Effective Non-deterministic Semantics for first-order LFIs

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A *paraconsistent logic* is a logic which allows non-trivial inconsistent theories. One of the best-known approaches to designing useful paraconsistent logics is da Costa's approach, which has led to the family of Logics of Formal Inconsistency (LFIs), where the notion of inconsistency is expressed at the object level. In this paper we use *non-deterministic matrices*, a generalization of standard multi-valued matrices, to provide simple and modular finite-valued semantics for a large family of first-order LFIs. We demonstrate that the modular approach of Nmatrices provides new insights into the semantic role of the studied axioms and the dependencies between them. Furthermore, we study the issue of *effectiveness* in Nmatrices, a property which is crucial for the usefulness of semantics. We show that all of the non-deterministic semantics provided in this paper are effective.

Key words: Many-valued logic, Paraconsistent logic, Non-deterministic matrices

1 INTRODUCTION

In classical logic any proposition can be inferred from an inconsistent set of assumptions. Thus classical logic fails to capture that information systems which contain some inconsistent information may still produce useful answers. For such cases one needs a *paraconsistent logic* ([6, 7]), which is a logic that allows contradictory yet non-trivial theories. There are several approaches to the problem of designing a useful paraconsistent logic. One of the best known is da Costa's approach ([11, 9]), which has led to the family of *Logics of Formal Inconsistency* (LFIs). This family is based on two main ideas. The first is to divide all propositions into two sorts: the "normal" (or "consistent") and the "abnormal" (or "inconsistent") ones. The second idea is to express the meta-theoretical notions of consistency/inconsistency at the object language level, by adding to the language a new connective $\bullet$, with the intended meaning of $\bullet\varphi$ being "φ is inconsistent". (Alternatively, the dual connective $\circ$, expressing consistency can be used, see e.g. [9]). Using the inconsistency operator, one can limit the applicability of the rule $\varphi, \neg\varphi \vdash \psi$ (which amounts to "a single contradiction entails everything" and leads to trivialization in case of contradictions in classical logic) to the case when φ is consistent (i.e., $\varphi, \neg\varphi, \neg\bullet\varphi \vdash \psi$).

Although the syntactic formulations of LFIs are relatively simple, already on the propositional level the problem of finding semantic interpretations for them is rather complicated: the vast majority of LFIs cannot be characterized by means of finite multi-valued matrices. Moreover, for the majority of them no useful infinite-valued matrices are known. Thus other types of semantics, like bivaluations semantics and possible translations semantics have been proposed ([9]). However, it is not clear how to extend these types of semantics to the first-order level.

An alternative framework for providing semantics for propositional paraconsistent logics was used in [3, 1, 2]. This framework is based on a generalization of the standard multi-valued matrices, called *non-deterministic matrices* (Nmatrices). Nmatrices are multi-valued structures, in which the value assigned by a valuation to a complex formula can be chosen *non-deterministically* out of a certain nonempty set of options. The framework of Nmatrices has a number of attractive properties. First of all, the semantics provided by Nmatrices is *modular*: the main effect of each of the rules of a proof system is reducing the degree of non-determinism of operations, by forbidding some options. The semantics of a proof system is obtained by combining the semantic constraints imposed by its rules in a rather straightforward way. Secondly, this semantics is *effective** , i.e. any partial valuation

* No general theorem of effectiveness is available for the semantics of bivaluations or for possible translations semantics described in [9] and has to be proven from scratch for any instance of these types of semantics. See [8] for details on the connection between Nmatrices and other types of abstract semantics.

closed under subformulas can be extended to a full valuation. This property is crucial for the usefulness of semantics, in particular for constructing counterexamples.

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In [4] the framework of Nmatrices was extended to the full first-order level. This extension was adapted in [17, 5] for first-order LFIs based on the consistency operator $\circ$. In this paper we further develop and apply this framework to provide semantics for a large family of first-order LFIs using the inconsistency operator $\bullet$. We analyze the semantic effects of 18 new axioms (capturing de Morgan principles and inconsistency propagation) and demonstrate that the modular approach of Nmatrices provides new insights into the dependencies between the axioms.

Another crucial topic that we study here in detail (which was touched upon in [5]) is the effectiveness of Nmatrices in the case when various congruences between formulas are involved. The motivation for the use of congruence relations lies in the lack of the IPE[‡] principle in LFIs. This is a well-known principle of classical logic; we say that the IPE principle holds in a system **S** if every two sentences which are equivalent in a system **S** are *logically indistinguishable* in **S** (i.e. the provability of $A \leftrightarrow B$ in **S** entails the provability

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[‡] IPE - Intersubstitutability of provable equivalents. See [9] for further details on the IPE principle in LFIs.

of $\psi(A) \leftrightarrow \psi(B)$ in $\mathbf{S}$ for any context ψ). Unfortunately, this principle does not hold for any of the LFIs studied in this paper. For instance, already on the propositional level one cannot infer $\neg(A \wedge B) \leftrightarrow \neg(B \wedge A)$ from $(A \wedge B) \leftrightarrow (B \wedge A)$ in any of these systems. The lack of the IPE principle becomes really harmful on the first-order level, as even the α-conversion principle[¶] does not hold, which is of course unacceptable in any reasonable logical system. For instance, although $\forall x p(x) \leftrightarrow \forall y p(y)$ is provable in the basic first-order LFI **QB** defined in the sequel, $\neg\forall x p(x) \leftrightarrow \neg\forall y p(y)$ is not. A similar problem arises in the case of vacuous quantification: the provability of $\forall x \exists y p(x) \leftrightarrow \forall x p(x)$ in **QB** does not imply the provability of $\neg(\forall x \exists y p(x)) \leftrightarrow \neg(\forall x p(x))$. The straightforward solution used by da Costa ([11]) to solve the last two problems is adding an explicit extra-postulate: $\vdash \psi \leftrightarrow \varphi$ whenever ψ and φ are congruent in the above sense, i.e φ can be obtained from ψ by renaming bound variables and deletion/addition of vacuous quantifiers. In a similar way, one can add other natural congruence-based postulates to LFIs to treat the above abnormalities. However, the effectiveness of Nmatrices characterizing systems with such postulates becomes more problematic. In fact, it is not always guaranteed. We formulate necessary and sufficient conditions for the effectiveness of Nmatrices for a case-study of a variety of congruences, based on α-conversion, vacuous quantification, commutativity and idempotency of $\wedge$ and $\vee$. We then show that all of the non-deterministic semantics provided in this paper are effective.

2 A TAXONOMY OF FIRST-ORDER LFIS

In what follows, L_C is a first-order language over $\{\bullet, \neg, \wedge, \vee, \supset, \forall, \exists\}$.

Definition 1 *Let $\mathbf{HCL}^+$ be some Hilbert-type system which has Modus Ponens as the only inference rule, and is sound and strongly complete for the positive fragment of classical propositional logic. The first-order system $\mathbf{HCL}^+_{FOL}$ is obtained from $\mathbf{HCL}^+$ by adding the axioms $\forall x\psi \supset \psi\{t/x\}$ and $\psi\{t/x\} \supset \exists x\psi$, where t is any term free for x in ψ, and the inference rules $\dfrac{(\varphi \supset \psi)}{(\varphi \supset \forall x\psi)}$ and $\dfrac{(\psi \supset \varphi)}{(\exists x\psi \supset \varphi)}$, where t is free for x in ψ and $x \notin Fv[\varphi]$. The system $\mathbf{QB}$[§] is obtained from $\mathbf{HCL}^+_{FOL}$ by adding the schemata:*

$$(\mathbf{t})\ \neg\varphi \vee \varphi \qquad (\mathbf{b})\ \neg \bullet \varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi)$$

[¶] The principle identifies syntactic objects differing only in the names of their bound variables.
[§] In [5] the name **QB** is used for a slightly different first-order system.

We obtain a large family of first-order LFIs by adding to the basic system $\mathbf{QB}$ different combinations of the following schemata:

Definition 2 *The set Ax consists of*[||] *:*

$(\mathbf{c})\ \neg\neg\varphi \supset \varphi$ $\qquad$ $(\mathbf{e})\ \varphi \supset \neg\neg\varphi$

$(\mathbf{i_1})\ \bullet\varphi \supset \varphi$ $\qquad$ $(\mathbf{i_2})\ \bullet\varphi \supset \neg\varphi$

$(\mathbf{Dm^1_\forall})\ \neg\forall x\psi \supset \exists x\neg\psi$ $\qquad$ $(\mathbf{Dm^2_\forall})\ \exists x\neg\psi \supset \neg\forall x\psi$

$(\mathbf{Dm^1_\exists})\ \neg\exists x\psi \supset \forall x\neg\psi$ $\qquad$ $(\mathbf{Dm^2_\exists})\ \forall x\neg\psi \supset \neg\exists x\psi$

$(\mathbf{Dm^1_\wedge})\ \neg(\psi \wedge \varphi) \supset (\neg\psi \vee \neg\varphi)$ $\qquad$ $(\mathbf{Dm^2_\wedge})\ (\neg\psi \vee \neg\varphi) \supset \neg(\psi \wedge \varphi)$

$(\mathbf{Dm^1_\vee})\ \neg(\psi \vee \varphi) \supset (\neg\psi \wedge \neg\varphi)$ $\qquad$ $(\mathbf{Dm^2_\vee})\ (\neg\psi \wedge \neg\varphi) \supset \neg(\psi \vee \varphi)$

$(\mathbf{J^1_\wedge})\ \bullet(\psi \wedge \varphi) \supset ((\bullet\psi \wedge \varphi) \vee (\bullet\varphi \wedge \psi))$

$(\mathbf{J^2_\wedge})\ ((\bullet\psi \wedge \varphi) \vee (\bullet\varphi \wedge \psi)) \supset (\bullet(\psi \wedge \varphi))$

$(\mathbf{J^1_\vee})\ \bullet(\psi \vee \varphi) \supset ((\bullet\psi \wedge \neg\varphi) \vee (\bullet\varphi \wedge \neg\psi))$

$(\mathbf{J^2_\vee})\ ((\bullet\psi \wedge \neg\varphi) \vee (\bullet\varphi \wedge \neg\psi)) \supset \bullet(\psi \vee \varphi)$

$(\mathbf{J^1_\supset})\ \bullet(\psi \supset \varphi) \supset (\psi \wedge \bullet\varphi)$

$(\mathbf{J^2_\supset})\ (\psi \wedge \bullet\varphi) \supset \bullet(\psi \supset \varphi)$

$(\mathbf{J^1_\forall}):\ \bullet\forall x\psi \supset (\exists x\bullet\psi \wedge \forall x\psi)$

$(\mathbf{J^2_\forall}):\ (\exists x\bullet\psi \wedge \forall x\psi) \supset \bullet\forall x\psi$

$(\mathbf{J^1_\exists}):\ \bullet\exists x\psi \supset (\exists x\bullet\psi \wedge \forall x\neg\psi)$

$(\mathbf{J^2_\exists}):\ (\exists x\bullet\psi \wedge \forall x\neg\psi) \supset \bullet\exists x\psi$

For $X \subseteq Ax$, the system $\mathbf{QB}[X]$ is obtained from $\mathbf{QB}$ by adding to it the schemata in X.

Notation: We denote $\mathbf{QB}[X]$ by $\mathbf{QB}s$, where s is a string consisting of the names of the schemata in X (thus we write $\mathbf{QBce}$ rather than $\mathbf{QB}[\{(\mathbf{c}),(\mathbf{e})\}]$. If both $(\mathbf{i_1})$ and $(\mathbf{i_2})$ are in X we abbreviate it by **i**. Also, if for $\mathbf{x} \in \{\mathbf{J}, \mathbf{Dm}\}$ $\mathbf{x^i_y}$ is in X for every $\mathbf{y} \in \{\supset, \wedge, \vee, \forall, \exists\}$ and $\mathbf{i} \in \{1, 2\}$, we simply write $\mathbf{x}$.

Remark: $\mathbf{QBcieJDm_\forall Dm_\exists}$ is the first-order system LFI1* designed in [9] for handling evolutionary databases.

It is important to note that all of the systems defined above lack the IPE principle (see Introduction). This problem is really alarming on the first-order level: from $\forall xp(x) \leftrightarrow \forall yp(y)$ one cannot infer $\neg\forall xp(x) \leftrightarrow \neg\forall yp(y)$ in $\mathbf{QB}$ or any of its extensions defined above, and so the α-conversion principle is not derivable. This is clearly not acceptable in any useful logical system. The case of vacuous quantification is similar: one cannot derive in **QB** $\neg\forall xp(c) \leftrightarrow \neg p(c)$

[||] The schemata are taken from [9, 10]. **(c)**, **(e)**, **(i$_1$)** and **(i$_2$)** were studied in [2, 17, 5] (with the dual operator $\circ$). The rest are studied in context of Nmatrices for the first time. (**Dm** stands for 'De Morgan'.)

from $\forall x p(c) \leftrightarrow p(c)$. In [11] da Costa has solved this problem by adding an explicit extra-postulate based on a congruence between formulas, capturing α-equivalence and void quantification. This method can be further extended to other natural congruences. As a case study, we shall consider in addition to the essential principles of α-conversion and void quantification also the natural (although less obvious) principles of commutativity and idempotency of $\wedge$ and $\vee$. We formalize this as follows:

Definition 3 *The set* **CNG** *includes the following binary relations between formulas of* L_C*:*

- *α-conversion:* $\alpha = \{\langle A, B\rangle \mid A, B \in Frm_L \textit{ and } A \equiv_\alpha B\}$.
- *Void quantification:* $V_Q = \{\langle QxA, A\rangle \mid A \in Frm_L \textit{ and } x \notin Fv(A)\}$ *for* $Q \in \{\forall, \exists\}$.
- *Commutativity:* $C_\diamond = \{\langle (A \diamond B), (B \diamond A)\rangle \mid A, B \in Frm_L\}$ *for* $\diamond \in \{\wedge, \vee\}$.
- *Idempotency:* $I_\diamond = \{\langle (A \diamond A), A\rangle \mid A \in Frm_L\}$ *for* $\diamond \in \{\wedge, \vee\}$.

For $Z \subseteq$**CNG**, CNG_Z *is the minimal congruence relation on* Frm_{L_C}*, such that for every* $R \in Z$*:* $R \subseteq \mathsf{CNG}_Z$.

Note that in da Costa's first-order C-systems, the congruence relation $\mathsf{CNG}_{\{\alpha, V_\forall, V_\exists\}}$ is used (see [11]).

Definition 4 *For* $X \subseteq Ax$ *and* $Z \subseteq \mathbf{CNG}$*, the system* $\mathbf{QB}^Z[X]$ *is obtained from the system* $\mathbf{QB}[X]$ *by adding the extra-postulate (Z)* $\psi \supset \psi'$ *for any* $\psi, \psi' \in Frm_{L_C}$*, such that* $\mathsf{CNG}_Z(\psi, \psi')$ *holds.*

3 NON-DETERMINISTIC MATRICES

Our main semantic tool will be the following generalization of the concept of a multi-valued matrix first introduced in [3] and used in [2, 1, 4, 17, 5].

Definition 5 (Non-deterministic matrix) *A non-deterministic matrix (Nmatrix) for a language* L *is a tuple* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$*, where:* $\mathcal{V}$ *is a non-empty set of truth values,* $\mathcal{D}$ *(designated truth values) is a non-empty proper subset of* $\mathcal{V}$ *and* $\mathcal{O}$ *includes the following interpretation functions:*

- $\tilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to P^+(\mathcal{V})$ *for every* n*-ary connective* $\diamond$.

- $\tilde{Q}_{\mathcal{M}} : P^{+}(\mathcal{V}) \rightarrow P^{+}(\mathcal{V})$ *for every unary quantifier* Q.

Definition 6 (L-structure) *Let $\mathcal{M}$ be an Nmatrix. An L-structure for $\mathcal{M}$ is a pair $S = \langle D, I \rangle$ where D is a (non-empty) domain and I is a function interpreting constants, function symbols, and predicate symbols of L, satisfying the following conditions: $I[c] \in D$ if c is a constant, $I[f] : D^n \rightarrow D$ if f is an n-ary function, and $I[p] : D^n \rightarrow \mathcal{V}$ if p is an n-ary predicate.*
I is extended to interpret closed terms of L as follows:

$$I[f(t_1, ..., t_n)] = I[f][I[t_1], ..., I[t_n]]$$

Here a note on our treatment of quantification in the framework of Nmatrices is in order. The standard approach to interpreting first-order formulas is by using *objectual* (or referential) semantics, where the variable is thought of as ranging over a set of objects from the domain (see. e.g. [12, 13]). An alternative approach is *substitutional* quantification ([14]), where quantifiers are interpreted substitutionally, i.e. a universal (an existential) quantification is true if and only if every one (at least one) of its substitution instances is true (see. e.g. [15, 16]). [4] explains the motivation behind choosing the substitutional approach for the framework of Nmatrices, and points out the problems of the objectual approach in this context. The substitutional approach assumes that every element of the domain has a closed term referring to it. Thus given a structure $S = \langle D, I \rangle$, we extend the language L with *individual constants*, one for each element of D.

Definition 7 (L(D)) *Let S=$\langle D, I \rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. $L(D)$ is the language obtained from L by adding to it the set of* individual constants $\{\overline{a} \mid a \in D\}$. *$S' = \langle D, I' \rangle$ is the $L(D)$-structure, such that I' is the extension of I satisfying: $I'[\overline{a}] = a$.*

Given an L-structure $S = \langle D, I \rangle$, we shall refer to the extended $L(D)$-structure $\langle D, I' \rangle$ as S and to I' as I when the meaning is clear from the context.

Definition 8 (Congruence[#]) *Let S be an L-structure for a Nmatrix $\mathcal{M}$. The relation $\sim^S$ between $L(D)$-terms is defined inductively as follows: (i) $x \sim^S x$, (ii) For closed terms t, t' of $L(D)$: $t \sim^S t'$ when $I[t] = I[t']$, (iii) If*

[#] The motivation for this definition is purely technical and is related to extending the language with the set of individual constants $\{\overline{a} \mid a \in D\}$. Suppose we have a closed term **t**, such that $I[\mathbf{t}] = a \in D$. But a also has an individual constant $\overline{a}$ referring to it. We would like to be able to substitute **t** for $\overline{a}$ in every context.

$t_1 \sim^S t'_1, ..., t_n \sim^S t'_n$, then $f(t_1, ..., t_n) \sim^S f(t'_1, ..., t'_n)$.
The relation $\sim^S$ between $L(D)$-formulas is defined as follows: (i) If $t_1 \sim^S t'_1, t_2 \sim^S t'_2, ..., t_n \sim^S t'_n$, then $p(t_1, ..., t_n) \sim^S p(t'_1, ..., t'_n)$. (ii) If $\psi_1 \sim^S \psi'_1, ..., \psi_n \sim^S \psi'_n$ then $\diamond(\psi_1, ..., \psi_n) \sim^S \diamond(\psi'_1, ..., \psi'_n)$ for any n-ary connective $\diamond$ of L, and (iii) If $\psi \sim^S \psi'$, then $Qx\psi \sim^S Qx\psi'$ for $Q \in \{\forall, \exists\}$.

Definition 9 (S-valuation) *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. An S-valuation is $v : Frm^{\mathsf{cl}}_L \to \mathcal{V}$* legal *in $\mathcal{M}$ if it satisfies the following conditions: (i) $v[\psi] = v[\psi']$ for every two L-sentences ψ, ψ', such that $\psi \sim^S \psi'$, (ii) $v[p(t_1, ..., t_n)] = I[p][I[t_1], ..., I[t_n]]$, (iii) $v[\diamond(\psi_1, ..., \psi_n)] \in \tilde{\diamond}_{\mathcal{M}}[v[\psi_1], ..., v[\psi_n]]$, and (iv) $v[Qx\psi] \in \tilde{Q}_{\mathcal{M}}[\{v[\psi\{\overline{a}/x\}] \mid a \in D\}]$.*

Definition 10 (Semantics) *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. An $\mathcal{M}$-legal S-valuation v is* a model of a formula ψ in $\mathcal{M}$, *denoted by $S, v \models_{\mathcal{M}} \psi$, if $v[\psi'] \in \mathcal{D}$ for every closed instance ψ' of ψ in $L(D)$. A formula ψ is $\mathcal{M}$-valid in S if for every S-valuation v legal in $\mathcal{M}$, $S, v \models_{\mathcal{M}} \psi$. ψ is $\mathcal{M}$-valid if ψ is $\mathcal{M}$-valid in every L-structure for $\mathcal{M}$. The consequence relation $\vdash_{\mathcal{M}}$ between sets of L-formulas and L-formulas is defined as follows: $\Gamma \vdash_{\mathcal{M}} \psi$ if for every L-structure S and every $\mathcal{M}$-legal S-valuation v: $S, v \models_{\mathcal{M}} \Gamma$ implies that $S, v \models_{\mathcal{M}} \psi$. An Nmatrix $\mathcal{M}$ is* sound *for a proof system $\mathbf{S}$ if $\vdash_{\mathbf{S}} \subseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is* complete *for $\mathbf{S}$ if $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathbf{S}}$. $\mathcal{M}$ is a characteristic Nmatrix for $\mathbf{S}$ if it is sound and complete for $\mathbf{S}$.*

In order to characterize first-order LFIs with congruence-based postulates (see Defn. 4), we refine the notion of a consequence relation $\vdash_{\mathcal{M}}$ induced by $\mathcal{M}$, by using only $\mathcal{M}$-legal valuations which respect the congruence relation CNG_Z:

Definition 11 *For any $Z \subseteq \mathbf{CNG}$, an S-valuation is $\langle \mathcal{M}, \mathsf{CNG}_Z\rangle$-legal if it is $\mathcal{M}$-legal and for every two $L(D)$-sentences ψ, ψ': $\mathsf{CNG}_Z(\psi, \psi')$ implies $v[\psi] = v[\psi']$. The consequence relation $\vdash^Z_{\mathcal{M}}$ induced by $\langle \mathsf{CNG}_Z, \mathcal{M}\rangle$ is defined as follows: $\Gamma \vdash^Z_{\mathcal{M}} \psi$ if for every L-structure S and every $\langle \mathcal{M}, \mathsf{CNG}_Z\rangle$-legal S-valuation v: $S, v \models_{\mathcal{M}} \Gamma$ implies that $S, v \models_{\mathcal{M}} \psi$.*

Remark: For some non-trivial[**] consequence relation $\vdash_{\mathcal{M}}$, $\vdash^Z_{\mathcal{M}}$ may be trivialized. Consider, for instance, the Nmatrix $\mathcal{M}_1 = \langle\{t, f\}, \{t\}, \mathcal{O}\rangle$ for L_C with the following (non-standard) interpretation of $\forall$: $\tilde{\forall}[\{f\}] = \{t\}$ and $\tilde{\forall}[\{t\}] = \{f\}$. It is easy to see that no $\mathcal{M}_1$-legal valuation can respect the

[**] We say that a consequence relation $\vdash$ is trivial if for every set of formulas Γ and a formula ψ: $\Gamma \vdash \psi$.

$\mathsf{CNG}_{\{V_\forall\}}$ relation, and so the consequence relation $\vdash^{\{V_\forall\}}_{\mathcal{M}_1}$ is trivialized, although $\vdash_{\mathcal{M}_1}$ is not trivial. In the next section we study the connection between the trivialization of $\vdash^{Z}_{\mathcal{M}}$ and the effectiveness of $\mathcal{M}$.

The following is an extension of Definition 2.9 and Theorem 2.10 from [1] to first-order languages:

Definition 12 (Reduction, refinement) *Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for L. A reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$ is a function $F : \mathcal{V}_1 \to \mathcal{V}_2$, such that: (i) For every $x \in \mathcal{V}_1$, $x \in \mathcal{D}_1$ iff $F(x) \in \mathcal{D}_2$, (ii) $F(y) \in \tilde{\diamond}_{\mathcal{M}_2}[F(x_1), ..., F(x_n)]$ for every n-ary connective $\diamond$ of L and every $x_1, ..., x_n, y \in \mathcal{V}_1$, such that $y \in \tilde{\diamond}_{\mathcal{M}_1}[x_1, ..., x_n]$, and (iii) $F(y) \in \tilde{Q}_{\mathcal{M}_2}[\{F(z) \mid z \in H\}]$ for $Q \in \{\forall, \exists\}$ and every $y \in \mathcal{V}_1$ and $H \subseteq P^+(\mathcal{V}_1)$, such that $y \in \tilde{Q}_{\mathcal{M}_1}[H]$.*
$\mathcal{M}_1$ is a refinement *of $\mathcal{M}_2$ if there exists a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$.*

Theorem 13 *If $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$, then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.*

Proof: a straightforward extension of the proof of theorem 2.10 from [1].

4 EFFECTIVENESS OF NMATRICES

One of the most important properties of the propositional framework of Nmatrices is its *effectiveness*, in the sense that for determining whether $\Gamma \vdash_{\mathcal{M}} \varphi$ (where $\mathcal{M}$ is an Nmatrix) it always suffices to check only *partial* valuations, defined only on *subformulas* of $\Gamma \cup \{\varphi\}$. This entails compactness and decidability in the finite case and allows for constructing counter-examples. Now we extend the notion of effectiveness to the first-order level.

Definition 14 (Closure under subformulas) *For an L-structure S, a set of sentences $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ is* S-closed under subformulas *if it satisfies the following conditions: (i) for every n-ary connective $\diamond$ of L: $\psi_1, ..., \psi_n \in W$ whenever $\diamond(\psi_1, ..., \psi_n) \in W$, and (ii) for $Q \in \{\forall, \exists\}$ and every $a \in D$: $\psi\{\overline{a}/x\} \in W$ whenever $Qx\psi \in W$.*

Definition 15 (Effectiveness) *An Nmatrix $\mathcal{M}$ for a first-order language L is* effective *if for every L-structure S and every set of $L(D)$-sentences W which is S-closed under subformulas: any partial $\mathcal{M}$-legal S-valuation on W can be extended to a full $\mathcal{M}$-legal S-valuation.*

The proof of effectiveness for propositional Nmatrices is very simple (see proposition 2 in [2]). The proof for the first-order case is less trivial because of the condition that every $\mathcal{M}$-legal valuation also needs to respect the $\sim^S$-relation (see Defn. 9), as we shall see shortly.
For the special case of Nmatrices for L_C (the language of the first-order LFIs studied in this paper) and the congruence-based postulates, we also define *effectiveness for* CNG_Z:

Definition 16 *For $Z \subseteq \mathbf{CNG}$, an Nmatrix $\mathcal{M}$ for L_C is* effective *for* CNG_Z *if for every L-structure S and every set of $L(D)$-sentences W which is S-closed under subformulas: any partial $\langle \mathcal{M}, \mathsf{CNG}_Z \rangle$-legal S-valuation on W can be extended to a full $\langle \mathcal{M}, \mathsf{CNG}_Z \rangle$-legal S-valuation.*

It is important to stress that the effectiveness of $\mathcal{M}$ for CNG_Z is in fact not always guaranteed. Consider, for instance, the Nmatrix $\mathcal{M}_1$ from the previous section, which interprets $\forall$ non-standardly: $\tilde{\forall}[\{f\}] = \{t\}$. Let $S = \langle \{a\}, I \rangle$ be the L-structure in which $I[c] = a$ and $I[p][a] = f$. Let $W = \{p(c)\}$ (obviously, W is S-closed under subformulas). Then no partial valuation on W can be extended to a full $\langle \mathcal{M}, \mathsf{CNG}_{\{V_\forall\}} \rangle$-legal valuation v, since any such valuation assigns f to $p(c)$ and t to $\forall x p(c)$.
Below we formulate necessary and sufficient conditions for the effectiveness of Nmatrices for L_C for CNG_Z.

Definition 17 *For $Z \subseteq \mathbf{CNG}$, an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for L_C is CNG_Z-suitable if: (i) If $V_Q \in Z$ then for every $a \in \mathcal{V}$: $a \in \tilde{Q}_{\mathcal{M}}[\{a\}]$, (ii) If $C_\diamond \in Z$ then for every $a, b \in \mathcal{V}$: $\tilde{\diamond}_{\mathcal{M}}[a, b] = \tilde{\diamond}_{\mathcal{M}}[b, a]$, (iii) If $I_\diamond \in Z$ then for every $a \in \mathcal{V}$: $a \in \tilde{\diamond}_{\mathcal{M}}[a, a]$.*

Remark: The Nmatrix $\mathcal{M}_1$ defined above is not suitable for $\mathsf{CNG}_{\{V_\forall\}}$, since $f \notin \tilde{\forall}_{\mathcal{M}_1}[\{f\}]$.

Proposition 18 *For $Z \subseteq \mathbf{CNG}$, an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for L_C is effective for CNG_Z iff it is CNG_Z-suitable*[††] .

Proof:
($\Leftarrow$): Let $Z \subseteq \mathbf{CNG}$ and suppose that $\mathcal{M}$ is a CNG_Z-suitable Nmatrix. Let S be an L_C-structure and suppose $W \subseteq Frm_L$ is a set of $L(D)$-sentences S-closed under subformulas. Let v_p be some partial $\langle \mathcal{M}, \mathsf{CNG}_Z \rangle$-legal S-valuation on W. We will construct an extension of v_p to a full $\langle \mathcal{M}, \mathsf{CNG}_Z \rangle$-legal valuation. For $\diamond \in \{\wedge, \vee\}$ and every $a, b \in \mathcal{V}$, choose a truth-value

[††] The special case of $\{\alpha, V_\forall, V_\exists\}$ was treated in [5].

$\mathbf{b}^{\diamond}_{a,b} \in \tilde{\diamond}[a,b]$, so that: (i) if $C_{\diamond} \in Z$, then $\mathbf{b}^{\diamond}_{a,b} = \mathbf{b}^{\diamond}_{b,a}$, and (ii) if $I_{\diamond} \in Z$, $\mathbf{b}^{\diamond}_{a,a} = a$. For $Q \in \{\forall, \exists\}$, and every $H \in P^{+}(\mathcal{V})$ choose a truth-value $\mathbf{b}^{Q}_{H}$, such that if $V_Q \in Z$, then for every $a \in \mathcal{V}$: $\tilde{Q}[\{a\}] = a$. It is easy to see that such choice is possible due to the CNG_Z-suitability of $\mathcal{M}$. Denote by H_{CNG_Z} the set of all equivalence classes of sentences[‡‡] of L_C under CNG_Z. Denote the equivalence class of ψ under CNG_Z by $[\![\psi]\!]$. We define the function $\chi : H_{\mathsf{CNG}_Z} \to \mathcal{V}$ as follows: $\chi([\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!]) = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$. For $\diamond \in \{\wedge, \vee\}$, if there is some $\varphi \in ([\![\psi_1 \diamond \psi_2]\!] \cap W)$, then $\chi([\![\psi_1 \diamond \psi_2]\!]) = v_p[\varphi]$ (note that this is well-defined since v_p respects CNG_Z), otherwise $\chi([\![\psi_1 \diamond \psi_2]\!]) = \mathbf{b}_{\chi([\![\psi_1]\!]), \chi([\![\psi_2]\!])}$. For $Q \in \{\forall, \exists\}$, if there is some $\varphi \in ([\![Qx\psi]\!] \cap W)$, then $\chi([\![Qx\psi]\!]) = v_p[\varphi]$, otherwise $\chi([\![Qx\psi]\!]) = \mathbf{b}^{Q}_{\{\chi([\![\psi\{\overline{a}/x\}]\!]) \mid a \in D\}}$. It is easy to check that χ is well-defined. Next we define the valuation v as follows: $v[\psi] = \chi([\![\psi]\!])$. Clearly, v is an extension of v_p and respects the CNG_Z relation. The proof that v is legal in $\mathcal{M}$ is not difficult and is left to the reader.
($\Rightarrow$): Suppose that $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is not CNG_Z-suitable. Then one of the conditions from Defn. 17 is violated. We show the proof for the first condition. Suppose that condition (i) is violated, and there is some $a \in \mathcal{V}$, so that $a \notin \tilde{Q}[\{a\}]$ for some $Q \in \{\forall, \exists\}$. Suppose for simplicity that a unary predicate p and a constant c is in L_C (this assumption is not needed but it simplifies the presentation). Let S be an L_C-structure, such that $I[p][I[c]] = a$. Let $W = \{p(c)\}$ and let v_p be a partial S-valuation on W, such that $v_p[p(c)] = a$. It is easy to see that v_p is $\langle \mathcal{M}, \mathsf{CNG}_Z \rangle$-legal. But for any $\mathcal{M}$-legal extension of v of v_p to $Qxp(c)$: $v[Qxp(c)] \neq a$ and so any extension of v_p to a full S-valuation is not $\langle \mathcal{M}, \mathsf{CNG}_Z \rangle$-legal.

Corollary 19 *1. Any Nmatrix $\mathcal{M}$ for any language L is effective.*

2. Any Nmatrix $\mathcal{M}$ for L_C is effective for $\mathsf{CNG}_{\{\alpha\}}$.

Intuitively, the α-equivalence principle is more basic than the rest of the studied principles, as it is purely syntactic and does not depend on the semantic interpretations of the connectives and quantifiers.
The effectiveness of an Nmatrix $\mathcal{M}$ for CNG_Z is directly related to the non-triviality of the refined consequence relation $\vdash^{Z}_{\mathcal{M}}$:

Proposition 20 *Let $\mathcal{M}$ be an Nmatrix for L_C, such that $\vdash_{\mathcal{M}}$ is non-trivial. For any $Z \subseteq \mathbf{CNG}$, if $\mathcal{M}$ is effective for CNG_Z, then $\vdash^{Z}_{\mathcal{M}}$ is non-trivial.*

[‡‡] It is easy to show that for any $Z \subseteq \mathbf{CNG}$, if ψ, ψ' are in the same equivalence class under CNG_Z and ψ is a sentence, then ψ' is also a sentence.

Proof: Let $Z \subseteq \mathbf{CNG}$ and suppose that $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is effective for CNG_Z. Assume (for simplicity of presentation) that L_C includes two 0-ary predicates p, q. Clearly, $\langle p, q \rangle \notin \mathsf{CNG}_Z$. Define an L_C-structure S, such that $I[p] \in \mathcal{D}$ and $I[q] \notin \mathcal{D}$ (this is possible since $\vdash_{\mathcal{M}}$ is non-trivial). The partial valuation v_p on $\{p, q\}$, such that $v_p[p] = I[p]$ and $v_p[q] = I[q]$ is $\langle \mathcal{M}, \mathsf{CNG}_Z \rangle$-legal. Since $\mathcal{M}$ is effective for CNG_Z, v_p can be extended to a full $\langle \mathcal{M}, \mathsf{CNG}_Z \rangle$-legal S-valuation v, such that $v \models_{\mathcal{M}} p$ and $v \not\models_{\mathcal{M}} q$. Thus, $p \not\vdash^Z_{\mathcal{M}} q$ and so $\vdash^Z_{\mathcal{M}}$ is non-trivial.

The opposite direction does not hold. For instance, consider the following Nmatrix $\mathcal{M}_2 = \langle \{t, f, I\}, \{t, I\}, \mathcal{O} \rangle$ for L_C, such that $\tilde{\forall}[\{I\}] = \{f\}$, $\tilde{\forall}[\{t\}] = \{t\}$ and $\tilde{\forall}[\{f\}] = \{f\}$. It is easy to see that $\mathcal{M}_2$ is not suitable for $\mathsf{CNG}_{\{V_\forall\}}$, since $I \notin \tilde{\forall}[\{I\}]$ and so by Proposition 18, $\mathcal{M}_2$ is not effective for $\mathsf{CNG}_{\{V_\forall\}}$. Assume for simplicity that two 0-ary predicate symbols p and q are in L_C. Let $S = \langle D, I \rangle$ be an L_C-structure, such that $I[p] = t$ and $I[q] = f$. It is easy to see that there exists an $\mathcal{M}_2$-legal full S-valuation v, such that $v(p) = t$ and $v(q) = f$. Thus $p \not\vdash^Z_{\mathcal{M}_2} q$ for $Z = \{V_\forall\}$ and so $\vdash^Z_{\mathcal{M}_2}$ is not trivial.

5 NON-DETERMINISTIC SEMANTICS FOR FIRST-ORDER LFIS

In this section we provide non-deterministic semantics for the first-order LFIs obtained from the basic system $\mathbf{QB}$ by adding various combinations of the schemata from Ax and congruence-based postulates. The results in this section are an extension of the results in [2, 17, 5].

The system $\mathbf{QB}$ treats the connectives $\wedge, \vee, \supset$ and the quantifiers $\forall, \exists$ similarly to classical logic. The treatment of $\bullet$ and $\neg$ is different: intuitively, the truth/falsity of $\neg\psi$ or $\bullet\psi$ is not completely determined by the truth/falsity of ψ. More data is needed for it. The central idea is to include all the relevant data concerning a sentence ψ in the truth-value which is assigned to ψ. In our case the relevant data beyond the truth/falsity of ψ is the truth/falsity of $\neg\psi$ and of $\bullet\psi$. This leads to the use of elements from $\{0, 1\}^3$ as truth-values, where the intended meaning of $v[\psi] = \langle x, y, z \rangle$ is as follows: $x = 1$ iff $v[\psi] \in \mathcal{D}$, $y = 1$ iff $v[\neg\psi] \in \mathcal{D}$ and $z = 1$ iff $v[\bullet\psi] \in \mathcal{D}$. Note that because of the schema $(\mathbf{t})$, not all tuples can be used as legal truth values. The schema $(\mathbf{t})$ means that at least one of $\varphi, \neg\varphi$ must be true. Thus, the truth values $\langle 0, 0, 0 \rangle$ and $\langle 0, 0, 1 \rangle$ are rejected. The schema $(\mathbf{b})$ means that if φ and $\neg\varphi$ are true, then $\neg\bullet\varphi$ must be false. Since for every $v \in \tilde{\bullet}[\langle 1, 1, 0 \rangle]$, $v = \langle 0, x, y \rangle$ (recall that the third element specifies the truth/falsity of $\bullet\psi$), it means that x must be 0, which yields an illegal truth value, and thus $\langle 1,1,0\rangle$ is also rejected. We are left with the following five truth values: $f = \langle 0,1,0\rangle$,$f_I = \langle 0,1,1\rangle$, $t = \langle 1,0,0\rangle$, $t_I = \langle 1,0,1\rangle$, $I = \langle 1,1,1\rangle$.

Definition 21 *The Nmatrix $\mathcal{QM}_5 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is defined as follows: $\mathcal{V} = \{t, f, I, t_I, f_I\}$, $\mathcal{D} = \{t, t_I, I\}$ and $\mathcal{F} = \mathcal{V} - \mathcal{D}$. The operations in $\mathcal{O}$ are:*

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & otherwise \end{cases} \quad a \tilde{\supset} b = \begin{cases} \mathcal{D} & a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & otherwise \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases} \quad \tilde{\neg} a = \begin{cases} \mathcal{F} & a \in \{t, t_I\} \\ \mathcal{D} & a \in \{f, f_I, I\} \end{cases}$$

$$\tilde{\bullet} a = \begin{cases} \mathcal{F} & a \in \{t, f\} \\ \mathcal{D} & a \in \{t_I, f_I\} \\ \{t, t_I\} & a = I \end{cases}$$

$$\tilde{\forall}[H] = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases} \quad \tilde{\exists}[H] = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

Note that the definition of $\tilde{\bullet}$ in $\mathcal{QM}_5$ is dictated by the schema **(b)**, according to which $\neg\bullet\varphi$, φ and $\neg\varphi$ cannot all be true at the same time. This is guaranteed by the condition $\tilde{\bullet}[I] \in \{t_I, t\}$.

Lemma 22 *For any $Z \subseteq \mathbf{CNG}$, $\mathcal{QM}_5$ is effective for CNG_Z.*

Proof: Let $Z \subseteq \mathbf{CNG}$. It is easy to check that $\mathcal{QM}_5$ is suitable for CNG_Z. The lemma follows from Prop. 18.

Theorem 23 *(**Soundness and completeness**) Let $Z \subseteq \mathbf{CNG}$. Then $\Gamma \vdash_{\mathbf{QB}^Z} \psi_0$ iff $\Gamma \vdash^Z_{\mathcal{QM}_5} \psi_0$.*

Proof: The proof of soundness is not hard and is left to the reader. For completeness, we first note that by definition of the interpretation of $\forall$ in $\mathcal{QM}_5$, $\forall x\varphi \vdash_{\mathcal{QM}_5} \varphi$ and $\varphi \vdash_{\mathcal{QM}_5} \forall x\varphi$ for every formula φ and every variable x. Obviously the same relations hold between φ and $\forall x\varphi$ in $\mathbf{HCL}^+_{FOL}$, and so in $\vdash^Z_{\mathbf{QB}}$. It follows that we may assume that all formulas in $\Gamma \cup \{\psi_0\}$ are sentences. It is also easy to see that we may restrict ourselves to L_r, the subset of L_C consisting of all the constants, function, and predicate symbols occurring in $\Gamma \cup \{\psi_0\}$. Now suppose that $\Gamma \not\vdash^Z_{\mathbf{QB}} \psi_0$. We will construct an L_C-structure S and an $\langle \mathcal{QM}_5, \mathsf{CNG_Z}\rangle$-legal S-valuation v, such that $S, v \models_{\mathcal{M}_5} \Gamma$, but

$S, v \not\models_{\mathcal{QM}_5} \psi_0$.
Let L' be the language obtained from L_r by adding a countably infinite set of new constants. It is a standard matter to show (using a usual Henkin-type construction) that Γ can be extended to a maximal set Γ^* of sentences in L', such that: (i) $\Gamma^* \not\vdash_{\mathbf{QB}} \psi_0$, (ii) $\Gamma \subseteq \Gamma^*$, (iii) for every L'-sentence $\exists x\psi \in \Gamma^*$ there is a constant $\mathbf{c}$ of L', such that $\psi\{\mathbf{c}/x\} \in \Gamma^*$, and for every L'-sentence $\forall x\psi \notin \Gamma^*$, there is a constant $\mathbf{c}$ of L', such that $\psi\{\mathbf{c}/x\} \notin \Gamma^*$. (iii) follows from (ii), the deduction theorem for $\mathbf{QB}$, and the fact that for any $x \notin Fv[\varphi]$, $(\forall x\psi \supset \varphi) \supset \exists x(\psi \supset \varphi)$ is provable in the positive fragment of first-order classical logic, and so also in $\mathbf{QB}$). It is easy to show that Γ^* has the following properties for L'-sentences ψ, φ, and $\forall x\theta$: (1) If $\psi \notin \Gamma^*$, then $\psi \supset \psi_0 \in \Gamma^*$, (2) If ψ and $\neg\psi$ are both in Γ^*, then $\neg \bullet \psi \notin \Gamma^*$, (3) Either $\psi \in \Gamma^*$ or $\neg\psi \in \Gamma^*$, (4) If $\psi \in \Gamma^*$, then for every L'-sentence ψ' such that $\mathsf{CNG}_Z(\psi', \psi)$: $\psi' \in \Gamma^*$, (5) $\psi \vee \varphi \in \Gamma^*$ iff either $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$, (6) Similarly for $\wedge$ and $\supset$, (7) If $\forall x\theta \in \Gamma^*$, then for every closed L'-term $\mathbf{t}$: $\theta\{\mathbf{t}/x\} \in \Gamma^*$, and if $\forall x\theta \notin \Gamma^*$, then there is some closed term $\mathbf{t}_\theta$ of L', such that $\theta\{\mathbf{t}_\theta/x\} \notin \Gamma^*$, (8) If $\exists x\theta \in \Gamma^*$, then there is some closed term $\mathbf{t}_\theta$ of L, such that $\theta\{\mathbf{t}_\theta/x\} \in \Gamma^*$, and if $\exists x\theta \notin \Gamma^*$, then for every closed term $\mathbf{t}$ of L': $\theta\{\mathbf{t}/x\} \notin \Gamma^*$.
Define the L'-structure $S = \langle D, I\rangle$ as follows: (i) D is the set of all the closed terms of L', (ii) for every constant c of L': $I[c] = c$, (iii) for every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$, (iv) for every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I[p][\mathbf{t}_1, ..., \mathbf{t}_n] = \langle x, y, z\rangle$, where $x, y, z \in \{0, 1\}$ and $x = 1$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$, $y = 1$ iff $\neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$, and $z = 1$ iff $\bullet p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$.
Note that in the extended language $L'(D)$ we now have an individual constant $\bar{\mathbf{t}}$ for every term $\mathbf{t} \in D$.
For any $L'(D)$-term $\mathbf{t}$ ($L'(D)$-sentence ψ), $\widetilde{\mathbf{t}}$ ($\widetilde{\psi}$) is the term (sentence) obtained from $\mathbf{t}$ (ψ) by replacing all the individual constants of the form $\bar{\mathrm{s}}$ occurring in $\mathbf{t}$ (ψ) by the respective (closed) term $\mathbf{s}$. Then the following properties can be easily proved by induction on ψ and ψ':
(prop1) If $\mathsf{CNG}_Z(\psi, \psi')$, then $\mathsf{CNG}_Z(\widetilde{\psi}, \widetilde{\psi'})$.
(prop2) If $\psi \sim^S \psi'$, then $\widetilde{\psi} = \widetilde{\psi'}$.
Next we define the refuting S-valuation $v : Frm^{\mathsf{cl}}_{L'(D)} \to \mathcal{V}$ as follows: $v[\psi] = \langle x_\psi, y_\psi, z_\psi\rangle$, where $x_\psi, y_\psi, z_\psi \in \{0, 1\}$ and $x_\psi = 1$ iff $\widetilde{\psi} \in \Gamma^*$, $y_\psi = 1$ iff $\widetilde{\neg\psi} \in \Gamma^*$, and $z_\psi = 1$ iff $\widetilde{\bullet\psi} \in \Gamma^*$.
Now let ψ, ψ' be two $L'(D)$-sentences and suppose that $\mathsf{CNG}_Z(\psi, \psi')$. Then also $\mathsf{CNG}_Z(\neg\psi, \neg\psi')$ and $\mathsf{CNG}_Z(\bullet\psi, \bullet\psi')$ (since CNG_Z is a congruence relation). By **(prop1)**, $\mathsf{CNG}_Z(\widetilde{\psi}, \widetilde{\psi'})$, $\mathsf{CNG}_Z(\neg\widetilde{\psi}, \neg\widetilde{\psi'})$ and $\mathsf{CNG}_Z(\bullet\widetilde{\psi}, \bullet\widetilde{\psi'})$.

By property (4) of Γ': $\widetilde{\psi} \in \Gamma'$ iff $\widetilde{\psi}' \in \Gamma'$, and similarly for $\neg\widetilde{\psi}$ and $\bullet\widetilde{\psi}$. By definition of v, $v[\psi] = v[\psi']$ and so v respects the CNG_Z-relation.
From **(prop2)** and the definition of v it follows that v also respects the $\sim^S$-relation. It remains to show that v respects the interpretations of the connectives and quantifiers in $\mathcal{QM}_5$. Let us show for the case of $\bullet$. Let $v[\psi] \in \{t, f\}$. Then $\widetilde{\bullet\psi} \notin \Gamma^*$ and so $v[\bullet\psi] \in \mathcal{F}$. Now let $v[\psi] \in \{t_I, f_I\}$. Then $\widetilde{\bullet\psi} \in \Gamma^*$ and so $v[\bullet\psi] \in \mathcal{D}$. Finally, let $v[\psi] = I$. Then $v[\widetilde{\psi}] \in \Gamma^*$, $v[\neg\widetilde{\psi}] \in \Gamma^*$ and $v[\bullet\widetilde{\psi}] \in \Gamma^*$. By property (2) of Γ^*, $\neg\bullet\widetilde{\psi} \notin \Gamma^*$. Since also $\bullet\widetilde{\psi} \in \Gamma^*$, $v[\bullet\psi] \in \{t_I, t\}$. The proof for the rest of the cases is quite similar and is left to the reader.
We have shown that v is $\langle \mathcal{QM}_5, \mathsf{CNG}_Z \rangle$-legal. Also, for every L'-sentence ψ: $v[\psi] \in \mathcal{D}$ iff $\psi \in \Gamma^*$. So $S, v \models_{\mathcal{QM}_5} \Gamma$ (recall that $\Gamma \subseteq \Gamma^*$), but $S, v \not\models_{\mathcal{QM}_5} \psi_0$. Hence, $\Gamma \not\vdash^Z_{\mathcal{QM}_5} \psi_0$.

Next we study the semantic effects of adding the schemata from Ax to a system $\mathbf{QB}^Z$ for $Z \subseteq \mathbf{CNG}$. The obtained semantics is modular, and the addition of a schema imposes a certain refining condition on the basic Nmatrix $\mathcal{QM}_5$, and the semantics of a system is obtained by simply combining all relevant refining conditions.

Definition 24 *The refining conditions for the schemata from Ax are:*
$\mathbf{Cond(c)}$*: $a \in \{f, f_I\} \Rightarrow \tilde{\neg}[a] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(e)}$ $\tilde{\neg}[I] \subseteq \{I\}$
$\mathbf{Cond(i_1)}$*: delete f_I*
$\mathbf{Cond(i_2)}$*: delete t_I.*
$\mathbf{Cond(Dm^1_\wedge)}$*: if $a, b \in \{t, t_I\}$, then $\tilde{\wedge}[a, b] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(Dm^2_\wedge)}$*: if a or b are in $\{f, f_I, I\}$, then $\tilde{\wedge}[a, b] \subseteq \{f, f_I, I\}$*
$\mathbf{Cond(Dm^1_\vee)}$*: if a or b are in $\{t, t_I\}$, then $\tilde{\vee}[a, b] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(Dm^2_\vee)}$*: if $a, b \in \{f, f_I, I\}$, then $\tilde{\vee}[a, b] \subseteq \{f, f_I, I\}$*
$\mathbf{Cond(Dm^1_\forall)}$*: if $H \cap \{f, f_I, I\} = \emptyset$, then $\tilde{\forall}[H] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(Dm^2_\forall)}$*: if $H \cap \{f, f_I, I\} \neq \emptyset$, then $\tilde{\forall}[H] \subseteq \{f, f_I, I\}$.*
$\mathbf{Cond(Dm^1_\exists)}$*: if $H \cap \{t, t_I\} \neq \emptyset$, then $\tilde{\exists}[H] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(Dm^2_\exists)}$*: if $H \cap \{t, t_I\} = \emptyset$, then $\tilde{\exists}[H] \subseteq \{f, f_I, I\}$.*
$\mathbf{Cond(J^1_\wedge)}$*: if $[a \in \{t, f\}$ or $b \in \{f, f_I\}]$ and $[b \in \{t, f\}$ or $a \in \{f, f_I\}]$, then $\tilde{\wedge}[a, b] \subseteq \{t, f\}$.*
$\mathbf{Cond(J^2_\wedge)}$*: if $[a \in \{t_I, f_I, I\}$ and $b \in \{t, I, t_I\}]$ or $[b \in \{t_I, f_I, I\}$ and $a \in \{t, I, t_I\}]$, then $\tilde{\wedge}[a, b] \subseteq \{I, f_I, t_I\}$.*
$\mathbf{Cond(J^1_\vee)}$*: if $[a \in \{t, f\}$ or $b \in \{t, t_I\}]$ and $[b \in \{t, f\}$ or $a \in \{t, t_I\}]$, then $\tilde{\vee}[a, b] \subseteq \{t, f\}$.*

$\mathbf{Cond}(\mathbf{J}^{\mathbf{2}}_{\vee})$: *if* $[a \in \{t_I, I, f_I\}$ *and* $b \in \{f, f_I, I\}]$ *or* $[b \in \{t_I, I, f_I\}$ *and* $a \in \{f, f_I, I\}]$, *then* $\tilde{\vee}[a, b] \subseteq \{t_I, f_I, I\}$.
$\mathbf{Cond}(\mathbf{J}^{\mathbf{1}}_{\supset})$: *if* $a \in \{t, f\}$ *or* $b \in \{f, f_I\}$, *then* $\tilde{\supset}[b, a] \subseteq \{t, f\}$.
$\mathbf{Cond}(\mathbf{J}^{\mathbf{2}}_{\supset})$: *if* $a \in \{t, t_I, I\}$ *and* $b \in \{f_I, t_I, I\}$, *then* $\tilde{\supset}[b, a] \subseteq \{I, t_I, f_I\}$.
$\mathbf{Cond}(\mathbf{J}^{\mathbf{1}}_{\forall})$: *if* $H \subseteq \{t, f\}$ *or* $H \cap \{f, f_I\} \neq \emptyset$, *then* $\tilde{\forall}[H] \subseteq \{t, f\}$.
$\mathbf{Cond}(\mathbf{J}^{\mathbf{2}}_{\forall})$: *for* $H \subseteq \{t, t_I, I\}$, *such that* $t_I \in H$ *or* $I \in H$: $\tilde{\forall}[H] \subseteq \{I, f_I, t_I\}$.
$\mathbf{Cond}(\mathbf{J}^{\mathbf{1}}_{\exists})$: *if* $H \cap \{t, t_I\} \neq \emptyset$ *or* $H \subseteq \{t, f\}$, *then* $\tilde{\exists}[H] \subseteq \{t, f\}$.
$\mathbf{Cond}(\mathbf{J}^{\mathbf{2}}_{\exists})$: *if* $H \subseteq \{I, f, f_I\}$ *and* $\{I, t_I, f_I\} \cap H \neq \emptyset$, *then* $\tilde{\exists}[H] \subseteq \{I, t_I, f_I\}$.
For $X \subseteq Ax$, $\mathcal{QM}_5[X]$ *is the weakest refinement of* $\mathcal{QM}_5$ *which satisfies the refining conditions of the schemata from* X. *In other words,* $\mathcal{QM}_5[X] = \langle \mathcal{V}_X, \mathcal{D}_X, \mathcal{O}_X \rangle$, *where* $\mathcal{V}_X$ *is the set of values from* $\{t, f, t_I, f_I, I\}$ *which are not deleted by any condition of a schema from* X, $\mathcal{D}_X = \mathcal{V}_X \cap \{t, t_I, I\}$, *for any connective* $\diamond$ *and any* $a_1, ..., a_n \in \mathcal{V}_X$, $\tilde{\diamond}_{\mathcal{QM}_5[X]}$ *assigns to* $\overrightarrow{a}$ *the set of all truth-values in* $\tilde{\diamond}_{\mathcal{QM}_5}$ *which are not forbidden by any condition of a schema from* X, *and for* $Q \in \{\forall, \exists\}$ *and any* $H \subseteq P^+(\mathcal{V}_X)$, $\tilde{Q}_{\mathcal{QM}_5[\mathbf{X}]}$ *assigns to* H *the set of all the truth-values in* $\tilde{Q}_{\mathcal{QM}_5}$ *which are not forbidden by any condition of a schema from* X.

Notation: We write $\mathcal{QM}_5\mathbf{s}$ instead of $\mathcal{QM}_5[X]$, where $\mathbf{s}$ is the string of all the names of the schemata from X. It is easy to check that for every $X \subseteq Ax$ the conditions in X are coherent, the interpretations of the connectives and quantifiers in $\mathcal{QM}_5[X]$ are non-empty and so $\mathcal{QM}_5[X]$ is well-defined.

Lemma 25 *Let* $X \subseteq Ax$ *and* $Z \subseteq \mathbf{CNG}$. $\mathcal{QM}_5[X]$ *is effective for* CNG_Z.

Proof: It is easy to see that for any $X \subseteq Ax$, $\mathcal{QM}_5[X]$ is suitable for CNG_Z. The lemma follows from Prop. 18.
Example 1: The interpretations of $\neg, \bullet$ in $\mathcal{QM}_5\mathbf{c}$ are as follows:

	$\mathbf{f}$	$\mathbf{f_I}$	$\mathbf{I}$	$\mathbf{t}$	$\mathbf{t_I}$
$\tilde{\neg}$	$\{t, t_I\}$	$\{t, t_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$
$\tilde{\bullet}$	$\{t, t_I, I\}$	$\{t, t_I, I\}$	$\{t, t_I\}$	$\{f, f_I\}$	$\{t, t_I, I\}$

Example 2: Consider the Nmatrix $\mathcal{QM}_5\mathbf{iJ}^{\mathbf{1}}_{\forall}\mathbf{J}^{\mathbf{1}}_{\exists}$. By **Cond(i)**, f_I and t_I are deleted and we are left with only three truth-values: t, f, I. The interpretations of $\forall$ and $\exists$ in this Nmatrix are:

$\mathbf{H}$	$\widetilde{\forall}[\mathbf{H}]$	$\widetilde{\exists}[\mathbf{H}]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t\}$
$\{t, I\}$	$\{t, I\}$	$\{t\}$
$\{f, I\}$	$\{f\}$	$\{t, I\}$
$\{t, f, I\}$	$\{f\}$	$\{t\}$

Example 3: In [9] it is shown that LFI1* can be characterized by a deterministic three-valued matrix. Note that the Nmatrix $\mathcal{QM}_5\mathbf{cieJDm}_\forall\mathbf{Dm}_\exists$ is indeed completely deterministic and matches the three-valued semantics of [9] (where the truth-values $0, \frac{1}{2}, 1$ are used instead of f, I, t respectively).

Theorem 26 (Soundness and completeness) *For* $X \subseteq Ax$ *and* $Z \subseteq \mathbf{CNG}$*:* $\Gamma \vdash_{\mathbf{QB}^Z[X]} \psi$ *iff* $\Gamma \vdash^Z_{\mathcal{QM}_5[X]} \psi$.

Proof: a straightforward modification of the proof of theorem 23. We only have to check that the conditions imposed by the schemata in X are respected by the valuation v defined in the proof. Let us show the proof for the case of $(\mathbf{Dm}^\mathbf{1}_\forall)$, for instance. Suppose that $(\mathbf{Dm}^\mathbf{1}_\forall) \in X$. Then from the definition of Γ^* it follows that $\exists x\neg\varphi \in \Gamma^*$ whenever $\neg\forall x\varphi \in \Gamma^*$. Now let $\forall x\psi$ be a sentence, such that for $H_\psi = \{\psi\{\overline{a}/x\} \mid a \in D\}$: $(*)$ $H_\psi \cap \{f, f_I, I\} = \emptyset$. Suppose by contradiction that v does not respect the interpretation of $\forall$ in $\mathcal{QM}_5[X]$ and so $v[\forall x\psi] \not\in \{t, t_I\}$. Then $\neg\widetilde{\forall x\psi} = \neg\forall x\widetilde{\psi} \in \Gamma^*$ and so $\exists x\neg\widetilde{\psi} \in \Gamma^*$. By property 8 of Γ^*, there is some closed term $\mathbf{t} \in D$, such that $\neg\widetilde{\psi}\{\mathbf{t}/x\} = \neg\widetilde{\psi\{\bar{\mathbf{t}}/x\}} \in \Gamma^*$. Then $v[\neg\psi\{\bar{\mathbf{t}}/x\}] \in \{f, f_I, I\}$, in contradiction to $(*)$.

The modular approach of Nmatrices provides some new insights into the semantic role of each of the schemata considered above and the dependencies between them. For instance, it is easy to see that for every $\mathbf{x} \in \{\wedge, \vee, \forall, \exists\}$ and $\mathbf{j} \in \{1, 2\}$, the semantic effects of $\mathbf{Cond}(\mathbf{J}^\mathbf{j}_\mathbf{x})$ and $\mathbf{Cond}(\mathbf{Dm}^\mathbf{j}_\mathbf{x})$ (see Defn. 24) on $\mathcal{QM}_5$ differ only in their behavior for the truth-values t_I and f_I. In the presence of **(i**$_1$**)** and **(i**$_2$**)**, t_I and f_I are deleted and the semantic effects of $\mathbf{Cond}(\mathbf{J}^\mathbf{j}_\mathbf{x})$ and $\mathbf{Cond}(\mathbf{Dm}^\mathbf{j}_\mathbf{x})$ on $\mathcal{QM}_5\mathbf{i}$ coincide. This leads to the following important observation:

Corollary 27 *For* $\mathbf{x} \in \{\wedge, \vee, \forall, \exists\}$ *and* $\mathbf{i} \in \{1, 2\}$*:* $\vdash_{\mathbf{QBiJ}^\mathbf{i}_\mathbf{x}} \mathbf{Dm}^\mathbf{i}_\mathbf{x}$ *and* $\vdash_{\mathbf{QBiDm}^\mathbf{i}_\mathbf{x}} \mathbf{J}^\mathbf{i}_\mathbf{x}$.

We conclude that $\mathbf{Dm^i_x}$ and $\mathbf{J^i_x}$ are equivalent in any extension of $\mathbf{QBi}$, and so two of the schemata in the axiomatization of LFI1* in [10] are derivable from the rest of the axioms.

Next we apply the effectiveness property of the semantics defined above to prove a non-trivial proof-theoretical property of a subclass of the studied LFIs. First we formalize the notion of *logical indistinguishability* for first-order languages.

Definition 28 *Given an L-sentence ψ, the set $SSF(\psi)$ of its* substitutable subformulas *is defined as follows: (i) $SSF(p(t_1,...,t_n)) = \{p(t_1,...,t_n)\}$, (ii) $SSF(\diamond(\psi_1,...,\psi_n)) = \{\diamond(\psi_1,...,\psi_n)\} \cup SSF(\psi_1) \cup ... \cup SSF(\psi_n)$, (iii) If $x \notin Fv[\psi]$, then $SSF(Qx\psi) = \{Qx\psi\} \cup SSF(\psi)$. Otherwise, $SSF(Qx\psi) = \{Qx\psi\}$.*
For L-sentences φ, ψ, denote by $\varphi(\psi)$ an L-sentence obtained by substituting some $\theta \in SSF(\varphi)$ for ψ in φ. We say that two L-sentences ψ_1, ψ_2 are logically indistinguishable *in a system* **S** *if for every L-sentence φ:*
$\vdash_{\mathbf{S}} \psi_1 \leftrightarrow \psi_2$ *implies* $\vdash_{\mathbf{S}} \varphi(\psi_1) \leftrightarrow \varphi(\psi_2)$.

Theorem 29 *Let* **S** *be a system over L_C, such that $\psi_1 \vdash_{\mathbf{S}} \psi_2$ whenever $\mathsf{CNG}_Z(\psi_1, \psi_2)$. Let $Z \subseteq \mathbf{CNG}$. If $\mathbf{QB}^Z\mathbf{ieJ}$ is an extension of* **S**, *then two L-sentences ψ, φ are logically indistinguishable in* **S** *iff $\mathsf{CNG}_Z(\psi, \varphi)$.*

Proof: Assume that $\mathsf{CNG}_Z(\psi, \varphi)$ does not hold. Let S be an L-structure and W - the minimal set of $L(D)$-sentences S-closed under subformulas, such that $\neg\neg\neg(\varphi \supset \varphi) \in W$. Define a partial S-valuation v on W, such that: $v[\varphi \supset \varphi] = t$, $v[\neg(\varphi \supset \varphi)] = f$, $v[\neg\neg(\varphi \supset \varphi)] = I$, $v[\neg\neg\neg(\varphi \supset \varphi)] = I$. Extend v, so that: $v[\psi \supset \psi] = t$, $v[\neg(\psi \supset \psi)] = f$, $v[\neg\neg(\psi \supset \psi)] = t$, $v(\neg\neg\neg(\psi \supset \psi)) = f$. It is easy to see that v is CNG_Z-legal in $\mathcal{QM}_5\mathbf{ieJ}$. By Lemma 25, v can be extended to a full valuation which is $\langle \mathcal{QM}_5\mathbf{ieJ}, \mathsf{CNG}_Z\rangle$-legal. By Theorem 26, $\neg\neg\neg(\varphi \supset \varphi) \nvdash_{\mathbf{QBieJ}[Z]} \neg\neg\neg(\psi \supset \psi)$. Since $\mathbf{QBieJ}[Z]$ is an extension of **S**, $\neg\neg\neg(\varphi \supset \varphi) \nvdash_{\mathbf{S}} \neg\neg\neg(\psi \supset \psi)$ Hence ψ and φ are not logically indistinguishable in **S**. The other direction is trivial.
Remark: This theorem extends the results in [2, 17] and Remark 4.8 in [9] (concerning the propositional fragment of **QBieJDm**, called LFI1 there) by covering all the *first-order* systems between **QBie** and **QBieJ** and by considering the congruence-based postulates Z for $Z \subseteq \mathbf{CNG}$.

6 SUMMARY AND FURTHER RESEARCH

Non-deterministic multi-valued matrices are an attractive semantic framework due to their modularity and effectiveness. In this paper we have applied Nmatrices to provide finite-valued non-deterministic semantics for a large useful family of first-order LFIs and explored their effectiveness. The modular approach provides new insights into the semantic roles of each of the studied schemata and the dependencies between them. For instance, we have shown that four of the schemata from the axiomatization of LFI1* given in [10] are derivable from the rest of its axioms.

One immediate research direction is extending the results of this paper to first-order paraconsistent systems with equality, as well as to higher-order languages. Another direction is developing a general theory of Nmatrices, determining the type of systems, for which the methods demonstrated in this paper are applicable, and formulating the conditions under which various schemata impose non-contradictory semantic conditions on the corresponding Nmatrix. The third direction is increasing the degree of non-determinism of Nmatrices by allowing also non-deterministic interpretations of predicate and function symbols of the language. Such extension of Nmatrices seems particularly promising for representing fuzzy notions.

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