots, \neg\psi_n\}$.
4. No restrictions on the side formulas.

Direct correspondence: the coherence criterion can be generalized.

Adding more axioms

$(\mathbf{n}^{\mathbf{l}}_{\wedge})$ $\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$ $(\mathbf{n}^{\mathbf{r}}_{\wedge})$ $(\neg\varphi \vee \neg\psi) \supset \neg(\varphi \wedge \psi)$

$(\mathbf{n}^{\mathbf{l}}_{\vee})$ $\neg(\varphi \vee \psi) \supset (\neg\varphi \wedge \neg\psi)$ $(\mathbf{n}^{\mathbf{r}}_{\vee})$ $(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$

$(\mathbf{n}^{\mathbf{l}}_{\supset})$ $\neg(\varphi \supset \psi) \supset (\varphi \wedge \neg\psi)$ $(\mathbf{n}^{\mathbf{r}}_{\supset})$ $(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$

Beware of conflicts!

In their presence the system is not paraconsistent, and may not even have a characteristic Nmatrix!

Example

- $Cond(\mathbf{o}^{\mathbf{1}}_{\wedge})$: for $b \in \{t, \top\}$, $t \wedge b = \{t\}$.
- $Cond(\mathbf{n}^{\mathbf{r}}_{\wedge})$: for $b \in \{t, \top\}$, $b \wedge \top = \top \wedge b = \{\top\}$.
- *Conflict in the case of $t \wedge \top$!*
- Exhaustive list:
  - $(\mathbf{o}^{\mathbf{1}}_{\wedge})$ and $(\mathbf{n}^{\mathbf{r}}_{\wedge})$
  - $(\mathbf{o}^{\mathbf{2}}_{\wedge})$ and $(\mathbf{n}^{\mathbf{r}}_{\wedge})$
  - $(\mathbf{o}^{\mathbf{1}}_{\vee})$ and $(\mathbf{n}^{\mathbf{r}}_{\vee})$
  - $(\mathbf{o}^{\mathbf{2}}_{\vee})$ and $(\mathbf{n}^{\mathbf{r}}_{\vee})$
  - $(\mathbf{o}^{\mathbf{1}}_{\supset})$ and $(\mathbf{n}^{\mathbf{r}}_{\supset})$.

Extension: paracomplete systems

The basic paracomplete system BP

The system **BP** extends the positive fragment of classical logic with

(N2) $(\psi \wedge \neg\psi) \supset \varphi$ (instead of **(N1)** $(\psi \vee \neg\psi)$)

(b) $\circ\psi \supset ((\psi \wedge \neg\psi) \supset \varphi)$

(k) $\circ\psi \vee (\psi \wedge \neg\psi)$

BP$[A]$ is obtained by adding to **BP** the axioms from $A \subseteq Ax'$.

Non-deterministic three-valued semantics and cut-free systems for paracomplete systems are obtained similarly to the paraconsistent case.

Further extension: paraconsistent systems without (k)

- Use more complex truth-values, which include the following data concerning a formula ψ:
    1. The truth/falsity of ψ
    2. The truth/falsity of $\neg\psi$
    3. The truth/falsity of $\circ\psi$
- This leads to the use of elements from $\{0, 1\}^3$ as truth-values, where the intended meaning of $v(\psi) = \langle x, y, z\rangle$ is as follows:

$$x = 1 \text{ iff } v(\psi) \in \mathcal{D}$$
$$y = 1 \text{ iff } v(\neg\psi) \in \mathcal{D}$$
$$z = 1 \text{ iff } v(\circ\psi) \in \mathcal{D}$$

- **(N1)** $(\psi \vee \neg\psi)$ leads to the deletion of $\langle 0, 0, 0\rangle$ and $\langle 0, 0, 1\rangle$.
- **(b)** $\circ\psi \supset ((\psi \wedge \neg\psi) \supset \varphi)$ leads to the deletion of $\langle 1, 1, 1\rangle$.

$t = \langle 1, 0, 1\rangle$, $t_I = \langle 1, 0, 0\rangle$, $I = \langle 1, 1, 0\rangle$, $f = \langle 0, 1, 1\rangle$, $f_I = \langle 0, 1, 0\rangle$

Problematic axioms: (**l**) and (**d**)

(**l**) $\neg(\psi \wedge \neg\psi) \supset \circ\psi$ (**d**) $\neg(\neg\psi \wedge \psi) \supset \circ\psi$

Problematic axioms: (l) and (d)

$$\textbf{(l)}\ \neg(\psi \wedge \neg\psi) \supset \circ\psi \quad \textbf{(d)}\ \neg(\neg\psi \wedge \psi) \supset \circ\psi$$

Theorem

If (**l**) $\in A$ *or* (**d**) $\in A$ *then* **BK**[A] *has no finite-valued characteristic Nmatrix.*

Problematic axioms: (l) and (d)

$$(\mathbf{l})\ \neg(\psi \wedge \neg\psi) \supset \circ\psi \quad (\mathbf{d})\ \neg(\neg\psi \wedge \psi) \supset \circ\psi$$

Theorem

If $(\mathbf{l}) \in A$ *or* $(\mathbf{d}) \in A$ *then* $\mathbf{BK}[A]$ *has no finite-valued characteristic Nmatrix.*

Luckily, they have infinitely-valued characteristic Nmatrices, which still:

- guarantee their decidability, and
- induce a method for a modular construction of cut-free sequent calculi for them.

Intuition for infinite-valuedness

- (**l**) and (**d**) involve a conjunction of a formula with its negation.
- We need to be able to isolate the case of a conjunction of an "inconsistent" formula ψ with its negation from the cases of conjunction of ψ with other formulas.
- This requires an infinite number of truth-values, corresponding to the infinitely many formulas of the language.
- The finite Nmatrix $\mathbf{M}^3_{\mathbf{BK}}$ is replaced by an infinite Nmatrix which uses three sets of truth-values:

$$\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\},\ \mathcal{I} = \{\top_i^j \mid i \geq 0, j \geq 0\},\ \mathcal{F} = \{f\}$$

The Nmatrix $\mathbf{M}_0$

$$\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\},\ \mathcal{I} = \{\top_i^j \mid i \geq 0, j \geq 0\},\ \mathcal{F} = \{f\}, \mathcal{D} = \mathcal{T} \cup \mathcal{I}$$

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{if } a \in \mathcal{F} \\ \{\top_i^{j+1}, t_i^{j+1}\} & \text{if } a = \top_i^j \end{cases} \qquad \widetilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

Semantic conditions for (l) and (d)

$$\textbf{(l)}\ \neg(\psi \wedge \neg\psi) \supset \circ\psi \quad \textbf{(d)}\ \neg(\neg\psi \wedge \psi) \supset \circ\psi$$

GC(l): For $a = \top_i^j$ and $b \in \{\top_i^{j+1}, t_i^{j+1}\}$, $a \wedge b \subseteq \mathcal{T}$.

GC(d): For $b = \top_i^j$ and $a \in \{\top_i^{j+1}, t_i^{j+1}\}$, $a \wedge b \subseteq \mathcal{T}$.

Semantic conditions for the rest of the axioms

- Derived similarly to the finite case (replacing **t** with $\mathcal{T}$, and $\top$ with $\mathcal{I}$).
- Example:

$$(\mathbf{a}_\wedge)\quad \circ\psi \wedge \circ\varphi \supset \circ(\psi \wedge \varphi)$$

$$\mathsf{Cond}(\mathbf{a}_\wedge)\text{: if } a, b \in \mathcal{T}\text{, then } a \tilde{\wedge} b \subseteq \mathcal{T}$$

BKIX is decidable

To check whether a given formula φ is provable in **BKIX** (where **X** $\subseteq Ax$), it suffices to check all legal partial valuations v in the corresponding Nmatrix $\mathcal{M}_{\mathbf{BKIX}}$ which assign to subformulas of φ values in

$$\{f\} \cup \{t_i^j \mid 0 \leq i \leq n(\varphi), 0 \leq j \leq k(\varphi)\} \cup$$

$$\{\top_i^j \mid 0 \leq i \leq n(\varphi), 0 \leq j \leq k(\varphi)\}$$

where $n(\varphi)$ is the number of subformulas of φ which do not begin with $\neg$, and $k(\varphi)$ is the maximal number of consecutive negation symbols occurring within φ. This is a finite process.

Example: semantics for da Costa's C_1

da Costa's system **$\mathbf{C_1}$** is decidable, and its semantics is as follows:

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \\ \{\top_i^{j+1}, t_i^{j+1}\} & \text{if } a = \top_i^j \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \text{ and } b \notin \mathcal{I} \\ \mathcal{T} & \text{if } b \in \mathcal{T} \text{ and } a \notin \mathcal{I} \\ \mathcal{D} & \text{otherwise} \end{cases} \qquad a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{T} \text{ and } b \in \mathcal{T} \\ \mathcal{T} & \text{if } a = \top_i^j \text{ and } b \in \{\top_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

No Improvements Possible

Theorem

No logic between **BKl** and **BKlcieo** can have a finite characteristic Nmatrix.

Corollary

$\mathbf{C_1}$ has no finite characteristic Nmatrix.

Gentzen-type rules for (l) and (d)

$$(\mathbf{l})\ \neg(\psi \wedge \neg\psi) \supset \circ\psi \quad (\mathbf{d})\ \neg(\neg\psi \wedge \psi) \supset \circ\psi$$

In **BKl** $\circ\varphi$ is weakly equivalent to $\neg(\varphi \wedge \neg\varphi)$
In **BKd** $\circ\varphi$ is weakly equivalent to $\neg(\neg\varphi \wedge \varphi)$

Accordingly:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \circ\varphi \Rightarrow \Delta}\ (\circ \Rightarrow)$$

$$\Downarrow$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \neg(\varphi \wedge \neg\varphi) \Rightarrow \Delta}\ \mathsf{Rl} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \neg(\neg\varphi \wedge \varphi) \Rightarrow \Delta}\ \mathsf{Rd}$$

Deriving Gentzen-type rules for other axioms

Let v be an $\mathbf{M}$-valuation, where $\mathbf{M}$ is a simple refinement of $\mathbf{M}_0$.

- $v(\psi) \in \mathcal{T}$ iff $\neg\psi \Rightarrow$ is true in v.
- $v(\psi) \in \mathcal{F}$ iff $\psi \Rightarrow$ is true in v.
- $v(\psi) \in \mathcal{I}$ iff $\Rightarrow \psi$ and $\Rightarrow \neg\psi$ are both true in v.
- $v(\psi) \in \mathcal{F} \cup \mathcal{I}$ iff $\Rightarrow \neg\psi$ is true in v.
- $v(\psi) \in \mathcal{T} \cup \mathcal{I}$ iff $\Rightarrow \psi$ is true in v.
- $v(\psi) \in \mathcal{F} \cup \mathcal{T}$ iff $\psi, \neg\psi \Rightarrow$ is true in v.

Reminder: A sequent $\Gamma \Rightarrow \Delta$ is true in v if $v(\psi) \notin \mathcal{D}$ for some $\psi \in \Gamma$ or $v(\psi) \in \mathcal{D}$ for some $\psi \in \Delta$.

Semantic Conditions and Their Induced Rules

	ax	GC(ax)	R(ax)
$(\mathbf{i_1})$	$\neg\circ\varphi \supset \varphi$	for $a \in \mathcal{F}$: $\circ a \subseteq \mathcal{T}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\circ\varphi \Rightarrow \Delta}$
$(\mathbf{i_2})$	$\neg\circ\varphi \supset \neg\varphi$	for $a \in \mathcal{T}$: $\circ a \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg\circ\varphi \Rightarrow \Delta}$
$(\mathbf{a}_\vee)$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi)$	for $a \in \mathcal{T}, b \in \mathcal{T} \cup \mathcal{F}$: $a \vee b \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
		for $a \in \mathcal{T}, b \in \mathcal{T} \cup \mathcal{F}$: $b \vee a \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{a}_\supset)$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \supset \psi)$	for $b \in \mathcal{F}, a \in \mathcal{T} \cup \mathcal{F}$: $b \supset a \subseteq \mathcal{T}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \neg\psi, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$
		for $b \in \mathcal{T}, a \in \mathcal{T} \cup \mathcal{F}$: $a \supset b \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\varphi, \varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$

Summary

- Using the framework of Nmatrices to provide non-deterministic semantics for C-systems.
- A method for a systematic construction of cut-free sequent calculi for C-systems.
  - *Generality:* the method applies to practically every C-system considered in the literature.
  - *Modularity:* each axiom corresponds to a set of Gentzen-type rules, which are easily computed from the semantic conditions induced by the axiom.
  - *Uniformity:* The rules of the obtained calculi have a simple, intuitive and uniform form.