Canonical Signed Calculi, Non-deterministic Matrices and Cut-elimination

Arnon Avron and Anna Zamansky

School of Computer Science
Tel-Aviv University, Ramat Aviv, 69978, Israel

Abstract. Canonical propositional Gentzen-type calculi are a natural class of systems which in addition to the standard axioms and structural rules have only logical rules where exactly one occurrence of a connective is introduced and no other connective is mentioned. Cut-elimination in such systems is fully characterized by a syntactic constructive criterion of coherence. In this paper we extend the theory of canonical systems to the considerably more general class of *signed calculi*. We show that the extended criterion of coherence fully characterizes only analytic cut-elimination in such calculi, while for characterizing strong and standard cut-elimination a stronger criterion of density is required. Modular semantics based on non-deterministic matrices are provided for every coherent canonical signed calculus.

1 Introduction

The possibility to eliminate cuts is a crucial property of useful sequent calculi. This property was first established by Gerhard Gentzen in his classical paper "Investigations Into Logical Deduction" ([12]) for sequent calculi for classical and intuitionistic logic. Since then many other cut-elimination[1] theorems, for many systems, have been proved by various methods. Now showing that a given sequent calculus admits cut-elimination is a difficult task, often carried out using heavy syntactic arguments and based on many case-distinctions. It is thus important to have some useful simple criteria that *characterize* cut-elimination (i.e., conditions which are both necessary and sufficient for having an appropriate cut-elimination theorem).

In the same seminal paper ([12]) Gentzen also established an important tradition in the philosophy of logic, according to which the syntactic rules of a proof system determine the semantic meaning of a logical connective in proof systems of an "ideal type". In [2] the idea of such "well-behaved" propositional Gentzen-type systems was formalized by defining "canonical rules and systems" in precise terms. These are systems which in addition to the standard axioms

[1] We note that by 'cut-elimination' we shall mean in this paper the *existence* of proofs without (certain forms of) cuts, rather than an algorithm to transform a given proof to a cut-free one (the term "cut-admissibility" is sometimes used for cases without non-logical axioms, but this notion is too weak for our purposes).

and structural rules have only logical rules where exactly one occurrence of a connective is introduced and no other connective is mentioned. In these systems cut-elimination is fully characterized by a constructive syntactic criterion of *coherence*. Moreover, the coherence of a canonical system is equivalent to the existence of a semantic characterization of this system in terms of two-valued *non-deterministic matrices* (Nmatrices), a natural generalization of the standard multi-valued matrices (see e.g. [14]).

In this paper we extend the theory of canonical systems, in particular the characterization of various forms of cut-elimination, to the class of *signed calculi* ([7, 10]), of which Gentzen-type systems are particular (two-signed) instances. We show that canonical signed calculi are indeed "well-behaved" in the two senses discussed above. First of all, simple and constructive criteria for characterizing various notions of cut-elimination can be defined for these calculi. Namely, we show that the criterion of *coherence*, extended to the context of signed calculi, fully characterizes only *analytic* cut-elimination in such calculi, while for characterizing strong and standard cut-elimination, a strictly stronger criterion of *density* is required. Secondly, we use finite Nmatrices to provide semantics for canonical signed calculi, and demonstrate that the principle of *modularity* of Nmatrices, which was studied in the context of various non-classical logics (see, e.g. [4]), but never discussed in the context of canonical systems, applies also in this context. We start by providing semantics for the most basic canonical system, and then proceed to show that the semantics of a more complex system is obtained by straightforwardly combining the semantic effects of each of the added rules. As a result, the semantic effect of each syntactic rule taken separately can be analyzed (which is impossible in standard multi-valued matrices). This provides a concrete interpretation of Gentzen's thesis that the meaning of a logical connective is dictated by its introduction rules.

2 Preliminaries

In what follows, $\mathcal{L}$ is a propositional language and $Frm_{\mathcal{L}}$ is the set of wffs of $\mathcal{L}$. $\mathcal{V}$ is a finite set of signs.
Signed calculi ([15, 7, 10]) manipulate sets of signed formulas, while the signs can be thought of as syntactic markers which keep track of the formulas in the course of a derivation.

Definition 1. *A* signed formula *for $\mathcal{L}$ and $\mathcal{V}$ is an expression of the form $s : \psi$, where $s \in \mathcal{V}$ and ψ is a formula of $\mathcal{L}$. A signed formula $s : \psi$ is atomic if ψ is an atomic formula. A* sequent *is a finite set of signed formulas. A* clause *is a sequent consisting of atomic signed formulas.*

Formulas will be denoted by φ, ψ, signed formulas - by $\alpha, \beta, \gamma, \delta$, sets of signed formulas - by Υ, Λ, sequents - by Ω, Σ, Π, sets of sets of signed formulas - by Φ, Ψ and sets of sequents - by Θ, Ξ. We write $s : \Delta$ instead of $\{s : \psi \mid \psi \in \Delta\}$, $S : \psi$ instead of $\{s : \psi \mid s \in S\}$, and $S : \Delta$ instead of $\{s : \psi \mid s \in S, \psi \in \Delta\}$.

Note 1. The usual (two-sided) sequent notation $\Gamma \Rightarrow \Delta$ can be interpreted as $\{f : \Gamma\} \cup \{t : \Delta\}$, i.e. a sequent in the sense of Definition 1 over $\{t, f\}$.

Definition 2. *Let v be a function from the set of formulas of $\mathcal{L}$ to $\mathcal{V}$.*

1. *v satisfies a signed formula $\gamma = (l : \psi)$, denoted by $v \models (l : \psi)$, if $v(\psi) = l$.*
2. *v satisfies a set of signed formulas Υ, denoted by $v \models \Upsilon$, if there is some $\gamma \in \Upsilon$, such that $v \models \gamma$.*

Thus sequents are interpreted as a *disjunction* of statements, saying that a particular formula takes a particular truth-value (interpreting sequents in a dual way corresponds to the method of analytic tableaux, see e.g. [6, 13])

Non-deterministic matrices are a natural generalization of the notion of a standard multi-valued matrix, in which the value of a complex formula can be chosen non-deterministically out of a non-empty set of options. Below we shortly reproduce the basic definitions from [2].

Definition 3. *A* non-deterministic matrix (Nmatrix) *for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:*

- *$\mathcal{V}$ is a non-empty set of truth values (signs).*
- *$\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$.*
- *For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$.*

A valuation $v : Frm_{\mathcal{L}} \to \mathcal{V}$ is legal in an Nmatrix $\mathcal{M}$ *if for every n-ary connective $\diamond$ of $\mathcal{L}$:*

$$v(\diamond(\psi_1, ..., \psi_n)) \in \widetilde{\diamond}(v(\psi_1), ..., v(\psi_n))$$

Note that in deterministic matrices the truth-value assigned by a valuation v to a complex formula is uniquely determined by the truth-values of its subformulas. This is not the case in Nmatrices, as v makes a non-deterministic choice out of the set of options $\tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$ and so the semantics defined above is not truth-functional.

Proposition 1. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$ and let v_p be an $\mathcal{M}$-legal partial valuation defined on a set S of $\mathcal{L}$-formulas closed under subformulas (i.e., $\psi_1, ..., \psi_n \in S$ whenever $\diamond(\psi_1, ..., \psi_n) \in S$). Then v_p can be extended to a full $\mathcal{M}$-legal valuation.*

Definition 4. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$[2] be some Nmatrix for $\mathcal{L}$. For a set of sequents Θ and a sequent Ω, $\Theta \vdash_{\mathcal{M}} \Omega$ if for every $\mathcal{M}$-legal valuation v: whenever v satisfies all the sequents in Θ, v also satisfies Ω.*

[2] The set of designated truth-values $\mathcal{D}$ in $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is needed for defining the consequence relation which is induced by $\mathcal{M}$ between *sets of $\mathcal{L}$-formulas* and *$\mathcal{L}$-formulas* (see e.g. [5]). In contrast, the consequence relation $\vdash_{\mathcal{M}}$ used in this paper is between *sets of signed sequents* and *signed sequents*, so the set of designated truth-values plays no role in this context. However, the former consequence relation can be fully characterized in terms of the latter, see e.g. [3].

Definition 5. *For a signed calculus G, we shall write $\Theta \vdash_G \Omega$ whenever a sequent Ω is derivable from a set of sequents Θ in G. We say that $\mathcal{M}$ is a strongly characteristic Nmatrix for G if $\Theta \vdash_G \Omega$ iff $\Theta \vdash_{\mathcal{M}} \Omega$.*

Note 2. Note that in the case of two-signed calculi (corresponding to Gentzen-type systems, recall Note 1), if $\mathcal{M}$ is strongly characteristic for a (Gentzen-type) system G, then $\mathcal{M}$ is also sound and complete for G in the standard sense.

3 Canonical Signed Calculi

We start by extending the notion of a "Gentzen-type canonical rule" from [2] to the context of signed calculi:

Definition 6. *A* signed canonical rule for a language $\mathcal{L}$ and a finite set of signs $\mathcal{V}$ *is an expression of the form $[\Theta/S : \diamond]$, where S is a non-empty subset of $\mathcal{V}$, $\diamond$ is an n-ary connective of $\mathcal{L}$ and $\Theta = \{\Sigma_1, ..., \Sigma_m\}$, where $m \geq 0$ and for every $1 \leq j \leq m$: Σ_j is a clause consisting of atomic signed formulas of the form $a : p_k$ for $a \in \mathcal{V}$ and $1 \leq k \leq n$.*
An application of a signed canonical rule $[\{\Sigma_1, ..., \Sigma_m\}/S : \diamond]$ *is an inference step of the following form:*

$$\frac{\Omega, \Sigma_1^* \quad ... \quad \Omega, \Sigma_m^*}{\Omega, S : \diamond(\psi_1, ..., \psi_n)}$$

where $\psi_1, ..., \psi_n$ are $\mathcal{L}$-formulas, Ω is a sequent, and for all $1 \leq i \leq m$: Σ_i^* is obtained from Σ_i by replacing p_j by ψ_j for every $1 \leq j \leq n$.

Example 1. 1. The standard Gentzen-style introduction rules for the classical conjunction are usually defined as follows:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

Using the notation from Note 1, we can write $\{f : \Gamma\} \cup \{t : \Delta\}$ (that is, ψ occurs with a sign 'f' if $\psi \in \Gamma$ and with a sign 't' if $\psi \in \Delta$), thus the canonical representation of the rules above is as follows:

$$[\{\{f : p_1, f : p_2\}\}/\{f\} : \wedge] \qquad [\{\{t : p_1\}, \{t : p_2\}\}/\{t\} : \wedge]$$

Applications of these rules have the forms:

$$\frac{\Omega \cup \{f : \psi_1, f : \psi_2\}}{\Omega \cup \{f : \psi_1 \wedge \psi_2\}} \qquad \frac{\Omega \cup \{t : \psi_1\} \quad \Omega \cup \{t : \psi_2\}}{\Omega \cup \{t : \psi_1 \wedge \psi_2\}}$$

2. Consider a calculus over $\mathcal{V} = \{a, b, c\}$ with the following introduction rules for a ternary connective $\circ$:

$$[\{\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}\}/\{a, c\} : \circ(p_1, p_2, p_3)]$$

$$[\{\{c : p_2\}, \{a : p_3, b : p_3\}, \{c : p_1\}\}/\{b, c\} : \circ(p_1, p_2, p_3)]$$

Their applications are of the forms:

$$\frac{\Omega \cup \{a : \psi_1, c : \psi_2\} \quad \Omega \cup \{a : \psi_3, b : \psi_2\}}{\Omega \cup \{a : \circ(\psi_1, \psi_2, \psi_3), c : \circ(\psi_1, \psi_2, \psi_3)\}}$$

$$\frac{\Omega \cup \{c : \psi_2\} \quad \Omega \cup \{a : \psi_3, b : \psi_3\} \quad \Omega \cup \{c : \psi_1\}}{\Omega \cup \{b : \circ(\psi_1, \psi_2, \psi_3), c : \circ(\psi_1, \psi_2, \psi_3)\}}$$

Definition 7. *Let $\mathcal{V}$ be a finite set of signs.*

1. *A* logical axiom *for $\mathcal{V}$ is a sequent of the form $\{l : \psi \mid l \in \mathcal{V}\}$.*
2. *The cut*[3] *and weakening rules for $\mathcal{V}$ are defined as follows:*

$$\frac{\Omega \cup \{l : \psi \mid l \in L_1\} \quad \Omega \cup \{l : \psi \mid l \in L_2\}}{\Omega \cup \{l : \psi \mid l \in L_1 \cap L_2\}} \ \textbf{cut}$$

$$\frac{\Omega}{\Omega, l : \psi} \ \textbf{weak}$$

where $L_1, L_2 \subseteq \mathcal{V}$ and $l \in \mathcal{V}$.

The following proposition follows from the completeness of many-valued resolution ([8]):

Proposition 2. *Let Θ be a set of clauses and Ω - a clause. Then Ω follows from Θ iff there is some $\Omega' \subseteq \Omega$, such that Ω' is derivable from Θ by cuts.*

Corollary 1. *For a set of clauses Θ, the empty sequent is derivable from Θ by cuts iff Θ is not satisfiable.*

Proof: Follows from the above proposition and the fact that Θ is unsatisfiable iff the empty sequent follows from Θ.

We are now ready to define the notion of a "canonical signed calculus":

Definition 8. *We say that a signed calculus over a language $\mathcal{L}$ and a finite set of signs $\mathcal{V}$ is* canonical *if it consists of:*

1. *All logical axioms for $\mathcal{V}$.*
2. *The rules of cut and weakening (see Defn. 7).*
3. *A finite number of signed canonical inference rules.*

[3] The cut is a variation of the basic resolution rule of [8].

Although we can define arbitrary canonical signed systems, our main quest is for systems, the syntactic rules of which determine the semantic meaning of the logical connectives they introduce. Thus we are interested in calculi with a "reasonable" (or "non-contradictory") set of rules, which allows for defining a sound and complete semantics for the system (we shall later see that this is also strongly related to cut-elimination). This can be captured by the following simple syntactic *coherence* criterion (which is a generalization of the criterion in [2] for canonical Gentzen-type systems).

Definition 9. *We say that a canonical calculus G is* coherent *if $\Theta_1 \cup ... \cup \Theta_m$ is unsatisfiable whenever $[\Theta_1/S_1 : \diamond], ..., [\Theta_m/S_m : \diamond]$ is a set of rules of G, such that $S_1 \cap ... \cap S_m = \emptyset$.*

Note 3. Obviously, coherence is a decidable property of canonical calculi. We also observe that by Proposition 1, a canonical calculus G is coherent if whenever $\{[\Theta_1/S_1 : \diamond], ..., [\Theta_m/S_m : \diamond]\}$ is a set of rules of G, and $S_1 \cap ... \cap S_m = \emptyset$, we have that $\Theta_1 \cup ... \cup \Theta_m$ is inconsistent (i.e. the empty sequent can be derived from it using cuts). Moreover, we will shortly see that it is not sufficient to check only pairs of rules in the definition above, as it can be the case that $S_1 \cap S_2 \neq \emptyset$ and $S_2 \cap S_3 \neq \emptyset$, but $S_1 \cap S_2 \cap S_3 = \emptyset$.

Example 2. 1. Consider the canonical calculus G_1 over $\mathcal{L} = \{\wedge\}$ and $\mathcal{V} = \{t, f\}$, the canonical rules of which are the two rules for $\wedge$ from Example 1. We can derive the empty sequent from $\{\{t : p_1\}, \{t : p_2\}, \{f : p_1, f : p_2\}\}$ as follows:

$$\dfrac{\dfrac{\{t : p_1\} \quad \{f : p_1, f : p_2\}}{\{f : p_2\}}\ \textbf{cut} \qquad \{t : p_2\}}{\emptyset}\ \textbf{cut}$$

Thus G_1 is coherent.

2. Consider the canonical calculus G_2 over $\mathcal{V} = \{a, b, c\}$ with the following introduction rules for the ternary connective $\circ$:

$$[\{\{a : p_1\}, \{b : p_2\}\}/\{a, b\} : \circ(p_1, p_2, p_3)]$$

$$[\{\{a : p_2, c : p_3\}\}/\{c\} : \circ(p_1, p_2, p_3)]$$

Clearly, the set $\{\{a : p_1\}, \{b : p_2\}, \{a : p_2, c : p_3\}\}$ is satisfiable, thus G_2 is not coherent.

Next we define some notions of cut-elimination in canonical calculi:

Definition 10. Let G be a canonical signed calculus and let Θ be some set of sequents.

1. A cut is called a *Θ-cut* if the cut formula occurs in Θ. We say that a proof is Θ-cut-free if the only cuts in it are Θ-cuts.

2. A cut is called *Θ-analytic* if the cut formula is a subformula of some formula occurring in Θ. A proof is called Θ-analytic[4] if all cuts in it are Θ-analytic. We say that a sequent Ω has a *proper proof from Θ in G* whenever Ω has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G.
3. A canonical calculus G admits *(standard) cut-elimination* if whenever $\vdash_G \Omega$, Ω has a cut-free proof in G. G admits *strong cut-elimination*[5] if whenever $\Theta \vdash_G \Omega$, Ω has in G a Θ-cut-free proof from Θ.
4. G admits *strong analytic cut-elimination* if whenever $\Theta \vdash_G \Omega$, Ω has in G a $\Theta \cup \{\Omega\}$-analytic proof from Θ. G admits *analytic cut-elimination* if whenever $\vdash_G \Omega$, Ω has in G a $\{\Omega\}$-analytic proof.

Example 3. Consider the following calculus G' for a language with a binary connective $\circ$ and $\mathcal{V} = \{a, b, c\}$. The rules of G' are as follows:

$$R_1 = [\{\{a : p_1\}\}/\{a, b\} : \circ] \qquad R_2 = [\{\{a : p_1\}\}/\{b, c\} : \circ]$$

The following proof of $\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}$ in G' is proper, as the cut in the final step is analytic:

$$\dfrac{\dfrac{\{a : p_1, b : p_1, c : p_1\}}{\{b : p_1, c : p_1, b : (p_1 \circ p_2), c : (p_1 \circ p_2)\}}\, R_2 \qquad \dfrac{\{a : p_1, b : p_1, c : p_1\}}{\{b : p_1, c : p_1, a : (p_1 \circ p_2), b : (p_1 \circ p_2)\}}\, R_1}{\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}}\, \mathbf{cut}$$

4 Modular Semantics for Canonical Calculi

We now describe a general method of providing modular semantics for canonical signed calculi based on Nmatrices. But first let us explain the intuition behind the need for non-determinism. Consider the standard Gentzen-type introduction rules for negation, which can be represented as follows in terms of signed calculi:

$$[\{t : p_1\}/\{f\} : \neg] \qquad [\{f : p_1\}/\{t\} : \neg]$$

The corresponding semantics is of course the classical truth-table for negation, according to which $\tilde{\neg}(t) = f$ (corresponding to the first rule) and $\tilde{\neg}(f) = t$ (corresponding to the second rule). Now suppose we would like to follow intuitionistic logic and discard the second rule, which corresponds to the law of excluded middle. We have a case of *underspecification*, as it is unclear what should now be the truth value of $\tilde{\neg}(f)$. Nmatrices deal with underspecification in a natural way: if $\tilde{\neg}(f)$ is underspecified, then it can be either 't' or 'f', and so we set $\tilde{\neg}(f) = \{t, f\}$.
We start by defining semantics for the most basic signed canonical calculus - the one without *any* canonical rules.

Definition 11. $G_0^{(\mathcal{L},\mathcal{V})}$ *is the canonical calculus for a language $\mathcal{L}$ and a finite set of signs $\mathcal{V}$, whose set of canonical rules is empty.*

[4] This is a generalization of the notion of analytic cut (see e.g. [11]).
[5] Strong cut-elimination was characterized for Gentzen-type systems in [5].

Henceforth we assume that our language $\mathcal{L}$ and the set of signs $\mathcal{V}$ are fixed, and so we shall write G_0 instead of $G_0^{(\mathcal{L},\mathcal{V})}$. It is easy to see that G_0 is (trivially) coherent.
We now define a strongly characteristic Nmatrix for G_0. It has the maximal degree of non-determinism in interpreting all of the connectives of $\mathcal{L}$.

Definition 12. *Let $\mathcal{M}_0 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be the Nmatrix for $\mathcal{L}$, in which for every n-ary connective $\diamond$ of $\mathcal{L}$: $\tilde{\diamond}(a_1, ..., a_n) = \mathcal{V}$ for every $a_1, ..., a_n \in \mathcal{V}$.*

Theorem 1. *$\mathcal{M}_0$ is a strongly characteristic Nmatrix for G_0.*

Proof: The proof is a simplified version of the proof of Theorem 2 in the sequel.

Now we turn to the modular effects of canonical rules: each rule added to the basic canonical calculus imposes a certain semantic condition on the basic Nmatrix $\mathcal{M}_0$, and coherence guarantees that these semantic conditions are not contradictory. For formalizing this we shall need the following technical propositions:

Definition 13. *Let $\mathcal{V}$ be a set of signs. For $\langle a_1, ..., a_n \rangle \in \mathcal{V}^n$, the set of clauses $\mathsf{C}_{\langle a_1,...,a_n \rangle}$ is defined as follows:*

$$\mathsf{C}_{\langle a_1,...,a_n \rangle} = \{\{a_1 : p_1\}, \{a_2 : p_2\}, ..., \{a_n : p_n\}\}$$

Definition 14. We say that a set of clauses Θ is n-canonical if the only atomic formulas occurring in Θ are in $\{p_1, ..., p_n\}$.

Corollary 2. *Let $\Theta_1, \Theta_2 ..., \Theta_m$ be some n-canonical clauses. If the sets of clauses $\mathsf{C}_{\langle a_1,...,a_n \rangle} \cup \Theta_1, ..., \mathsf{C}_{\langle a_1,...,a_n \rangle} \cup \Theta_m$ are satisfiable, then so is the set $\Theta_1 \cup \Theta_2 ... \cup \Theta_m$.*

Corollary 3. *Let Θ be an n-canonical clause. $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n \rangle}$ is consistent iff for every $\Omega \in \Theta$ there is some $1 \leq i \leq n$, such that $a_i : p_i \in \Omega$.*

Now we define the semantic condition corresponding to each canonical rule:

Definition 15. *Let R be a canonical rule of the form $[\Theta/S : \diamond]$.* **Cond(R)**, *the refining condition induced by R is defined as follows:*

Cond(R): For $a_1, ..., a_n \in \mathcal{V}$: if $\mathsf{C}_{\langle a_1,...,a_n \rangle} \cup \Theta$ is consistent, then $\tilde{\diamond}(a_1, ..., a_n) \subseteq S$.

Intuitively, a rule $[\Theta/S : \diamond]$ causes the deletion of all the truth-values which are not in S. Whenever some rules $[\Theta_1/S_1 : \diamond], ..., [\Theta_m/S_2 : \diamond]$ "overlap", their overall effect leads to $S_1 \cap ... \cap S_m$ (as we will see below, the coherence of a calculus guarantees that $S_1 \cap ... \cap S_m$ is not empty in such a case).

Definition 16. *Let G be a signed canonical calculus.*

1. *Define an application of a rule $[\Theta/S : \diamond]$ of G for some n-ary connective $\diamond$ on $\overrightarrow{a} = \langle a_1, ..., a_n \rangle \in \mathcal{V}^n$ as follows:*

$$[\Theta/S : \diamond](\overrightarrow{a}) = \begin{cases} S & \textit{if } \Theta \cup \mathsf{C}_{\overrightarrow{a}} \textit{ is consistent} \\ \mathcal{V} & \textit{otherwise} \end{cases}$$

2. $\mathcal{M}_G = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *is the Nmatrix, in which for every* n*-ary connective* $\diamond$ *for* $\mathcal{L}$ *and every* $\overrightarrow{a} = \langle a_1, ..., a_n \rangle \in \mathcal{V}^n$*:*

$$\tilde{\diamond}_{\mathcal{M}_G}(\overrightarrow{a}) = \bigcap \{[\Theta/S : \diamond](\overrightarrow{a}) \mid [\Theta/S : \diamond] \in G\}$$

Proposition 3. *If* G *is coherent, then* $\mathcal{M}_G$ *is well-defined.*

Proof: It suffices to check that for every n-ary connective $\diamond$ and every $a_1, ..., a_n \in \mathcal{V}$, $\tilde{\diamond}_{\mathcal{M}_G}(a_1, ..., a_n)$ is not empty. Suppose by contradiction that for some n-ary connective $\diamond$ and some $a_1, ..., a_n \in \mathcal{V}$, $\tilde{\diamond}(a_1, ..., a_n) = \emptyset$. But then there are some rules $[\Theta_1/S_1 : \diamond], ..., [\Theta_1/S_m : \diamond]$, such that $S_1 \cap ... \cap S_m = \emptyset$ and $\Theta_1 \cup \mathsf{C}_{\langle a_1,...,a_n \rangle}, ..., \Theta_m \cup \mathsf{C}_{\langle a_1,...,a_n \rangle}$ are consistent. By Corollary 2, $\Theta_1 \cup ... \Theta_m \cup \mathsf{C}_{\langle a_1,...,a_n \rangle}$ is consistent, and so is $\Theta_1 \cup ... \cup \Theta_m$, in contradiction to our assumption about the coherence of G.

Example 4. Consider a calculus G with the following canonical rules for a unary connective $\bullet$ for $\mathcal{V} = \{t, f, \top, \bot\}$:

$$[\{t : p_1\}/\{t\} : \bullet] \quad [\{f : p_1\}/\{f, \bot\} : \bullet]$$

$$[\{f : p_1, \bot : p_1\}/\{t, \bot\} : \bullet]$$

and the rule for conjunction from Example 1:

$$[\{\{f : p_1, f : p_2\}\}/\{f\} : \wedge]$$

Then the interpretations of $\wedge$ and $\bullet$ in $\mathcal{M}_G$ are as follows:

$\wedge$	t	f	$\top$	$\bot$
t	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\mathcal{V}$
f	$\{f\}$	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\mathcal{V}$
$\bot$	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\mathcal{V}$

$\bullet$	
t	$\{t\}$
f	$\{\bot\}$
$\top$	$\mathcal{V}$
$\bot$	$\{t, \bot\}$

Let us explain how the truth-tables are obtained. We start with the basic Nmatrix $\mathcal{M}_0$, for which $\tilde{\bullet}_{\mathcal{M}_0}(x) = \mathcal{V}$ and $\tilde{\wedge}_{\mathcal{M}_0}(x, y) = \mathcal{V}$ for every $x, y \in \mathcal{V}$. Consider the first rule for $\bullet$. Since $\{\{t : p_1\}\}$ is only consistent with $\mathsf{C}_{\langle t \rangle}$, this rule affects $\tilde{\bullet}_{\mathcal{M}_G}(t)$ by deleting the truth-values $f, \top, \bot$ from $\tilde{\bullet}_{\mathcal{M}_0}(t)$, and so $\tilde{\bullet}_{\mathcal{M}_G}(t) = \{t\}$. The second and the third rules both affect the set $\tilde{\bullet}_{\mathcal{M}_G}(f)$ (since the sets $\{\{f : p_1\}\}$ and $\{\{f : p_1, \bot : p_1\}\}$ are both consistent with $\mathsf{C}_{\langle f \rangle}$): the second rule deletes the truth-values $t, \top$, while the third deletes $\top, f$ from $\tilde{\bullet}_{\mathcal{M}_0}$. Thus we are left with $\tilde{\bullet}_{\mathcal{M}_G}(f) = \{\bot\}$. The third rule also dictates $\tilde{\bullet}_{\mathcal{M}_G}(\bot) = \{t, \bot\}$. Finally, as we have underspecification concerning $\tilde{\bullet}_{\mathcal{M}_G}(\top)$, in this case $\tilde{\bullet}_{\mathcal{M}_G}(\top) = \{t, f, \top, \bot\}$. As for the the rule for $\wedge$, the set $\{\{f : p_1, f : p_2\}\}$ is consistent with $\mathsf{C}_{\langle x,y \rangle}$ whenever at least one of $x, y \in \mathcal{V}$ is 'f', and so the rule deletes $t, \top, \bot$ from $\tilde{\wedge}_{\mathcal{M}_0}(x, y)$ for every such x, y.

Now suppose that we obtain a new calculus G' by adding the following rule for $\wedge$ to G (clearly, G' is still coherent):

$$[\{\{t : p_1, \top : p_1\}, \{\bot : p_2, f : p_2\}\}/\{f, \bot\} : \wedge]$$

This rule deletes the truth-values $t, \top$ from $\tilde{\wedge}_{\mathcal{M}_G}(x, y)$ for every $x \in \{t, \top\}$ and $y \in \{f, \bot\}$. Thus the truth-table for $\wedge$ in $\mathcal{M}_{G'}$ is now modified as follows:

$\wedge$	t	f	$\top$	$\bot$
t	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\{f, \bot\}$
f	$\{f\}$	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\{f, \bot\}$
$\bot$	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\mathcal{V}$

Note 4. It is easy to see that for a coherent calculus G, $\mathcal{M}_G$ is the weakest refinement of $\mathcal{M}_0$, in which all the conditions induced by the rules of G are satisfied. Thus if G' is a coherent calculus obtained from G by adding a new canonical rule, $\mathcal{M}'_G$ can be straightforwardly obtained from $\mathcal{M}_G$ by some deletions of options as dictated by the condition corresponding to the new rule.

Note 5. It is easy to verify that for the two-sided case studied in [2], the Nmatrix $\mathcal{M}_G$ defined above is similar to the two-valued Nmatrix constructed there. However, our construction of $\mathcal{M}_G$ above is much simpler: a canonical calculus in [2] is first transformed into an equivalent normal form calculus, which is then used to construct the characteristic Nmatrix. The idea is to transform the calculus so that each rule dictates the interpretation for only one tuple $\langle a_1, ..., a_n \rangle$. However, the above definition shows that the transformation into normal form is not necessary and $\mathcal{M}_G$ can be constructed directly from G.

Theorem 2. *For every coherent canonical calculus G, $\mathcal{M}_G$ is a strongly characteristic Nmatrix for G.*

Proof: The proof of strong soundness is not hard and is left to the reader. For strong completeness, suppose that Ω has no proper proof from Θ in G. We will show that this implies $\Theta \not\vdash_{\mathcal{M}_G} \Omega$. It is a standard matter to show that Ω can be extended to a maximal set Ω^*, such that (i) no $\Omega' \subseteq \Omega^*$ has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G, and (ii) all formulas occurring in Ω^* are subformulas of formulas from $\Theta \cup \{\Omega\}$. We now show that Ω^* has the following properties:

1. If $\tilde{\diamond}(a_1, ..., a_n) = \{b_1, ..., b_k\}$ and it also holds that $b_1 : \diamond(\psi_1, ..., \psi_n), ..., b_k : \diamond(\psi_1, ..., \psi_n) \in \Omega^*$, then $a_i : \psi_i \in \Omega^*$ for some $1 \leq i \leq n$.
2. For every formula ψ which is a subformula of some formula from Θ, there is exactly one $l \in \mathcal{V}$, such that $l : \psi \notin \Omega^*$.

Let us prove the first property. Suppose by contradiction that for some $a_1, ..., a_n \in \mathcal{V}$, $\tilde{\diamond}(a_1, ..., a_n) = \{b_1, ..., b_k\}$ and $b_1 : \diamond(\psi_1, ..., \psi_n), ..., b_k : \diamond(\psi_1, ..., \psi_n) \in \Omega^*$, but for every $1 \leq i \leq n$, $a_i : \psi_i \notin \Omega^*$. By the maximality of Ω^*, for every $1 \leq i \leq n$ there is some $\Omega'_i \subseteq \Omega^*$, such that $\Omega'_i \cup \{a_i : \psi_i\}$ has a $\Theta \cup \{\Omega\}$-analytic proof from

Θ in G. First observe that $\{b_1, ..., b_k\} \neq \mathcal{V}$ (otherwise Ω^* would contain a logical axiom, in contradiction to property (i) of Ω^*). Then by definition of $\mathcal{M}_G$ there are some rules in G of the form $R_1 = [\Xi_1/S_1 : \diamond], ..., R_m = [\Xi_m/S_m : \diamond]$, such that $\Xi_1 \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}, ..., \Xi_m \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ are consistent and $S_1 \cap ... \cap S_m = \{b_1, ..., b_k\}$. Now let $1 \leq j \leq m$ and $\Sigma \in \Xi_j$. By Corollary 3, there is some $1 \leq k_\Sigma \leq n$, such that $(a_{k_\Sigma} : p_{k_\Sigma}) \in \Sigma$ (since $\Xi_j \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent). Now by our assumption, $\Omega'_{k_\Sigma} \cup \{a_{k_\Sigma} : \psi_{k_\Sigma}\}$ has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G. By applying weakening we get a $\Theta \cup \{\Omega\}$-analytic proof of $\Omega'_{k_\Sigma} \cup \Sigma^*$ from Θ in G for every $\Sigma \in \Xi_j$, where Σ^* is obtained from Σ by replacing p_r by ψ_r for all $1 \leq r \leq n$. By applying weakening and the canonical rule R_j, we get a $\Theta \cup \{\Omega\}$-analytic proof of $\bigcup_{\Sigma \in \Xi_j} \Omega'_{k_\Sigma} \cup S_j : \diamond(\psi_1, ..., \psi_n)$ from Θ in G. Thus for all $1 \leq j \leq n$, there is some $\Omega_j \subseteq \Omega$, such that $\Omega_j \cup \{S_j : \diamond(\psi_1, ..., \psi_n)\}$ has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G. Now by applying $\Theta \cup \{\Omega\}$-analytic cuts (recall that we assumed that $\{b_1, ..., b_k\} : \diamond(\psi_1, ..., \psi_n) \in \Omega^*$ and so $\diamond(\psi_1, ..., \psi_n)$ is a subformula of some formula from $\Theta \cup \{\Omega\}$), we get a $\Theta \cup \{\Omega\}$-analytic proof of $\Omega_1 \cup ... \cup \Omega_m \cup \{S_1 \cap ... \cap S_m : \diamond(\psi_1, ..., \psi_n)\} = \Omega_1 \cup ... \cup \Omega_m$ from Θ in G , in contradiction to property (i) of Ω^*.
Now we prove the second property. Let ψ be a subformula of some formula from Θ. Then there must be some $l \in \mathcal{V}$, such that $l : \psi \notin \Omega^*$ (otherwise Ω^* contains a logical axiom). Suppose by contradiction that there are some $l_1 \neq l_2$, such that $l_1 : \psi, l_2 : \psi \notin \Omega^*$. By the maximality of Ω^*, then there are some $\Omega'_1, \Omega'_2 \subseteq \Omega^*$, such that $\Omega'_1 \cup \{l_1 : \psi\}$ and $\Omega'_2 \cup \{l_2 : \psi\}$ have $\Theta \cup \{\Omega\}$-analytic proofs from Θ in G. But then by applying ($\Theta \cup \{\Omega\}$-analytic) cuts, we get a $\Theta \cup \{\Omega\}$-analytic proof of $\Omega'_1 \cup \Omega'_2 \subseteq \Omega^*$ from Θ in G, in contradiction to property (i) of Ω^*.

Next we define the partial valuation v on the subformulas of $\Theta \cup \{\Omega\}$ as follows by induction on the complexity of formulas. According to our goal, v is defined so that $v(\psi) \neq s$ for every $(s : \psi) \in \Omega^*$. First, let p be an atomic formula. As Ω^* cannot contain a logical axiom, there must be some $s_0 \in \mathcal{V}$, such that $(s_0 : p) \notin \Omega^*$. Define $v(p) = s_0$. Suppose we have defined v for formulas with complexity up to l, and let $\psi = \diamond(\psi_1, ..., \psi_n)$, where each ψ_i is of complexity at most l. Hence $v(\psi_i)$ is already defined for each i. Now suppose that for every $1 \leq i \leq n$: $v(\psi_i) = a_i$ and $\tilde{\diamond}(a_1, ..., a_n) = \{b_1, ..., b_k\}$. Then there must be some $b \in \{b_1, ..., b_k\}$, such that $(b : \psi) \notin \Omega^*$ (otherwise by property 1 there would be some j, such that $(a_j : \psi_j) \in \Omega^*$, contradicting the induction hypothesis). Define $v(\psi) = b$. By the above construction, v is $\mathcal{M}_G$-legal and $v \not\models_{\mathcal{M}_G} \Omega^*$. Now let $\Sigma \in \Theta$. Then there must be some $a : \psi \in \Sigma$, such that $a : \psi \notin \Omega^*$ (otherwise $\Sigma \subseteq \Omega^*$, while Σ has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G). By property 2, for every $l \in \mathcal{V} \setminus \{a\}$, $(l : \psi) \in \Omega^*$. By the property of v proven above, $v(\psi) \neq l$ for every $\mathcal{V} \setminus \{a\}$. Thus $v(\psi) = a$, and so $v \models_{\mathcal{M}_G} \Sigma$. By Proposition 1, the partial valuation v can be extended to a full $\mathcal{M}_G$-legal valuation v_f. Thus we have constructed an $\mathcal{M}_G$-legal valuation v_f, such that $v_f \models_{\mathcal{M}_G} \Theta$, but $v_f \not\models_{\mathcal{M}_G} \Omega$. Hence, $\Theta \not\vdash_{\mathcal{M}_G} \Omega$.

From the proof of Theorem 2 we also have the following corollary:

Corollary 4. *Any coherent canonical calculus admits strong analytic cut-elimination.*

Note 6. [3] provides a full axiomatization of finite Nmatrices: a canonical coherent signed calculus is constructed there for every finite Nmatrix. Theorem 2 provides the complementary link between canonical calculi and Nmatrices: every canonical coherent signed calculus has a corresponding finite Nmatrix.

5 Characterization of Cut-elimination

In this section we provide a characterization of the notions of cut-elimination from Defn. 10. We start with the following theorem, which establishes an *exact correspondence* between coherence of canonical calculi, non-deterministic matrices and strong analytic cut-elimination:

Theorem 3. *Let G be a canonical calculus. The following statements concerning G are equivalent.*

1. *G is coherent.*
2. *G has a strongly characteristic Nmatrix.*
3. *G admits strong analytic cut-elimination.*
4. *G admits analytic cut-elimination.*

Proof: (1) $\Rightarrow$ (2) by Theorem 2.
(1) $\Rightarrow$ (3) by Corollary 4.
(3) $\Rightarrow$ (4) by definition of strong analytic cut-elimination (Defn. 10).
Next we prove that (2) $\Rightarrow$ (1). Suppose that G has a strongly characteristic Nmatrix $\mathcal{M}$ and suppose for contradiction that G is not coherent. Then there are some rules $R_1 = [\Theta_1 : /S_1 : \diamond], ..., R_m = [\Theta_m : /S_m : \diamond]$ in G, such that $\Theta = \Theta_1 \cup ... \cup \Theta_m$ is consistent and $S_1 \cap ... \cap S_m = \emptyset$. By applying the rule R_j on Θ_j for all $1 \leq j \leq m$, we get a proof of $S_j : \diamond(p_1, ..., p_n)$. Then by applying cuts we derive the empty sequent from $\Theta_1 \cup ... \cup \Theta_m$. Let v be any $\mathcal{M}$-legal valuation which satisfies Θ (such valuation exists since Θ is consistent). But by the strong soundness of $\mathcal{M}$ for G, v must then satisfy the empty set, reaching a contradiction.
Finally, we prove that (4) $\Rightarrow$ (1). Suppose that G admits analytic cut-elimination but is not coherent. Then again there are some rules $[\Theta_1 : /S_1 : \diamond], ..., [\Theta_m : /S_m : \diamond]$ in G, such that $\Theta = \Theta_1 \cup ... \cup \Theta_m$ is consistent and $S_1 \cap ...S_m = \emptyset$. Let v be a valuation which satisfies Θ (such valuation exists since Θ is consistent). Let Π be the set of all signed formulas $a : p_i$ (for $1 \leq i \leq n$), such that $v(p_i) \neq a$. Then for every $\Omega \in \Theta$: $\Pi \cup \Omega$ is a logical axiom (indeed, since v satisfies Ω there is some $1 \leq j \leq n$, such that $v(p_j) : p_j \in \Omega$. Then for every $a \in \mathcal{V} \setminus \{v(p_j)\}$, $a : p_j \in \Pi$). By applying the above rules and then cuts Π is provable in G:

$$\dfrac{\dfrac{\Pi \cup \Omega_1^1 \quad ... \quad \Pi \cup \Omega_{k_1}^1}{\Pi \cup S_1 : \diamond(p_1, ..., p_n)} \quad ... \quad \dfrac{\Pi \cup \Omega_1^m \quad ... \quad \Pi \cup \Omega_{k_m}^m}{\Pi \cup S_m : \diamond(p_1, ..., p_n)}}{\Pi}\ \textbf{cut}$$

where for all $1 \leq j \leq m$: $\Theta_j = \{\Omega^j_1, ..., \Omega^j_{k_j}\}$. Π consists of atomic formulas only and does not contain a logical axiom, and so it has no $\Theta \cup \{\Omega\}$-analytic proof in G (from $\emptyset$), in contradiction to our assumption that G admits analytic cut-elimination.

Next we turn to strong cut-elimination. The following example shows that the quadruple correspondence from Theorem 3 fails if one wants to eliminate also analytic cuts:

Example 5. Consider the calculus G' from Example 3. Clearly, G' is coherent. An analytic proof of the sequent $\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}$ is given in that example. However, it is easy to show that this sequent has no cut-free proof in G'. We will shortly see that by adding the rule $R_3 = [\{\{a : p_1\}\}/\{b\} : \circ]$ to G' cut-elimination is guaranteed.

Thus coherence is not a sufficient condition for strong cut-elimination. Therefore a stronger condition is provided in the next definition:

Definition 17. *A canonical calculus G is* dense *if for every $a_1, ..., a_n \in \mathcal{V}$ and every two rules of G $[\Theta_1/S_1 : \diamond]$ and $[\Theta_2/S_2 : \diamond]$, such that $\Theta_1 \cup \Theta_2 \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent, there is some rule $[\Theta/S : \diamond]$ in G, such that $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent and $S \subseteq S_1 \cap S_2$.*

Note 7. It is easy to see that the density of a canonical calculus is decidable (recall also the analogous Note 3 on coherence).

Lemma 1. *Let G be a dense canonical calculus. Let $[\Theta_1/S_1 : \diamond], ..., [\Theta_m/S_m : \diamond]$ be such rules in G, that $\Theta_1 \cup ... \cup \Theta_m$ is consistent. Then for every $a_1, ..., a_n \in \mathcal{V}$, for which $\Theta_1, ..., \Theta_m \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent, there is some $S \subseteq S_1 \cap ... \cap S_m$, such that $[\Theta/S : \diamond]$ is a rule in G and $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent.*

Corollary 5. *Every dense canonical calculus is coherent.*

Proof: Suppose that G is dense and let $[\Theta_1/S_1 : \diamond], ..., [\Theta_m/S_m : \diamond]$ be such rules of G, that $S_1 \cap ... \cap S_m = \emptyset$. Suppose by contradiction that $\Theta_1 \cup ... \cup \Theta_m$ is consistent. By Lemma 1, there is some canonical rule $[\Theta/S : \diamond]$ in G, such that $S \subseteq S_1 \cap ... \cap S_m$. By definition of a canonical rule (recall Defn. 6) S is non-empty, in contradiction to our assumption. Thus G is coherent.

Now we can provide an exact characterization of canonical systems which admit standard and strong cut-elimination:

Theorem 4. *Let G be a canonical calculus. The following statements concerning G are equivalent:*

1. *G is dense.*
2. *G admits cut-elimination.*
3. *G admits strong cut-elimination.*

To prove the theorem, first we need the following propositions:

Proposition 4. *Let G be a dense calculus. If Ω has no cut-free proof from Θ in G, then $\Theta \not\vdash_{\mathcal{M}_G} \Omega$.*

Proof: Similar to the method used in the proof of Theorem 2.

Lemma 2. *Let G be a coherent calculus over a language with an n-ary connective $\diamond$. Assume that G has at least one canonical rule $[\Theta/S_0 : \diamond]$, such that $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent. Let Ω be a set of signed formulas of the form $a : \psi$, where $\psi \in \{p_1, ..., p_n, \diamond(p_1, ..., p_n)\}$. If Ω has a cut-free proof in G, then either (i) Ω is a logical axiom, or (ii) $\diamond(p_1, ..., p_n) \in \Omega$ and there is a rule $[\Xi/S : \diamond]$ in G, such that $\Xi \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent and $S \subseteq \{a \mid a : \diamond(p_1, ..., p_n) \in \Omega\}$.*

Proof of Theorem 4:
$(1 \Rightarrow 3)$: Let G be a dense calculus. Then by Proposition 5, it is also coherent and so $\mathcal{M}_G$ is well-defined. If $\Theta \vdash_G \Omega$, then $\Theta \vdash_{\mathcal{M}_G} \Omega$. Thus by Proposition 4, Ω has a cut-free proof from Θ. Clearly, also $(3 \Rightarrow 2)$ holds. It remains to show that $(2 \Rightarrow 1)$. Suppose that G admits cut-elimination and assume by contradiction that G is not dense. Then there are some $a_1, ..., a_n \in \mathcal{V}$ and some rules $R_1 = [\Theta_1/S_1 : \diamond]$ and $R_2 = [\Theta_2/S_2 : \diamond]$, such that $\Theta_1 \cup \Theta_2 \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent and $S_1 \cap S_2 \neq \emptyset$, but there is no rule $[\Theta/S : \diamond]$ in G, such that $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent and $S \subseteq S_1 \cap S_2$. Now let $\Omega_0 = \bigcup_{1 \le i \le n} \{\mathcal{V} \setminus \{a_i\} : p_i\}$. By lemma 3, for every $\Omega \in \Theta_1 \cup \Theta_2$, there is some $1 \le i \le n$, such that $a_i : p_i \in \Omega$. Thus for every $\Omega \in \Theta_1 \cup \Theta_2$, $\Omega \cup \Omega_0$ is a logical axiom. Let $\Theta_1 = \{\Omega_1^1, ..., \Omega_k^1\}$ and $\Theta_2 = \{\Omega_1^2, ..., \Omega_m^2\}$. By applying the two canonical rules and then cuts we get a proof of $\Omega_0 \cup S_1 \cap S_2 : \diamond(p_1, ..., p_n)$ in G:

$$\dfrac{\dfrac{\Omega_1^1 \cup \Omega_0 \quad ... \quad \Omega_k^1 \cup \Omega_0}{\Omega_0 \cup S_1 : \diamond(p_1, ..., p_n)}\, R_1 \qquad \dfrac{\Omega_1^2 \cup \Omega_0 \quad ... \quad \Omega_m^2 \cup \Omega_0}{\Omega_0 \cup S_2 : \diamond(p_1, ..., p_n)}\, R_2}{\Omega_0 \cup (S_1 \cap S_2) : \diamond(p_1, ..., p_n)}\, \textbf{cut}$$

However, since $\Omega_0 \cup (S_1 \cap S_2) : \diamond(p_1, ..., p_n)$ is not a logical axiom, by Lemma 2 it has no cut-free proof in G, in contradiction to our assumption.

Note 8 (A constructive proof). The semantic proof of cut-elimination in Theorem 4 is not constructive, i.e. it does not provide an algorithm for eliminating cuts in a derivation. A syntactic constructive proof can be obtained by an adaptation of the proof of Theorem 4.1 of [7] to the context of canonical calculi.

Finally, we turn to the special case of canonical calculi with two signs (this includes the canonical Gentzen-type calculi of [2]) the following proposition can be easily shown:

Proposition 5. *A canonical calculus with two signs is dense iff it is coherent.*

Corollary 6. *The following statements concerning a two signed canonical calculus G are equivalent[6]:*

[6] This is a generalization of the triple correspondence from Theorem 4.7 of [2] to strong, analytic and strong analytic cut-elimination, which were not handled there.

1. *G is coherent.*
2. *G is dense.*
3. *G has a strongly characteristic Nmatrix.*
4. *G admits strong analytic cut-elimination.*
5. *G admits analytic cut-elimination.*
6. *G admits strong cut-elimination.*
7. *G admits standard cut-elimination.*

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