Tutorial on Non-Deterministic Semantics
Part I: Introduction

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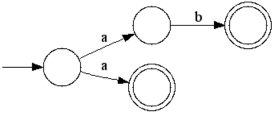
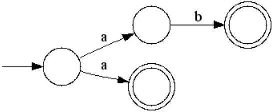


What this tutorial is about

Non-deterministic Matrices:

Incorporating the notion of *"non-deterministic computations"* from automata and computability theory into logical truth-tables.



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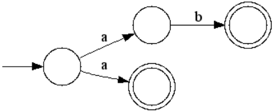

We would like to show:

Non-deterministic semantics is a natural and useful paradigm.

Let's start with a simple question

- What are the roots of the equation $x^2 = 1$?

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Answer:

1 and -1

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Naive answer:

there are no ordinary (real) numbers...

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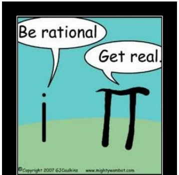
Not satisfied?

introduce imaginary numbers!



Another simple question: what is the semantics of LK?

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

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	$\neg$
t	**f**
f	**t**

		$\vee$
t	**t**	**t**
t	**f**	**t**
f	**t**	**t**
f	**f**	**f**

And of this system?

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

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t	**f**
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t	**t**	**t**
t	**f**	**t**
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	$\neg$
t	**f**
f	**?**

		$\vee$
t	**t**	**t**
t	**f**	**t**
f	**t**	**t**
f	**f**	**f**

Theorem:

The above system has no finite-valued characteristic matrix.

Not satisfied? Introduce non-deterministic matrices!

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

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	$\neg$
t	**f**
f	{**t**,**f**}

		$\vee$
t	**t**	**t**
t	**f**	**t**
f	**t**	**t**
f	**f**	**f**

The idea of non-determinististic semantics

- Shütte (1960) and Girard (1987)
- Ivlev (1998)
- Batens (1998)
- Crawford and Etherington (1998)
- Avron and Lev (2001)

What is the non-deterministic paradigm useful for?

- Modular semantic characterization of non-classical logics (*Logics of Formal (In)consistency*)
- Systematic construction of cut-free sequent systems
- Semantic tools for investigation of proof systems

Overview of Part I - Introduction

Linguistic ambiguity

In many natural languages, "or" has both inclusive and exclusive meanings.

If a mathematician says:

I shall either attack problem A or attack problem B

He may solve both problems, but in some situations he actually means *"...but don't expect me to solve both"*.

Linguistic ambiguity

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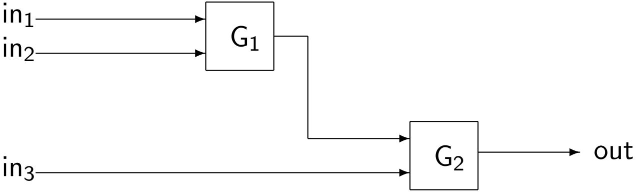
Capturing both meanings:

a	b	a OR b
t	t	{t, f}
t	f	{t}
f	t	{t}
f	f	{f}

Electrical circuits

- Problems in the manufacturing process
- Disturbing noise sources, temperature, etc.
- Adversary operations

$\star$	**t**	**f**
t	$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{t}\}$
f	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$



Evaluation with unknown computation models

- Send $A \vee B$ for evaluation to a distant computer C
- C performs either a parallel or a sequential computation on $A \vee B$.

$\vee$	**f**	**I**	**t**
f	**f**	**I**	**t**
I	**I**	**I**	**t**
t	**t**	**t**	**t**

Kleene

$\vee$	**f**	**I**	**t**
f	**f**	**I**	**t**
I	**I**	**I**	**I**
t	**t**	**t**	**t**

McCarthy

$\vee$	**f**	**I**	**t**
f	{**f**}	{**I**}	{**t**}
I	{**I**}	{**I**}	{**t**, **I**}
t	{**t**}	{**t**}	{**t**}

How we will use non-deterministic truth tables

To characterize interesting logics which had so far no *useful* semantic characterizations.

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To characterize interesting logics which had so far no *useful* semantic characterizations.

Some important questions:

- what is a logic?
- how are logics characterized?
- what characterizations are *useful*?

What is a logic?

1. A formal language $\mathcal{L}$.
2. A consequence relation $\vdash$ for $\mathcal{L}$.

consequence relation (cr) for $\mathcal{L}$

a binary relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas with:

strong reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
transitivity (cut): if $\Gamma \vdash \psi$ and $\Gamma, \psi \vdash \varphi$ then $\Gamma \vdash \varphi$.

Properties of CR $\vdash$ for $\mathcal{L}$

- Structurality: for every uniform $\mathcal{L}$-substitution σ and every Γ and ψ: if $\Gamma \vdash \psi$ then $\sigma(\Gamma) \vdash \sigma(\psi)$.
  Example: $p \wedge q \vdash q$ implies $\varphi \wedge \psi \vdash \psi$ for every $\varphi, \psi \in F_{\mathcal{L}}$.

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Propositional logic:

A pair $\langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a structural, consistent and finitary cr.

How are logics characterized?

- Semantically: $\Gamma \vdash_{\mathsf{S}} \psi$ if every "model" of Γ is a "model" of ψ in the semantics S.
- Syntactically: $\Gamma \vdash_{\mathsf{D}} \psi$ if ψ has a proof from Γ in the deduction system D.

Classical truth tables

$\tilde{\neg}$	
t	**f**
f	**t**

$\tilde{\supset}$	**t**	**f**
t	**t**	**f**
f	**t**	**t**

$\tilde{\wedge}$	**t**	**f**
t	**t**	**f**
f	**f**	**f**

$\tilde{\vee}$	**t**	**f**
t	**t**	**t**
f	**t**	**f**

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f	**t**	**t**

$\tilde{\wedge}$	**t**	**f**
t	**t**	**f**
f	**f**	**f**

$\tilde{\vee}$	**t**	**f**
t	**t**	**t**
f	**t**	**f**

A classical valuation:

any function $v : F_{cl} \to \{t, f\}$, such that

$$v(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

for any connective $\diamond \in \{\neg, \supset, \vee, \wedge\}$.

Semantic way of defining classical logic

- A classical valuation v is a model of an $\mathcal{L}$-formula ψ if $v(\psi) = \mathbf{t}$. v is a model of a theory Γ if v is a model of every $\psi \in \Gamma$.
- $\Gamma \vdash_{CPL} \psi$ if every classical model of Γ is a model of ψ.

Hilbert-style proof systems

A Hilbert-style proof system for $\mathcal{L}$

- an (effective) set of axioms
- an (effective) set of inference rules

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A proof of ψ from Γ in **H**:

a finite sequence of $\mathcal{L}$-formulas, where the last formula is ψ, and each formula is:

- an axiom of **H**, or
- a member of Γ, or
- is obtained from previous formulas in the sequence by applying some inference rule of **H**.

HCL

- Axiom schemata:

I1 $\varphi \supset (\psi \supset \varphi)$
I2 $(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$
I3 $((\psi \supset \varphi) \supset \psi) \supset \psi$
C1 $\varphi \wedge \psi \supset \varphi$
C2 $\varphi \wedge \psi \supset \psi$
C3 $\varphi \supset (\psi \supset \varphi \wedge \psi)$
D1 $\varphi \supset \varphi \vee \psi$
D2 $\psi \supset \varphi \vee \psi$
D3 $(\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$
N1 $\neg\varphi \vee \varphi$
N2 $(\varphi \wedge \neg\varphi) \supset \psi$

- Inference Rule:

$$\frac{\psi \quad \psi \supset \varphi}{\varphi}\ MP$$

HCL

- Axiom schemata:

I1 $\varphi \supset (\psi \supset \varphi)$
I2 $(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$
I3 $((\psi \supset \varphi) \supset \psi) \supset \psi$
C1 $\varphi \wedge \psi \supset \varphi$
C2 $\varphi \wedge \psi \supset \psi$
C3 $\varphi \supset (\psi \supset \varphi \wedge \psi)$
D1 $\varphi \supset \varphi \vee \psi$
D2 $\psi \supset \varphi \vee \psi$
D3 $(\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$
N1 $\neg\varphi \vee \varphi$
N2 $(\varphi \wedge \neg\varphi) \supset \psi$

- Inference Rule:

$$\frac{\psi \quad \psi \supset \varphi}{\varphi}\ MP$$

Soundness and completeness theorem for CPL:

$$\Gamma \vdash_{HCL} \psi \iff \Gamma \vdash_{CPL} \psi$$

Gentzen-style proof systems

- Hilbert-style systems operate on $\mathcal{L}$-formulas. Gentzen-style systems operate on *sequents*.
- A sequent: an expression of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of $\mathcal{L}$-formulas.

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A standard Gentzen-type system for $\mathcal{L}$:

1. Standard axioms: $\psi \Rightarrow \psi$.
2. Structural Weakening and Cut rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathit{Weakening}) \qquad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}\ (\mathit{Cut})$$

3. Logical introduction rules for the connectives of $\mathcal{L}$.

The system GCPL

$$\psi \Rightarrow \psi$$

(*Weakening*) $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$ (*Cut*) $\dfrac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}$

$(\neg \Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$ $(\Rightarrow \neg)$ $\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$

$(\supset\Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta}$ $(\Rightarrow\supset)$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$

$(\wedge \Rightarrow)$ $\dfrac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$ $(\Rightarrow \wedge)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$

$(\vee \Rightarrow)$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$ $(\Rightarrow \vee)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$

Completeness and cut-elimination

A Gentzen-style system admits cut-elimination if whenever $\Gamma \Rightarrow \Delta$ is provable in G, $\Gamma \Rightarrow \Delta$ also has a cut-free proof in G.

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Completeness Theorem for Classical Logic:

- $\Gamma \vdash_{GCPL} \psi$ iff $\Gamma \vdash_{CPL} \psi$
- GCPL admits cut-elimination

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Completeness Theorem for Classical Logic:

- $\Gamma \vdash_{GCPL} \psi$ iff $\Gamma \vdash_{CPL} \psi$
- GCPL admits cut-elimination

Corollary: the subformula property

If $\Gamma \Rightarrow \Delta$ has a derivation in GCPL, then all the formulas in this derivation are subformulas of the formulas in $\Gamma \Rightarrow \Delta$.

Basic principles of classical logic

- Bivalence:
  Every proposition is either true or false (there are exactly two truth-values).
- Inconsistency Intolerance:
  A proposition and its negation cannot be both true.
- Truth-Functionality:
  The truth-value of a complex proposition is uniquely defined by the truth-values of its constituents.

Many-valued logic - motivation

- Sometimes incomplete information prevents us from telling if something is true or not.
- Łukasiewicz, "On Determinism", 1922: *If statements about future events are already true or false, then the future is as much determined as the past and differs from the past only in so far as it has not yet come to pass.*
- The idea: reject Bivalence by adding a third truth-value **I**, to be read as "possible".

Three-valued Łukasiewicz logic

$\supset$	**f**	**I**	**t**
f	**t**	**t**	**t**
I	**I**	**t**	**t**
t	**f**	**I**	**t**

$\neg$	
f	**t**
I	**I**
t	**f**

- A legal valuation v is a Łuk-model of a formula ψ if $v(\psi) = \mathbf{t}$. v is a Łuk-model of a theory Γ if v is a Łuk-model of every $\psi \in \Gamma$.
- $\Gamma \vdash_{Luk} \psi$ if every Łuk-model of Γ is a Łuk-model of ψ.

Kleene and McCarthy logics

- Modelling parallel vs. sequential computation
- The third truth-value **I** - for "undefined"
- Negation is defined like in Łukasiewicz three-valued logic.
- The notion of a model of a formula and the associated cr are defined like in Łukasiewicz three-valued logic.

$\vee$	**f**	**I**	**t**
f	**f**	**I**	**t**
I	**I**	**I**	**t**
t	**t**	**t**	**t**

Kleene

$\vee$	**f**	**I**	**t**
f	**f**	**I**	**t**
I	**I**	**I**	**I**
t	**t**	**t**	**t**

McCarthy

General semantic method for defining logics

A denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$

- S - a non-empty set of "valuations" (usually mappings from formulas to "truth values")
- $\models_S$ - a "satisfaction" relation between "valuations" and formulas

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- $\models_S$ - a "satisfaction" relation between "valuations" and formulas

- For $v \in S$, v is a S-model of ψ if $v \models_S \psi$. v is an S-model of Γ if $v \models_S \psi$ for every $\psi \in \Gamma$.
- $\Gamma \vdash_{\mathsf{S}} \psi$ if every S-model of Γ is an S-model of ψ.
- For any denotational semantics $\mathsf{S} = \langle S, \models_{\mathsf{S}} \rangle$ for $\mathcal{L}$, $\vdash_{\mathsf{S}}$ is a cr.

Many-valued matrices as denotational semantics

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a matrix for a propositional language $\mathcal{L}$ if:

- $\mathcal{V}$ is a nonempty set of truth-values,
- $\emptyset \neq \mathcal{D} \subset \mathcal{V}$ (the set of designated truth-values),
- for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes an operation $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$

Examples

- In classical logic: $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}$, while $\mathcal{D} = \{\mathbf{t}\}$.

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- In classical logic: $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}$, while $\mathcal{D} = \{\mathbf{t}\}$.
- In the 3-valued logics of Łukasiewicz, Kleene, and McCarthy: $\mathcal{V} = \{\mathbf{t}, \mathbf{f}, \mathbf{I}\}$, while $\mathcal{D} = \{\mathbf{t}\}$.

Valuations

- A valuation v in a matrix $\mathbf{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is any function from the set of $\mathcal{L}$-formulas to $\mathcal{V}$ such that:

$$v(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$$

- v is a model of an $\mathcal{L}$-formula ψ in $\mathbf{M}$, denoted by $v \models_{\mathbf{M}} \psi$, if $v(\psi) \in \mathcal{D}$.
- v is a model of a theory Γ in $\mathbf{M}$, denoted by $v \models_{\mathbf{M}} \Gamma$, if v is a model of every $\psi \in \Gamma$.

Logics induced by matrices

- $\Gamma \vdash_{\mathbf{M}} \psi$ if for every valuation v in $\mathbf{M}$: $v \models_{\mathbf{M}} \Gamma$ implies $v \models_{\mathbf{M}} \psi$.

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- $\Gamma \vdash_{\mathbf{M}} \psi$ if for every valuation v in **M**: $v \models_{\mathbf{M}} \Gamma$ implies $v \models_{\mathbf{M}} \psi$.
- Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be some logic.
  L is sound for a matrix **M** if $\vdash \subseteq \vdash_{\mathbf{M}}$.
  L is complete for **M** if $\vdash_{\mathbf{M}} \subseteq \vdash$.
  M is a characteristic matrix for **L** if **L** is sound and complete for **M**.
- For any matrix **M** for $\mathcal{L}$, $\langle \mathcal{L}, \vdash_{\mathbf{M}} \rangle$ is a propositional logic.

Logics induced by matrices

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  $\mathbf{L}$ is sound for a matrix $\mathbf{M}$ if $\vdash \subseteq \vdash_{\mathbf{M}}$.
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  $\mathbf{M}$ is a characteristic matrix for $\mathbf{L}$ if $\mathbf{L}$ is sound and complete for $\mathbf{M}$.
- For any matrix $\mathbf{M}$ for $\mathcal{L}$, $\langle \mathcal{L}, \vdash_{\mathbf{M}} \rangle$ is a propositional logic.
- *Converse direction: is every propositional logic induced by a matrix?*

Logics induced by matrices and their families

Logic induced by family C of matrices:

$\Gamma \vdash_C \psi$ if for every $\mathbf{M} \in C$: $\Gamma \vdash_{\mathbf{M}} \psi$.

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- Every propositional logic is induced by some family of matrices.
- A set of $\mathcal{L}$-formulas (theory) Γ is $\vdash$-consistent if there exists some $\mathcal{L}$-formula ψ such that $\Gamma \not\vdash \psi$.
- A logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is uniform if for every two theories Γ_1, Γ_2 and an $\mathcal{L}$-formula ψ: $\Gamma_1 \vdash \psi$ whenever $\Gamma_1, \Gamma_2 \vdash \psi$ and Γ_2 is a $\vdash$-consistent theory with no atoms in common with $\Gamma_1 \cup \{\psi\}$.

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Theorem (Łos & Suszko)

A finitary propositional logic has a characteristic matrix iff it is uniform.

Non-deterministic matrices

A non-deterministic matrix (Nmatrix) for $\mathcal{L}$ is a tuple $\mathbf{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$:

- $\mathcal{V}$ - the set of truth-values,
- $\mathcal{D}$ - the set of designated truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to P^+(\mathcal{V})$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

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Ordinary matrices are a special case:

each $\widetilde{\diamond}$ is a function taking singleton values only (*can be treated as a function* $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$).

Example 1: two truth values

$$\mathcal{L} = \{\vee, \wedge, \supset, \neg\}$$

$$\mathcal{V} = \{\mathbf{f}, \mathbf{t}\}, \mathcal{D} = \{\mathbf{t}\}$$

- $\vee, \wedge$ and $\supset$ are interpreted classically
- $\neg$ satisfies law of excluded middle ($\neg\varphi \vee \varphi$), but not law of contradiction ($\neg(\varphi \wedge \neg\varphi)$).
- The Nmatrix $\mathbf{M}^2 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$:

		$\widetilde{\vee}$	$\widetilde{\wedge}$	$\widetilde{\supset}$
t	**t**	{**t**}	{**t**}	{**t**}
t	**f**	{**t**}	{**f**}	{**f**}
f	**t**	{**t**}	{**f**}	{**t**}
f	**f**	{**f**}	{**f**}	{**t**}

	$\widetilde{\neg}$
t	{**t**, **f**}
f	{**t**}

Example 2: three truth values

- Two 3-valued Nmatrices with $\mathcal{V} = \{\mathbf{f}, \mathbf{I}, \mathbf{t}\}, \quad \mathcal{D} = \{\mathbf{I}, \mathbf{t}\}$

$$\mathcal{M}^3_L = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_L \rangle \qquad \mathcal{M}^3_S = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_S \rangle$$

- Standard interpretations of disjunction, conjunction and implication:

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a = b = \mathbf{f} \end{cases} \qquad a \tilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if either } a = \mathbf{f} \text{ or } b = \mathbf{f} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a = \mathbf{f} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a \in \mathcal{D} \text{ and } b = \mathbf{f} \end{cases}$$

- Negation:

$\mathcal{M}^3_L$:

	$\tilde{\neg}$
t	$\{\mathbf{f}\}$
I	$\mathcal{V}$
f	$\{\mathbf{t}\}$

$\mathcal{M}^3_S$:

	$\tilde{\neg}$
t	$\{\mathbf{f}\}$
I	$\mathcal{D}$
f	$\{\mathbf{t}\}$

Non-deterministic matrices as denotational semantics

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ - an Nmatrix for $\mathcal{L}$.

- An **M**-valuation v is any function from $\mathcal{L}$-formulas to $\mathcal{V}$ which satisfies:

$$v(\diamond(\psi_1, ..., \psi_n)) \in \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$$

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- $v \models_{\mathbf{M}} \psi$ if $v(\psi) \in \mathcal{D}$.
- $v \models_{\mathbf{M}} \Gamma$ if $v(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma$.

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- $v \models_{\mathbf{M}} \Gamma$ if $v(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma$.

Consequence relation induced by a Nmatrix

$\Gamma \vdash_{\mathbf{M}} \psi$ if for every **M**-valuation v: $v \models_{\mathbf{M}} \Gamma$ implies $v \models_{\mathbf{M}} \psi$.

The Nmatrix $\mathbf{M}^2$ revisited

		$\widetilde{\vee}$	$\widetilde{\wedge}$	$\widetilde{\supset}$
t	**t**	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
t	**f**	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
f	**t**	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$

	$\widetilde{\neg}$
t	$\{\mathbf{t}, \mathbf{f}\}$
f	$\{\mathbf{t}\}$

- Any $\mathbf{M}^2$-valuation satisfies $\neg\psi \vee \psi$ but not necessarily $\psi \supset \neg\neg\psi$:

$$v(p) = \mathbf{t} \quad v(\neg p) = \mathbf{t} \quad v(\neg\neg p) = \mathbf{f}$$

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		$\tilde{\vee}$	$\tilde{\wedge}$	$\tilde{\supset}$
t	**t**	{**t**}	{**t**}	{**t**}
t	**f**	{**t**}	{**f**}	{**f**}
f	**t**	{**t**}	{**f**}	{**t**}
f	**f**	{**f**}	{**f**}	{**t**}

	$\tilde{\neg}$
t	{**t**, **f**}
f	{**t**}

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- The logic generated by $\mathbf{M}^2$ is known as the basic adaptive paraconsistent logic *CLuN* introduced by Batens.

Reminder: **HCL**

- Axiom schemata:

I1	$\varphi \supset (\psi \supset \varphi)$
I2	$(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$
I3	$((\psi \supset \varphi) \supset \psi) \supset \psi$
C1	$\varphi \wedge \psi \supset \varphi$
C2	$\varphi \wedge \psi \supset \psi$
C3	$\varphi \supset (\psi \supset \varphi \wedge \psi)$
D1	$\varphi \supset \varphi \vee \psi$
D2	$\psi \supset \varphi \vee \psi$
D3	$(\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$
N1	$\neg\varphi \vee \varphi$
N2	$(\varphi \wedge \neg\varphi) \supset \psi$

- Inference Rule:

$$\frac{\psi \quad \psi \supset \varphi}{\varphi} \; MP$$

$\mathbf{H}_{CluN}$

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N1	$\neg\varphi \vee \varphi$
N2	$(\varphi \wedge \neg\varphi) \supset \psi$

- Inference Rule:

$$\frac{\psi \quad \psi \supset \varphi}{\varphi}\ MP$$

Soundness and completeness theorem:

$$\Gamma \vdash_{\mathbf{H}_{CLuN}} \psi \iff \Gamma \vdash_{\mathbf{M}^2} \psi$$

The system GCPL

$$\psi \Rightarrow \psi$$

(*Weakening*) $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$ (*Cut*) $\dfrac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}$

$(\neg \Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$ $(\Rightarrow \neg)$ $\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$

$(\supset\Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta}$ $(\Rightarrow\supset)$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$

$(\wedge \Rightarrow)$ $\dfrac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$ $(\Rightarrow \wedge)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$

$(\vee \Rightarrow)$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$ $(\Rightarrow \vee)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$

The system G_{CLuN}

$$\psi \Rightarrow \psi$$

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$(\supset\Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta}$ $(\Rightarrow\supset)$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$

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Soundness and completeness theorem:

$$T \vdash_{G_{CLuN}} \psi \;\Leftrightarrow\; T \vdash_{\mathbf{M}^2} \psi$$

Reminder: $T \vdash_G \psi$ *if there is some finite* $\Gamma \subseteq T$, *such that* $\vdash_G \Gamma \Rightarrow \psi$.

$\mathcal{M}^3_L$ and $\mathcal{M}^3_S$ revisited

$$\mathcal{V} = \{\mathbf{f}, \mathbf{I}, \mathbf{t}\}, \quad \mathcal{D} = \{\mathbf{I}, \mathbf{t}\}$$

$$\mathcal{M}^3_L = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_L \rangle \qquad \mathcal{M}^3_S = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_S \rangle$$

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a = b = \mathbf{f} \end{cases} \qquad a \widetilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if either } a = \mathbf{f} \text{ or } b = \mathbf{f} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a = \mathbf{f} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a \in \mathcal{D} \text{ and } b = \mathbf{f} \end{cases}$$

$\mathcal{M}^3_L$:

	$\widetilde{\neg}$
t	$\{\mathbf{f}\}$
I	$\mathcal{V}$
f	$\{\mathbf{t}\}$

$\mathcal{M}^3_S$:

	$\widetilde{\neg}$
t	$\{\mathbf{f}\}$
I	$\mathcal{D}$
f	$\{\mathbf{t}\}$

Proof system for $\mathcal{M}_L^3$ and $\mathcal{M}_S^3$

- Let the Gentzen-style system GC_{min} be obtained by replacing the rule $(\neg \Rightarrow)$ in GCPL with the rule:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$$

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- Let the Gentzen-style system GC_{min} be obtained by replacing the rule $(\neg \Rightarrow)$ in GCPL with the rule:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$$

Theorem

$T \vdash_{GC_{min}} \psi$ iff $T \vdash_{\mathbf{M}^3_L} \psi$ iff $T \vdash_{\mathbf{M}^3_S} \psi$.

The logic $\mathbf{C_{min}}$

- As the negation rules

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Rightarrow \Delta, \neg\varphi}$$

  of GC_{min} translate to $(\neg\neg\varphi \supset \varphi)$ and $(\neg\varphi \vee \varphi)$, respectively, GC_{min} is equivalent to the system HC_{min} obtained by adding the above axiom schemes to HCL^+.
- The system HC_{min} represents the paraconsistent logic $\mathbf{C_{min}}$ (studied by Carnielli and Marcos).

Expressive power of Nmatrices

M - a two-valued Nmatrix with at least one proper non-deterministic operation.

- There is no finite family of finite ordinary matrices C, such that $\vdash_{\mathbf{M}} = \vdash_C$.
- If **M** includes the classical implication, then there is even no finite family of finite ordinary matrices C such that $\vdash_{\mathbf{M}} \psi$ iff $\vdash_C \psi$.

Analyticity, decidability and compactness

- An obvious, yet crucial fact: any partial **M**-legal valuation defined on a set of formulas closed under subformulas can be extended to a full **M**-legal valuation.

Analyticity, decidability and compactness

- An obvious, yet crucial fact: any partial **M**-legal valuation defined on a set of formulas closed under subformulas can be extended to a full **M**-legal valuation.
- If **M** is finite then this entails that $\vdash_{\mathbf{M}}$ is:
  - Decidable
  - Finitary (the compactness theorem obtains)

Refinements

Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}$.

- $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$ if:
    1. $\mathcal{V}_1 \subseteq \mathcal{V}_2$
    2. $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$
    3. $\widetilde{\diamond}_{\mathcal{M}_1}(x_1 \dots x_n) \subseteq \widetilde{\diamond}_{\mathcal{M}_2}(x_1 \dots x_n)$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $x_1 \dots x_n \in \mathcal{V}_1$.
- If $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

Examples of refinements

- $\mathcal{M}_S^3$ is a refinement of $\mathcal{M}_L^3$.
- The classical two-valued matrix is a refinement of $\mathcal{M}_S^3$.
- The classical two-valued matrix is also a refinement of $\mathcal{M}^2$.

$\mathcal{M}_L^3$:

	$\tilde{\neg}$
t	$\{\mathbf{f}\}$
I	$\mathcal{V}$
f	$\{\mathbf{t}\}$

$\mathcal{M}_S^3$:

	$\tilde{\neg}$
t	$\{\mathbf{f}\}$
I	$\mathcal{D}$
f	$\{\mathbf{t}\}$

$\mathcal{M}^2$:

	$\tilde{\neg}$
t	$\{\mathbf{t}, \mathbf{f}\}$
f	$\{\mathbf{t}\}$

Canonical Gentzen-type systems - outline

- Canonical Gentzen-type systems: abstract systems with all the standard structural rules and logical rules of a natural form.
- Two-valued Nmatrices allow to provide for these systems:
  - simple semantics
  - constructive semantic characterization of proof-theoretical properties: *cut-elimination*, *invertibility of rules*, *axiom-expansion*.

What is a canonical rule?

- An "ideal" logical rule: an introduction rule for *exactly one connective*, on *exactly one side of a sequent*.
- In its formulation: *exactly one occurrence* of the introduced connective, no other occurrences of other connectives.
- The rule should also be *pure* (i.e. context-independent): no side conditions limiting its application.
- Its active formulas: *immediate subformulas* of its principal formula.

What is a canonical rule?

Stage 1.

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

Stage 2.

$$\frac{\psi, \varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi}$$

Stage 3.

$$\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \ ; \ \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

Example 1

Conjunction rules:

$$\{p_1, p_2 \Rightarrow\} \ / \ p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \ ; \ \Rightarrow p_2\} \ / \ \Rightarrow p_1 \wedge p_2$$

Their applications:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

Example 2

Implication rules:

$$\{p_1 \Rightarrow p_2\} \;/\; \Rightarrow p_1 \supset p_2 \qquad \{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; p_1 \supset p_2 \Rightarrow$$

Their applications:

$$\frac{\Gamma, \psi \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$$

Example 3

Semi-implication rules:

$$\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \rightsquigarrow p_2 \Rightarrow \qquad \{\Rightarrow p_2\} \ / \ \Rightarrow p_1 \rightsquigarrow p_2$$

Their applications:

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \rightsquigarrow \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \rightsquigarrow \varphi}$$

Example 4

"Tonk" rules:

$$\{p_2 \Rightarrow\} \ / \ p_1 T p_2 \Rightarrow \qquad \{\Rightarrow p_1\} \ / \ \Rightarrow p_1 T p_2$$

Their applications:

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi T \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi T \psi}$$

Canonical systems

Reminder: a standard Gentzen-type system for $\mathcal{L}$:

1. Standard axioms: $\psi \Rightarrow \psi$.
2. Structural Weakening and Cut rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\textit{Weakening}) \qquad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}\ (\textit{Cut})$$

3. Logical introduction rules for the connectives of $\mathcal{L}$.

Canonical systems

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1. Standard axioms: $\psi \Rightarrow \psi$.
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3. Logical introduction rules for the connectives of $\mathcal{L}$.

- A standard Gentzen-type system is canonical if each of its logical (i.e. non-structural) rules is canonical.
- Examples: GCPL, G_{CLuN} (but not G_{Cmin})

What sets of rules are acceptable?

- If G is a canonical system, then $\vdash_G$ is a structural and finitary cr.

What sets of rules are acceptable?

- If G is a canonical system, then $\vdash_G$ is a structural and finitary cr.
- But is it a logic? i.e., is it also consistent?

Coherence

- A canonical calculus G is *coherent* if for every pair of rules $\Theta_1 / \Rightarrow \diamond(p_1, \ldots, p_n)$ and $\Theta_2 / \diamond(p_1, \ldots, p_n) \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically unsatisfiable (and so inconsistent, i.e., the empty sequent can be derived from it using only cuts)
- For a canonical calculus G, $\vdash_G$ is a logic iff G is coherent.

Coherent calculi:

$\{p_1, p_2 \Rightarrow\} \ / \ p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \ ; \ \Rightarrow p_2\} \ / \ \Rightarrow p_1 \wedge p_2$

$\{p_1 \Rightarrow p_2\} \ / \ \Rightarrow p_1 \supset p_2 \qquad \{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \supset p_2 \Rightarrow$

$\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \leadsto p_2 \Rightarrow \qquad \{\Rightarrow p_2\} \ / \ \Rightarrow p_1 \leadsto p_2$

$\{p_1 \Rightarrow\} \ / \ \Rightarrow \neg p_1 \qquad \{\Rightarrow p_1\} \ / \ \neg p_1 \Rightarrow$

Non-coherent: "Tonk"!

$$\{p_2 \Rightarrow\} \ / \ p_1 T p_2 \Rightarrow \qquad \{\Rightarrow p_1\} \ / \ \Rightarrow p_1 T p_2$$

From these rules, we can derive $p \Rightarrow q$ for any p, q:

$$\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow pTq} \qquad \dfrac{q \Rightarrow q}{pTq \Rightarrow q}}{p \Rightarrow q}$$

Every coherent calculus has a characteristic 2Nmatrix

G_0 - the canonical calculus over $\{\wedge, \leadsto\}$ with no canonical rules whatsoever.

ψ	φ	$\psi \wedge \varphi$
t	t	$\{\mathbf{t},\mathbf{f}\}$
t	f	$\{\mathbf{t},\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t},\mathbf{f}\}$

ψ	φ	$\psi \leadsto \varphi$
t	t	$\{\mathbf{t},\mathbf{f}\}$
t	f	$\{\mathbf{t},\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t},\mathbf{f}\}$

Every coherent calculus has a characteristic 2Nmatrix

Add the rule $\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$, which can be split in $\{p_1 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$ and $\{p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$

ψ	φ	$\psi \wedge \varphi$
t	t	$\{\mathbf{t},\mathbf{f}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{f}\}$
f	f	$\{\mathbf{f}\}$

ψ	φ	$\psi \rightsquigarrow \varphi$
t	t	$\{\mathbf{t},\mathbf{f}\}$
t	f	$\{\mathbf{t},\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t},\mathbf{f}\}$

Every coherent calculus has a characteristic 2Nmatrix

Add the rule $\{\Rightarrow p_1; \Rightarrow p_2\} / \Rightarrow p_1 \wedge p_2$

ψ	φ	$\psi \wedge \varphi$
t	t	$\{\mathbf{t}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{f}\}$
f	f	$\{\mathbf{f}\}$

ψ	φ	$\psi \leadsto \varphi$
t	t	$\{\mathbf{t},\mathbf{f}\}$
t	f	$\{\mathbf{t},\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t},\mathbf{f}\}$

Every coherent calculus has a characteristic 2Nmatrix

Add the rule $\{\Rightarrow p_2\} / \Rightarrow p_1 \rightsquigarrow p_2$

ψ	φ	$\psi \wedge \varphi$
t	t	$\{\mathbf{t}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{f}\}$
f	f	$\{\mathbf{f}\}$

ψ	φ	$\psi \rightsquigarrow \varphi$
t	t	$\{\mathbf{t}\}$
t	f	$\{\mathbf{t},\mathbf{f}\}$
f	t	$\{\mathbf{t}\}$
f	f	$\{\mathbf{t},\mathbf{f}\}$

Every coherent calculus has a characteristic 2Nmatrix

Add the rule $\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\}/p_1 \rightsquigarrow p_2 \Rightarrow$

ψ	φ	$\psi \wedge \varphi$
t	t	$\{\mathbf{t}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{f}\}$
f	f	$\{\mathbf{f}\}$

ψ	φ	$\psi \rightsquigarrow \varphi$
t	t	$\{\mathbf{t}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{t}\}$
f	f	$\{\mathbf{t},\mathbf{f}\}$

Every 2Nmatrix has a corresponding coherent calculus

p_1	p_2	$p_1 \circ p_2$
t	t	$\{\mathbf{f}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t}\}$

$$\{\Rightarrow p_1 \ ; \ \Rightarrow p_2\} \ / \ p_1 \circ p_2 \Rightarrow$$

$$\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \circ p_2 \Rightarrow$$

$$\{p_1 \Rightarrow \ ; \ p_2 \Rightarrow\} \ / \ \Rightarrow p_1 \circ p_2$$

Exact correspondence

Theorem: If G is a canonical calculus, then the following statements are equivalent:

1. $\vdash_G$ is consistent (and so it is a logic).
2. G is coherent.
3. G has a characteristic 2Nmatrix.
4. G admits cut-elimination.

Every 2Nmatrix has a corresponding coherent calculus

p_1	p_2	$p_1 \circ p_2$
t	t	$\{\mathbf{f}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t}\}$

$$\{\Rightarrow p_1 \ ; \ \Rightarrow p_2\} \ / \ p_1 \circ p_2 \Rightarrow$$

$$\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \circ p_2 \Rightarrow$$

$$\{p_1 \Rightarrow \ ; \ p_2 \Rightarrow\} \ / \ \Rightarrow p_1 \circ p_2$$

This is not the most efficient form - this calculus can be "normalized".

Canonical calculi in normal form

- A canonical calculus G is in normal form if each connective G has at most one left introduction rule and at most one right introduction rule.
- Any canonical calculus can be transformed into an equivalent normal form.

Example

Consider the following two rules for the binary connective X (representing XOR):

$$\{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; \Rightarrow p_1 X p_2 \quad \{\Rightarrow p_2 \;;\; p_1 \Rightarrow\} \;/\; \Rightarrow p_1 X p_2$$

$$\Downarrow$$

$$\{\Rightarrow p_1, p_2 \;;\; p_1 \Rightarrow p_1 \;;\; p_2 \Rightarrow p_2 \;;\; p_1, p_2 \Rightarrow\} \;/\; \Rightarrow p_1 X p_2$$

$$\Downarrow$$

$$\{\Rightarrow p_1, p_2 \;;\; p_1, p_2 \Rightarrow\} \;/\; \Rightarrow p_1 X p_2$$

The XOR connective

p_1	p_2	$p_1 X p_2$
t	t	$\{\mathbf{f}\}$
t	f	$\{\mathbf{t}\}$
f	t	$\{\mathbf{t}\}$
f	f	$\{\mathbf{f}\}$

$$\{\Rightarrow p_1, p_2 \ ; \ p_1, p_2 \Rightarrow\} / \Rightarrow p_1 X p_2$$

$$\{p_1 \Rightarrow p_2 \ ; \ p_2 \Rightarrow p_1\} / \ p_1 X p_2 \Rightarrow$$

When are canonical rules invertible?

Invertibility of a rule R of G

R is *invertible in* G if for every application of R it holds that whenever its conclusion is provable in G, also each of its premises is provable in G.

Example:

$$R_1 = \{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad R_2 = \{\Rightarrow p_1 \ ; \ \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

Invertibility of R_1:

$$\frac{\Gamma, \psi_1 \wedge \psi_2 \Rightarrow \Delta \qquad \dfrac{\Gamma, \psi_1 \Rightarrow \Delta, \psi_1 \quad \Gamma, \psi_2 \Rightarrow \Delta, \psi_2}{\Gamma, \psi_1, \psi_2 \Rightarrow \Delta, \psi_1 \wedge \psi_2}}{\Gamma, \psi_1, \psi_2 \Rightarrow \Delta}$$

When are canonical rules invertible?

- Calculus G_1:

$$\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \ ; \ \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

- An (equivalent) calculus G_2:

$$\{p_1 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$$

$$\{\Rightarrow p_1 \ ; \ \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

The first rules are NOT invertible: $\vdash_{G_2} \psi_1 \wedge \psi_2 \Rightarrow \psi_1$, but $\not\vdash_{G_2} \psi_1 \Rightarrow \psi_2$.

The reason: G_2 is not in normal form!

Axiom expansion

- Axiom expansion in a calculus allows for a reduction of logical axioms to the atomic case.
- An n-ary connective $\diamond$ admits axiom expansion in a calculus G if whenever $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ is provable in G, it has a cut-free derivation in G from atomic axioms of the form $\{p_i \Rightarrow p_i\}_{1 \leq i \leq n}$.

A yet another correspondence

Theorem

Let G be a coherent canonical calculus in normal form. The following statements are equivalent:

1. *The rules of G are invertible,*
2. *G has a characteristic two-valued deterministic matrix.*
3. *Every connective of $\mathcal{L}$ admits axiom expansion in G.*

Canonical Gentzen-type systems - summary

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- 2Nmatrices correspond to the class of canonical systems which are coherent and admit cut-elimination.

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Canonical Gentzen-type systems - summary

- 2Nmatrices correspond to the class of canonical systems which are coherent and admit cut-elimination.
- Deterministic 2Nmatrices correspond to the class of normal canonical systems with invertible rules and axiom-expansion.
- Modular semantics
- Simple constructive characterization of syntactic properties

Next part

Another application of the framework of Nmatrices for Logics of Formal (In)consistency:

- Modular non-deterministic semantics
- Systematic construction of cut-free proof systems

Thank you for your attention!