Modular Construction of Cut-free Sequent Calculi for Paraconsistent Logics

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***Abstract*—This paper makes a substantial step towards automatization of paraconsistent reasoning by providing a general method for a systematic and modular generation of cut-free calculi for thousands of paraconsistent logics known as Logics of Formal (In)consistency. The method relies on the use of non-deterministic semantics for these logics.**

I. Introduction

It is a fact of life that large knowledge bases are inherently inconsistent, in the same way that large programs are inherently buggy. However, classical logic (CL) fails to accommodate the fact that knowledge bases containing contradictory data may still produce useful answers to queries. This is because in CL a single inconsistency leads to trivialization of the whole knowledge base. Accordingly, over the last decades there has been a growing interest in computer science applications of paraconsistent logics — logics which allow non-trivial inconsistent theories. Integration of information from multiple sources in large knowledge bases, negotiations among agents with conflicting goals, and complex software specifications in which different stake-holders have incompatible requirements are just a few cases in point. Recently, suggestions have even been made (see, e.g., [19]) to adopt paraconsistent logic as a foundational concept for future information systems engineering.

One of the oldest and best known approaches to paraconsistency is that by da Costa ([17], [18]), which seeks to allow the use of classical logic whenever this is safe, but behaves completely differently when contradictions are involved. This approach has led to the introduction of the family of *Logics of Formal (In)consistency (LFIs)* (see [12], [13] for surveys and [14] for an application of LFIs for integration of inconsistent information in evolutionary databases). This family is based on two key ideas. The first is that propositions should be divided into two sorts: the "normal" (or consistent), and the "abnormal" (or inconsistent) ones. While classical logic can be applied freely to normal propositions, its use for the abnormal ones is restricted. The second idea is to reflect this classification within the language used. In the most important class of LFIs called *C-systems* ([13]), this is done by employing a special (either primitive or defined) connective $\circ$, where the intuitive meaning of $\circ\varphi$ is "φ is consistent".

Since their introduction in terms of Hilbert-style systems in the 1960s, the main obstacle to efficient use of C-systems has been the lack of analytic calculi for them. Efforts towards finding such calculi were at first concentrated on da Costa's historical system C_1. After an aborted attempt by Raggio in the 1960s ([24]), Beziau proposed in [10] somewhat peculiar sequent rules for C_1. Establishing cut-elimination for Beziau's system was a non-trivial task, which was accomplished only much later (in [11]). At about the same time, Carnielli et al. introduced a tableau system for C_1 ([15], [12], [16]). Recently some analytic calculi have been introduced also for a few other C-systems ([23], [22]). However, since each of these calculi was tailored to some specific system, their rules were introduced in a sort of an ad-hoc manner, and so they have no uniform structure. Therefore, even a slight modification in any of them means starting the search for a corresponding analytic calculus almost from scratch.

In this paper we provide a *uniform* and *modular* method for a *systematic* generation of cut-free sequent calculi for a large family of paraconsistent logics, which practically includes every C-system ever studied in the literature. The method is based on the semantics provided in [4] for this family. This semantics is given in terms of *non-deterministic matrices* (Nmatrices), a natural generalization of standard multi-valued matrices obtained by importing the notion of *non-deterministic computations* from computer science into the truth-tables of logical connectives. For some of these systems, the corresponding semantics is three-valued. In this case (as was explained in [6] using a concrete example of one such LFI), one can exploit the algorithm given in [5] for constructing cut-free Gentzen-type systems for logics which have a characteristic finite-valued Nmatrix (and whose language is sufficiently expressive). However, this method does not apply to any LFI which has no finite semantic characterization in terms of Nmatrices. Unfortunately, some of the most important LFIs, including da Costa's original C_1, cannot have such a characterization. Nevertheless, they do have infinitely-valued characterizations of this kind (sufficient to guarantee their decidability). In this paper we show that these characterizations can be used to extract cut-free sequent calculi for these logics, while preserving the crucial property of *modularity* of the method. We believe that these results can

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open the door to construction and implementation of efficient theorem provers based on this type of paraconsistent logics, which in turn will lead to their useful new applications for reasoning under uncertainty.

II. Preliminaries

In what follows, $\mathcal{L}$ is a propositional language, and $Frm_{\mathcal{L}}$ is its set of wffs.

Definition 1. A *propositional logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a structural[1] and finitary[2] non-trivial (Tarskian) consequence relation for $\mathcal{L}$.

The notion of paraconsistency (with respect to $\neg$) is usually defined as follows (see, e.g., [12]):

Definition 2. Let $\mathcal{L}$ be a language which includes a unary connective $\neg$. A propositional logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is *paraconsistent* (with respect to $\neg$) if there are formulas $\psi, \phi \in Frm_{\mathcal{L}}$ such that $\psi, \neg\psi \not\vdash \phi$.[3]

A. Taxonomy of C-systems

Let $\mathcal{L}_{cl} = \{\wedge, \vee, \supset, \neg\}$, $\mathcal{L}^{+}_{cl} = \{\wedge, \vee, \supset\}$, and $\mathcal{L}_{C} = \{\wedge, \vee, \supset, \neg, \circ\}$.

Definition 3. Let HCL^{+} be a standard Hilbert-style system which has MP as the only inference rule, and is sound and strongly complete for the positive fragment (i.e., the $\mathcal{L}^{+}_{cl}$-fragment) of classical propositional logic.

1) The system $\mathbf{B}$ for $\mathcal{L}_{C}$ is obtained by adding to HCL^{+} the axioms $(\mathbf{t})$ $\neg\varphi \vee \varphi$ and $(\mathbf{b})$ $\circ\varphi \supset (\varphi \wedge \neg\varphi \supset \psi)$.
2) The system **BK** (for $\mathcal{L}_{C}$) is obtained by adding to $\mathbf{B}$ the axiom $(\mathbf{k})$ $\circ\varphi \vee (\varphi \wedge \neg\varphi)$.

Logics of Formal (In)consistency (LFIs) form a large family of paraconsistent logics, in which the notion of consistency is internalized. Namely, a paraconsistent logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is an LFI if there is an atomic variable p and a set $X(p)$ of $\mathcal{L}$-formulas containing only the variable p such that $\psi, \neg\psi, X\{\psi/p\} \vdash \varphi$[4] for every $\psi, \varphi \in Frm_{\mathcal{L}}$. A particularly useful subclass of LFIs is that of *C-systems*, in which $X(p)$ is a singleton:

Definition 4. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with $\mathcal{L}$ containing $\mathcal{L}_{cl}$. We say that $\mathbf{L}$ is a *C-system* if (i) $\mathbf{L}$ contains the $\mathcal{L}^{+}_{cl}$-fragment of classical logic, (ii) $\mathbf{L}$ is paraconsistent, and (iii) $\mathbf{L}$ has a (primitive or defined) unary connective $\circ$, for which $(\mathbf{b})$, $(\mathbf{t})$ and $(\mathbf{k})$ are valid in $\mathbf{L}$.

In what follows, we take $\circ$ to be a primitive connective of the language, leaving handling of C-systems which do not have such a connective to Section V.

Remark 5. The original definition of C-systems in [12], [13] is slightly different from the above one: it does not require the validity of $(\mathbf{k})$. Indeed, in [12], [13] the system $\mathbf{B}$ (called **mbC** there) is considered to be the most basic C-system. Nevertheless, we find it much more appropriate to choose **BK** for this role. First of all, given the intended meaning of $\circ\varphi$ as "φ is consistent", the meaning of axiom **(b)** is that no formula is both consistent and contradictory. Axiom **(k)** complements this by saying that every formula is either consistent or contradictory. This last principle seems to be as essential for the intended meaning of $\circ\varphi$ as that expressed by axiom **(b)**. Another strong indication that **BK** is the most natural basic C-system is that in the Gentzen-type system for this logic the right and left introduction rules for all the connectives other than $\neg$ are dual: including one of them guarantees the invertibility (Definition 34) of the other in **BK** (see Proposition 35 below). *This includes the rules for* $\circ$ *corresponding to axioms* $(\mathbf{b})$ *and* $(\mathbf{k})$. Finally, $(\mathbf{k})$ is anyway a theorem of almost every important C-system ever studied. This is due to the fact that it is derivable in $\mathbf{B}$ from each of the three most important axioms concerning $\circ$ which have been studied in the literature: those denoted below by $(\mathbf{i})$, $(\mathbf{l})$, and $(\mathbf{d})$. These dependencies are easily established (see Example 33 for the case of $(\mathbf{l})$).

Next we provide a list of plausible axioms with which **BK** can be extended. They are divided into two parts: one dealing with combinations of negation with classical connectives, and the other involving the connective $\circ$.

Definition 6. Let $\mathbf{ACL}$ be the following set of axioms:

$(\mathbf{c})$ $\neg\neg\varphi \supset \varphi$	$(\mathbf{e})$ $\varphi \supset \neg\neg\varphi$
$(\mathbf{n}^{\mathbf{l}}_{\wedge})$ $\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$	$(\mathbf{n}^{\mathbf{r}}_{\wedge})$ $(\neg\varphi \vee \neg\psi) \supset \neg(\varphi \wedge \psi)$
$(\mathbf{n}^{\mathbf{l}}_{\vee})$ $\neg(\varphi \vee \psi) \supset (\neg\varphi \wedge \neg\psi)$	$(\mathbf{n}^{\mathbf{r}}_{\vee})$ $(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$
$(\mathbf{n}^{\mathbf{l}}_{\supset})$ $\neg(\varphi \supset \psi) \supset (\varphi \wedge \neg\psi)$	$(\mathbf{n}^{\mathbf{r}}_{\supset})$ $(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$

Let $\mathbf{ADC}$ be the following set of axioms for $\sharp \in \{\wedge, \vee, \supset\}$:

$(\mathbf{i})$ $\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)$	
$(\mathbf{o}^{\mathbf{1}}_{\sharp})$ $\circ\varphi \supset \circ(\varphi\sharp\psi)$	$(\mathbf{o}^{\mathbf{2}}_{\sharp})$ $\circ\psi \supset \circ(\varphi\sharp\psi)$
$(\mathbf{a}_{\sharp})$ $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi\sharp\psi)$	$(\mathbf{a}_{\neg})$ $\circ\varphi \supset \circ\neg\varphi$
$(\mathbf{l})$ $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$	$(\mathbf{d})$ $\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$

Let $\mathbf{A} = \mathbf{ACL} \cup \mathbf{ADC}$ and $\mathbf{A_0} = \mathbf{A} \setminus \{(\mathbf{l}), (\mathbf{d})\}$.

Definition 7. For $A \subseteq \mathbf{A}$, $\mathbf{BK}[A]$ is the system obtained from **BK** by adding to it all the axioms in A.

Remark 8. The set $\mathbf{ADC} \cup \{(\mathbf{c}), (\mathbf{e})\}$ includes the axioms most frequently used in the literature on LFIs ([12], [13]). Instead of our $(\mathbf{o}^{\mathbf{1}}_{\sharp})$ and $(\mathbf{o}^{\mathbf{2}}_{\sharp})$, one usually finds there the axiom $(\mathbf{o}_{\sharp})$ equivalent in **BK** to the conjunction of $(\mathbf{o}^{\mathbf{1}}_{\sharp}), (\mathbf{o}^{\mathbf{2}}_{\sharp})$. The $(\mathbf{n})$-axioms are important classical tautologies related to negation. The family of logics obtained by adding some of them to the $\mathcal{L}^{+}_{cl}$-fragment of classical logic was studied in [3]. By adding *all* the axioms from **ACL** to that fragment, we get the three-valued logic PAC (first introduced in [9]). By further addition of the axioms $(\mathbf{b})$ and $(\mathbf{i})$ (which implies $(\mathbf{k})$), the famous three-valued logic J_3 ([21], [20]) is obtained. The extension of $\mathbf{B}$ with the axiom $(\mathbf{c})$ was denoted in [22] by $\mathbf{bC}$, and with both $(\mathbf{c})$ and $(\mathbf{i})$ – by $\mathbf{Ci}$.

[1] $\vdash$ is structural if, for every uniform substitution θ, $T \vdash \psi$ implies $\theta(T) \vdash \theta(\psi)$.

[2] $\vdash$ is finitary if $T \vdash \psi$ implies that there is a finite $\Gamma \subseteq T$ such that $\Gamma \vdash \psi$.

[3] As $\vdash$ is structural, it is enough to require that there are atoms p, q such that $p, \neg p \not\vdash q$.

[4] The substitution $X\{\psi/p\}$ is understood in the standard way.

Notation 9. In the sequel we shall usually omit the various brackets, and write, e.g., $\mathbf{BKco}^{\mathbf{1}}_{\wedge}$ instead of $\mathbf{BK}[\{(\mathbf{c}),(\mathbf{o}^{\mathbf{1}}_{\wedge})\}]$. Moreover, we shall write, e.g., $\mathbf{BKa}$ instead of $\mathbf{BK}[\{(\mathbf{a}_{\wedge}),(\mathbf{a}_{\vee}),(\mathbf{a}_{\supset})\}]$, and use similar abbreviations in the cases of $(\mathbf{o})$ and $(\mathbf{n})$.

Remark 10. da Costa's original system C_1 ([17], [18]) is known to be equivalent to the $\circ$-free fragment of $\mathbf{BKcila}$. It follows from our results (see Corollary 57 below) that it is also equivalent to the $\circ$-free fragment of $\mathbf{BKcla}$.

Remark 11. A more modular treatment of the axioms from $\mathbf{A}$ becomes possible if we split some of them. Thus $(\mathbf{i})$ is equivalent to the conjunction of $(\mathbf{i_1})$ $\neg\circ\varphi \supset \varphi$ and $(\mathbf{i_2})$ $\neg\circ\varphi \supset \neg\varphi$, and $(\mathbf{n}^{\mathbf{r}}_{\wedge})$ — to the conjunction of $(\mathbf{n}^{\mathbf{r},\mathbf{1}}_{\wedge})$ $\neg\varphi \supset \neg(\varphi\wedge\psi)$ and $(\mathbf{n}^{\mathbf{r},\mathbf{2}}_{\wedge})$ $\neg\psi \supset \neg(\varphi\wedge\psi)$. Similar splitting can be done for $(\mathbf{n}^{\mathbf{l}}_{\vee})$ and $(\mathbf{n}^{\mathbf{l}}_{\supset})$. However, due to the lack of space, we shall stick to the current presentation.

Remark 12. Not all of the systems of the form $\mathbf{BK}[A]$ for $A \subseteq \mathbf{A}$ are different from each other. Thus $(\mathbf{a}_{\neg})$ is equivalent in $\mathbf{BK}$ to $(\mathbf{c})$, and $(\mathbf{a}_{\wedge})$ is equivalent to $(\mathbf{n}^{\mathbf{l}}_{\wedge})$. Moreover, not all these systems are paraconsistent: as explained below, any conflict between some axioms in A causes the collapse of $\mathbf{BK}[A]$ to classical logic. Using the semantics provided below, one can check mechanically all the dependencies and conflicts between the axioms of $\mathbf{A}$. Their full list is provided in the sequel.

Remark 13. It is easy to see that the "converse" of $(\mathbf{i})$ (i.e, $\varphi\wedge\neg\varphi \supset \neg\circ\varphi$) and the converse of $(\mathbf{l})$ (i.e, $\circ\varphi \supset \neg(\varphi\wedge\neg\varphi)$) are theorems of $\mathbf{B}$. Together, the four implications intuitively mean that $\circ\varphi$ and $\neg(\varphi\wedge\neg\varphi)$ "have the same meaning". On the other hand, $(\mathbf{d})$, its converse, and $(\mathbf{i})$ taken together mean that $\circ\varphi$ and $\neg(\neg\varphi\wedge\varphi)$ "have the same meaning". This intuition will be exploited in the sequel to provide Gentzen-type rules corresponding to $(\mathbf{l})$ and $(\mathbf{d})$. It should be emphasized that $(\mathbf{l})$ and $(\mathbf{d})$ are *not* equivalent in $\mathbf{BK}$ (this follows directly from the semantics provided below).

Remark 14. A note is in order here on the relationship between the Gentzen-type systems provided below and the corresponding Hilbert-style systems originally used to formulate the C-systems discussed above. Namely, using cuts one can show in a standard way that each such Gentzen-type system $\mathbf{G}$ is equivalent to the corresponding Hilbert-type system H in the sense that $T\vdash_H \psi$ iff $T\vdash_G \psi$ (where the consequence relation $\vdash_{\mathbf{G}}$ is defined, as usual, by: $T\vdash_{\mathbf{G}}\psi$ if there is a finite $\Gamma\subseteq T$ such that $\vdash_{\mathbf{G}} \Gamma\Rightarrow\psi$). In particular, ψ is a theorem of H iff $\vdash_{\mathbf{G}}\Rightarrow\psi$.

B. Non-deterministic Matrices

Our main semantic tool in what follows will be *non-deterministic multi-valued matrices* (Nmatrices), introduced in [7]. These structures are a natural generalization of the concept of a many-valued matrix, in which the truth-value assigned to a complex formula is chosen *non-deterministically* out of a given non-empty set of options.

Definition 15.
1) A non-deterministic matrix (Nmatrix) for a language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle\mathcal{V},\mathcal{D},\mathcal{O}\rangle$, where: $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$, and $\mathcal{O}$ includes an interpretation function $\tilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to P^+(\mathcal{V})$ for every n-ary connective $\diamond$. We say that $\mathcal{M}$ is *finite* if so is $\mathcal{V}$.
2) Let $\mathcal{M} = \langle\mathcal{V},\mathcal{D},\mathcal{O}\rangle$ be an Nmatrix, and let F be a set of $\mathcal{L}$-formulas closed under subformulas. An $\mathcal{M}$-valuation on F is a function $v : F \to \mathcal{V}$ which, for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1,\dots,\psi_n \in F$ such that $\diamond(\psi_1,\dots,\psi_n) \in F$, satisfies the condition: $v(\diamond(\psi_1,\dots,\psi_n)) \in \tilde{\diamond}_{\mathcal{M}}(v(\psi_1),\dots,v(\psi_n))$. A *full $\mathcal{M}$-valuation* is an $\mathcal{M}$-valuation on $Frm_{\mathcal{L}}$.
3) Let F be as above, and let $\psi\in F$. An $\mathcal{M}$-valuation v on F *satisfies* ψ, in symbols $v\models_{\mathcal{M}}\psi$, if $v(\psi)\in\mathcal{D}$. A valuation v satisfies a set $\Gamma\subseteq F$ of formulas, in symbols $v\models_{\mathcal{M}}\Gamma$, if it satisfies every formula in Γ.
4) Let F be as above, and let v be an $\mathcal{M}$-valuation on F. A sequent $\Gamma\Rightarrow\Delta$ such that $\Gamma\cup\Delta\subseteq F$ is *true under* v if either there is some $\psi\in\Delta$ such that $v\models_{\mathcal{M}}\psi$, or $v\not\models_{\mathcal{M}}\psi$ for every $\psi\in\Gamma$. A sequent is *valid in* $\mathcal{M}$ if it is true under every full $\mathcal{M}$-valuation.
5) $\vdash_{\mathcal{M}}$, the consequence relation induced by $\mathcal{M}$, is defined by: $T\vdash_{\mathcal{M}}\psi$ if $v\models_{\mathcal{M}}\psi$ for every full $\mathcal{M}$-valuation v such that $v\models_{\mathcal{M}} T$.

Nmatrices enjoy the most attractive properties of usual (deterministic) finite-valued matrices. This includes the following:

Proposition 16. (Compactness) *([7]) If $\mathcal{M}$ is a finite Nmatrix, then $T\vdash\psi$ iff there is a finite $\Gamma\subseteq T$ such that $\Gamma\vdash_{\mathcal{M}}\psi$.*

Proposition 17. (Semantic Analyticity) *[4] Let F be a set of $\mathcal{L}$-formulas closed under subformulas, and let $\mathcal{M}$ be an Nmatrix for $\mathcal{L}$. Then any $\mathcal{M}$-valuation on F can be extended to a full $\mathcal{M}$-valuation.*

Corollary 18. (Decidability) *For every finite Nmatrix $\mathcal{M}$, the question whether $\Gamma\vdash_{\mathcal{M}}\psi$ is decidable for every finite set of formulas Γ and every formula ψ.*

Definition 19. We say that an Nmatrix $\mathcal{M}$ is *characteristic* for a Gentzen-type system $\mathbf{G}$ if, for every Γ and Δ, $\vdash_{\mathbf{G}}\Gamma\Rightarrow\Delta$ holds iff $\Gamma\Rightarrow\Delta$ is valid in $\mathcal{M}$.

Remark 20. If $\mathcal{M}$ is characteristic for $\mathbf{G}$, then $\vdash_{\mathbf{G}}\Gamma\Rightarrow\psi$ iff $\Gamma\vdash_{\mathcal{M}}\psi$. By the compactness theorem (Proposition 16), if $\mathcal{M}$ is finite, then the latter implies that $\vdash_{\mathcal{M}}=\vdash_{\mathbf{G}}$.

The following notion of a *simple refinement* will be useful in the sequel:

Definition 21. Let $\mathcal{M}_1 = \langle\mathcal{V}_1,\mathcal{D}_1,\mathcal{O}_1\rangle$ and $\mathcal{M}_2 = \langle\mathcal{V}_2,\mathcal{D}_2,\mathcal{O}_2\rangle$. $\mathcal{M}_2$ is a *simple refinement* of $\mathcal{M}_1$ if $\mathcal{V}_1=\mathcal{V}_2$, $\mathcal{D}_1=\mathcal{D}_2$ and for every n-ary connective $\diamond$ and every $a_1,\dots,a_n\in\mathcal{V}_1$, $\tilde{\diamond}_{\mathcal{M}_2}(a_1,\dots,a_n)\subseteq\tilde{\diamond}_{\mathcal{M}_1}(a_1,\dots,a_n)$.

Proposition 22. ([4]) *If $\mathcal{M}_2$ is a simple refinement of $\mathcal{M}_1$, then $\vdash_{\mathcal{M}_1}\subseteq\vdash_{\mathcal{M}_2}$.*

III. Calculi with Quasi-Canonical Rules

A method for a systematic construction of cut-free Gentzen-type calculi for C-systems which have a finite characterization in terms of Nmatrices was proposed in [6], using an example of a particular system. In this section we apply this method for all the systems of the form $\mathbf{BK}[A]$ for $A \subseteq \mathbf{A_0}$ (systems which include the axioms $(\mathbf{l})$ and $(\mathbf{d})$ will be handled in the next section). As explained at the end of this section, the obtained calculi have rules of a certain uniform *quasi-canonical* form. Now the method is based on the following two key facts. First and foremost, all systems of the form $\mathbf{BK}[A]$ for $A \subseteq \mathbf{A_0}$ have a semantic characterization in terms of finite-valued (in fact, three-valued) Nmatrices. These characterizations can be obtained in a modular way within the finite-valued non-deterministic semantic framework developed in [4]. Secondly, [5] provides an algorithm for constructing cut-free Gentzen-type systems for logics which have a characteristic finite-valued Nmatrix $\mathcal{M}$ and whose language is *sufficiently expressive* with respect to $\mathcal{M}$. The latter can be easily shown to be the case in our context (see the proof of Proposition 14 in [6]).

Below we provide non-deterministic three-valued semantics for every system $\mathbf{BK}[A]$ such that $A \subseteq \mathbf{A_0}$, and then introduce their corresponding Gentzen-type systems.

A. Non-deterministic Three-valued Semantics

Our non-deterministic semantics is based on the following four truth-values, the intuition being that a formula φ is assigned a truth-value of the form $\langle x, y\rangle$, where $x = 1$ iff φ is "true", and $y = 1$ iff $\neg\varphi$ is "true":

$$t = \langle 1,0\rangle, f = \langle 0,1\rangle, \top = \langle 1,1\rangle, \bot = \langle 0,0\rangle$$

First we note that the axiom $(\mathbf{t})$ $\varphi \vee \neg\varphi$, included already in $\mathbf{B}$, rules out the fourth truth-value $\bot$ (as it intuitively means that φ and $\neg\varphi$ cannot be both "false"), and so we are left with three truth-values: t, f and $\top$. Semantics for systems without the axiom **(t)** (which are obtained from the positive fragment of classical logic by adding some axioms from $\mathbf{A_0}$) can be provided in a similar way using the above four truth-values (see, e.g., [2]).

We start by defining the Nmatrix $\mathcal{M}^3$ for $\mathbf{BK}$:

Definition 23. $\mathcal{M}^3 = (\{t, f, \top\}, \{t, \top\}, \mathcal{O})$ is an Nmatrix for $\mathcal{L}_C$ defined as follows:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{f\}$
f	$\{t,\top\}$	$\{t,\top\}$

$\wedge$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

$\vee$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$

$\supset$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$

Proposition 24. ([1]) *$T \vdash_{\mathcal{M}^3} \psi$ iff $T \vdash_{\mathbf{BK}} \psi$.*

We now turn to providing non-deterministic semantics for the extensions of $\mathbf{BK}$ with axioms from $\mathbf{A_0}$. The semantics are *modular* in the following sense: each axiom $ax \in \mathbf{A_0}$ corresponds to certain semantic conditions $\mathsf{C}(ax)$, which are automatically extracted from ax. These conditions lead to simple refinements of the basic Nmatrix $\mathcal{M}^3$ (which amount to reducing its level of non-determinism). The semantics of $\mathbf{BK}[A]$ is then obtained by straightforwardly combining the semantic effects of all the schemata from A.
Tables I and II include the various semantic conditions that correspond to the axioms in $\mathbf{A_0}$. Note that $(\mathbf{a}_\neg)$ and $(\mathbf{a}_\wedge)$ are not included in the tables, as they are equivalent in $\mathbf{BK}$ to $(\mathbf{c})$ and $(\mathbf{n}^{\mathbf{l}}_\wedge)$ respectively (recall Remark 12). We also include in these tables a reformulation $\mathsf{GC}(ax)$ of each semantic condition $\mathsf{C}(ax)$ using the sets $\mathcal{T} = \{t\}$, $\mathcal{I} = \{\top\}$ and $\mathcal{F} = \{f\}$. This reformulation will be useful later, for handling axioms $(\mathbf{l})$ and $(\mathbf{d})$. Note that to ensure uniformity with $\mathsf{GC}(ax)$, in the formulation of $\mathsf{C}(ax)$ we use inclusion instead of equality (e.g., we write $\circ f \subseteq \{t\}$ instead of $\circ f = \{t\}$).

Example 25. To see how the semantic conditions are derived, consider, e.g., the schema $(\mathbf{a}_\vee)$. To guarantee its validity, we must ensure that $(*)$ $v(\circ\varphi\wedge\circ\psi) \in \{t,\top\}$ implies $v(\circ(\varphi\vee\psi)) \in \{t,\top\}$. In any simple refinement of $\mathcal{M}^3$, $v(\circ(\varphi\vee\psi)) \in \{t,\top\}$ iff $v(\varphi \vee \psi) \in \{f, t\}$. Moreover, $v(\circ\varphi \wedge \circ\psi) \in \{t,\top\}$ iff $v(\circ\varphi), v(\circ\psi) \in \{t,\top\}$ iff $v(\varphi), v(\psi) \in \{t, f\}$. Since $v(\varphi \vee \psi) \in \{f, t\}$ is already guaranteed if $v(\varphi) = v(\psi) = f$, this and the truth table of $\vee$ in $\mathcal{M}^3$ together entail the two conditions for $b \in \{t, f\}$ which are given in Table II: (i) $t \vee b \subseteq \{t\}$ and (ii) $b \vee t \subseteq \{t\}$.

Definition 26. For $A \subseteq \mathbf{A_0}$, the Nmatrix $\mathcal{M}^3[A]$ is the weakest simple refinement of $\mathcal{M}^3$ in which $\mathsf{C}(ax)$ (from Tables I and II) holds for every $ax \in A$.

Proposition 27. *For $A \subseteq \mathbf{A_0}$, $T \vdash_{\mathcal{M}^3[A]} \psi$ iff $T \vdash_{\mathbf{BK}[A]} \psi$.*

Proof: A slight modification of Theorem 3 in [4]. ☐

All the inconsistencies among the axioms in $\mathbf{A_0}$ can be found using a mechanical check based on the semantic conditions from Tables I and II. Detection of a conflict implies that $\mathbf{BK}[A]$ is not paraconsistent. Consider, for instance, a set $A \subseteq \mathbf{A_0}$ such that $(\mathbf{o}^{\mathbf{1}}_\wedge), (\mathbf{n}^{\mathbf{r}}_\wedge) \in A$. The semantic condition corresponding to $(\mathbf{n}^{\mathbf{r}}_\wedge)$ implies that, for any $a \in \{f, \top\}$ and $x \in \{t, \top, f\}$, $x \wedge a \subseteq \{f, \top\}$. On the other hand, the semantic condition corresponding to $(\mathbf{o}^{\mathbf{1}}_\wedge)$ is: for $b \in \{t, \top\}$, $t \wedge b \subseteq \{t\}$. There is an obvious conflict in the case of $t \wedge \top$. This conflict is resolved only by deleting $\top$ from the set of available truth-values. It can be shown that this implies that the system under discussion is equivalent to classical logic, and so is not paraconsistent.

Definition 28. Let $A \subseteq \mathbf{A_0}$. We say that A is *coherent* if it does not contain any of the following pairs of axioms: (i) $(\mathbf{o}^{\mathbf{1}}_\wedge)$ and $(\mathbf{n}^{\mathbf{r}}_\wedge)$; (ii) $(\mathbf{o}^{\mathbf{2}}_\wedge)$ and $(\mathbf{n}^{\mathbf{r}}_\wedge)$; (iii) $(\mathbf{o}^{\mathbf{1}}_\vee)$ and $(\mathbf{n}^{\mathbf{r}}_\vee)$; (iv) $(\mathbf{o}^{\mathbf{2}}_\vee)$ and $(\mathbf{n}^{\mathbf{r}}_\vee)$; (v) $(\mathbf{o}^{\mathbf{1}}_\supset)$ and $(\mathbf{n}^{\mathbf{r}}_\supset)$.

Proposition 29. *For $A \subseteq \mathbf{A_0}$, $\mathbf{BK}[A]$ is paraconsistent iff A*

TABLE I

ACL AXIOMS AND THEIR CORRESPONDING SEMANTIC CONDITIONS AND GENTZEN-TYPE RULES (WHERE $x \in \{t, \top, f\}$, $y \in \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$)

	ax	C(ax)	GC(ax)	R(ax)
(c)	$\neg\neg\varphi \supset \varphi$	$\neg f \subseteq \{t\}$	for $a \in \mathcal{F}$: $\neg a \subseteq \mathcal{T}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$
(e)	$\varphi \supset \neg\neg\varphi$	$\neg\top \subseteq \{\top\}$	for $a \in \mathcal{I}$: $\neg a \subseteq \mathcal{I}$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$
$(\mathbf{n}^{\mathbf{r}}_{\wedge})$	$(\neg\varphi \vee \neg\psi) \supset \neg(\varphi \wedge \psi)$	for $a \in \{f, \top\}$: $a \wedge x, x \wedge a \subseteq \{f, \top\}$	for $a \in \mathcal{F} \cup \mathcal{I}$: $a \wedge y, y \wedge a \subseteq \mathcal{F} \cup \mathcal{I}$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$
$(\mathbf{n}^{\mathbf{l}}_{\wedge})$	$\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$	$t \wedge t \subseteq \{t\}$	for $a, b \in \mathcal{T}$: $a \wedge b \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\mathbf{n}^{\mathbf{r}}_{\vee})$	$(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$	for $a, b \in \{f, \top\}$: $a \vee b \subseteq \{\top, f\}$	for $a, b \in \mathcal{F} \cup \mathcal{I}$: $a \vee b \subseteq \mathcal{F} \cup \mathcal{I}$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$
$(\mathbf{n}^{\mathbf{l}}_{\vee})$	$\neg(\varphi \vee \psi) \supset (\neg\varphi \wedge \neg\psi)$	$t \vee x, x \vee t \subseteq \{t\}$	for $a \in \mathcal{T}$: $a \vee y, y \vee a \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{n}^{\mathbf{r}}_{\supset})$	$(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$	for $a \in \{t, \top\}$ and $b \in \{f, \top\}$: $a \supset b \subseteq \{f, \top\}$	for $a \in \mathcal{D}, b \in \mathcal{F} \cup \mathcal{I}$: $a \supset b \in \mathcal{F} \cup \mathcal{I}$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \supset \psi)}$
$(\mathbf{n}^{\mathbf{l}}_{\supset})$	$\neg(\varphi \supset \psi) \supset (\varphi \wedge \neg\psi)$	$f \supset x, x \supset t \subseteq \{t\}$	for $a \in \mathcal{F}, b \in \mathcal{T}$: $a \supset y, y \supset b \subseteq \mathcal{T}$	$\dfrac{\Gamma, \varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$

is coherent.

Remark 30. The dependencies among the axioms in $\mathbf{A_0}$ can also be mechanically computed using the conditions from Tables I and II. The following is their exhaustive list in $\mathbf{BK}$ for $\diamond \in \{\supset, \vee\}$: (i) $(\mathbf{a}_{\diamond})$ follows from the conjunction of $(\mathbf{o}^{\mathbf{1}}_{\diamond})$ and $(\mathbf{o}^{\mathbf{2}}_{\diamond})$; (ii) $(\mathbf{n}^{\mathbf{l}}_{\wedge})$ follows from the disjunction of $(\mathbf{o}^{\mathbf{1}}_{\wedge})$ and $(\mathbf{o}^{\mathbf{2}}_{\wedge})$; (iii) $(\mathbf{n}^{\mathbf{l}}_{\diamond})$ follows from $(\mathbf{o}^{\mathbf{i}}_{\diamond})$ for $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$. It should be also noted that $(\mathbf{k})$ follows from $(\mathbf{i})$ in $\mathbf{B}$ (we do not add this dependency to the list since $(\mathbf{k}) \notin \mathbf{A_0}$).

B. The Corresponding Gentzen-type Systems

The method of [5] for constructing a cut-free, sound and complete Gentzen-type system for a given finite Nmatrix $\mathcal{M}$ involves two stages. At the first stage, every entry of every truth-table of $\mathcal{M}$ is translated into a rule. At the second stage, certain streamlining principles are used to combine and simplify the rules obtained at the first stage in order to get an optimal set of rules.

Using the above method, we obtain the following system for $\vdash_{\mathcal{M}^3}$:

Definition 31. The system $\mathbf{G_K}$ consists of the identity axiom $\psi \Rightarrow \psi$, the structural rules of cut and weakening, and the logical rules given in Table III. $\mathbf{G}_0$ is the system obtained from $\mathbf{G_K}$ by deleting the rule $(\Rightarrow \circ)$.

Now from the results of [5] we directly obtain the following:

Theorem 32. *1)* $\mathbf{BK}$ *is equivalent to* $\mathbf{G_K}$*.*
2) $\mathbf{G_K}$ *enjoys cut-admissibility.*

Example 33. Below we show a proof of $(\mathbf{l}) \Rightarrow (\mathbf{k})$ in $\mathbf{G}_0$.

$$\dfrac{\dfrac{\dfrac{\varphi \wedge \neg\varphi \Rightarrow \circ\varphi, \varphi \wedge \neg\varphi}{\Rightarrow \neg(\varphi \wedge \neg\varphi), \circ\varphi, \varphi \wedge \neg\varphi}\ (\Rightarrow \neg) \qquad \circ\varphi \Rightarrow \circ\varphi, \varphi \wedge \neg\varphi}{\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi \Rightarrow \circ\varphi, \varphi \wedge \neg\varphi}\ (\supset\Rightarrow)}{\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi \Rightarrow \circ\varphi \vee (\varphi \wedge \neg\varphi)}\ (\Rightarrow \vee)$$

TABLE II

ADC AXIOMS AND THEIR CORRESPONDING SEMANTIC CONDITIONS AND GENTZEN-TYPE RULES (FOR $x \in \{t, \top, f\}$, $y \in \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$)

	ax	$\mathsf{C}(ax)$	$\mathsf{GC}(ax)$	$\mathsf{R}(ax)$
$(\mathbf{i})$	$\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)$	$\circ f \subseteq \{t\}$ and $\circ t \subseteq \{t\}$	for $a \in \mathcal{F} \cup \mathcal{T}$: $\circ a \subseteq \mathcal{T}$	$\dfrac{\Gamma, \varphi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg\circ\varphi \Rightarrow \Delta}$
$(\mathbf{a}_{\vee})$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi)$	for $b \in \{t, f\}$, $t \vee b \subseteq \{t\}$ for $b \in \{t, f\}$, $b \vee t \subseteq \{t\}$	for $a \in \mathcal{T}, b \in \mathcal{T} \cup \mathcal{F}$: $a \vee b \subseteq \mathcal{T}$ for $a \in \mathcal{T}, b \in \mathcal{T} \cup \mathcal{F}$: $b \vee a \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{a}_{\supset})$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \supset \psi)$	for $a \in \{t, f\}$, $f \supset a \subseteq \{t\}$ for $a \in \{t, f\}$, $a \supset t \subseteq \{t\}$	for $b \in \mathcal{F}, a \in \mathcal{T} \cup \mathcal{F}$: $b \supset a \subseteq \mathcal{T}$ for $b \in \mathcal{T}, a \in \mathcal{T} \cup \mathcal{F}$: $a \supset b \subseteq \mathcal{T}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \neg\psi, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \neg\varphi, \varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{1}}_{\wedge})$	$\circ\varphi \supset \circ(\varphi \wedge \psi)$	for $b \in \{t, \top\}$, $t \wedge b \subseteq \{t\}$	for $a \in \mathcal{T}$ and $b \in \mathcal{D}$: $a \wedge b \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{2}}_{\wedge})$	$\circ\psi \supset \circ(\varphi \wedge \psi)$	for $b \in \{t, \top\}$: $b \wedge t \subseteq \{t\}$	for $a \in \mathcal{T}$ and $b \in \mathcal{D}$: $b \wedge a \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{1}}_{\vee})$	$\circ\varphi \supset \circ(\varphi \vee \psi)$	$t \vee x \subseteq \{t\}$ for $b \in \{t, \top\}$, $f \vee b \subseteq \{t\}$	for $a \in \mathcal{T}$: $a \vee y \subseteq \mathcal{T}$ for $c \in \mathcal{F}$ and $b \in \mathcal{D}$: $c \vee b \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{2}}_{\vee})$	$\circ\psi \supset \circ(\varphi \vee \psi)$	$x \vee t \subseteq \{t\}$ for $b \in \{t, \top\}$: $b \vee f \subseteq \{t\}$	for $a \in \mathcal{T}$, $y \vee a \subseteq \mathcal{T}$ for $c \in \mathcal{F}$ and $b \in \mathcal{D}$: $b \vee c \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{1}}_{\supset})$	$\circ\varphi \supset \circ(\varphi \supset \psi)$	for $b \in \{t, \top\}$: $t \supset b \subseteq \{t\}$ $f \supset x \subseteq \{t\}$	for $a \in \mathcal{T}$ and $b \in \mathcal{D}$: $a \supset b \subseteq \mathcal{T}$ for $c \in \mathcal{F}$: $c \supset y \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{2}}_{\supset})$	$\circ\psi \supset \circ(\varphi \supset \psi)$	$x \supset t \subseteq \{t\}$ $f \supset f \subseteq \{t\}$	for $a \in \mathcal{T}$: $y \supset a \subseteq \mathcal{T}$ for $b, c \in \mathcal{F}$: $b \supset c \subseteq \mathcal{T}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$

TABLE III
THE LOGICAL RULES OF $\mathbf{G_K}$

$(\wedge\Rightarrow)$	$\dfrac{\Gamma,\psi,\phi\Rightarrow\Delta}{\Gamma,\psi\wedge\phi\Rightarrow\Delta}$	$(\Rightarrow\wedge)$	$\dfrac{\Gamma\Rightarrow\Delta,\psi\quad\Gamma\Rightarrow\Delta,\phi}{\Gamma\Rightarrow\Delta,\psi\wedge\phi}$
$(\vee\Rightarrow)$	$\dfrac{\Gamma,\psi\Rightarrow\Delta\quad\Gamma,\phi\Rightarrow\Delta}{\Gamma,\psi\vee\phi\Rightarrow\Delta}$	$(\Rightarrow\vee)$	$\dfrac{\Gamma\Rightarrow\Delta,\psi,\phi}{\Gamma\Rightarrow\Delta,\psi\vee\phi}$
$(\supset\Rightarrow)$	$\dfrac{\Gamma\Rightarrow\psi,\Delta\quad\Gamma,\phi\Rightarrow\Delta}{\Gamma,\psi\supset\phi\Rightarrow\Delta}$	$(\Rightarrow\supset)$	$\dfrac{\Gamma,\psi\Rightarrow\phi,\Delta}{\Gamma\Rightarrow\psi\supset\phi,\Delta}$
		$(\Rightarrow\neg)$	$\dfrac{\Gamma,\psi\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg\psi}$
$(\circ\Rightarrow)$	$\dfrac{\Gamma\Rightarrow\psi,\Delta\quad\Gamma\Rightarrow\neg\psi,\Delta}{\Gamma,\circ\psi\Rightarrow\Delta}$	$(\Rightarrow\circ)$	$\dfrac{\Gamma,\psi,\neg\psi\Rightarrow\Delta}{\Gamma\Rightarrow\circ\psi,\Delta}$

The rules of $\mathbf{G_K}$, except for the rule for negation, are particularly well-behaved in the following sense:

Definition 34. An introduction rule is *invertible in a Gentzen-type system* $\mathbf{G}$ if each of its premises has a derivation from its conclusion in $\mathbf{G}$.

Proposition 35. *The rules for the positive connectives ($\wedge,\vee,\supset$ and $\circ$) are invertible in* $\mathbf{BK}$.

Proof: We show the proof for the rules for $\circ$. The following is a derivation of $\Gamma,\psi,\neg\psi\Rightarrow\Delta$ from $\Gamma\Rightarrow\circ\psi,\Delta$ (i.e. the converse of $(\Rightarrow\circ)$) in $\mathbf{G_K}$:

$$\dfrac{\Gamma\Rightarrow\circ\psi,\Delta\qquad\dfrac{\Gamma,\psi\Rightarrow\Delta,\psi\quad\Gamma,\neg\psi\Rightarrow\Delta,\neg\psi}{\Gamma,\circ\psi,\psi,\neg\psi\Rightarrow\Delta}\,(\circ\Rightarrow)}{\Gamma,\psi,\neg\psi\Rightarrow\Delta}\,cut$$

The derivation of $\Gamma\Rightarrow\psi,\Delta$ from $\Gamma\Rightarrow\psi,\circ\psi,\Delta$ (i.e., one converse of $(\circ\Rightarrow)$) in $\mathbf{G_K}$ is as follows:

$$\dfrac{\Gamma,\circ\psi\Rightarrow\Delta\qquad\dfrac{\Gamma,\psi,\neg\psi\Rightarrow\psi,\Delta}{\Gamma\Rightarrow\psi,\circ\psi,\Delta}\,(\Rightarrow\circ)}{\Gamma\Rightarrow\psi,\Delta}\,cut$$

The derivation of $\Gamma\Rightarrow\neg\psi,\Delta$ from $\Gamma\Rightarrow\psi,\circ\psi,\Delta$ is symmetric to the last derivation. ☐

The method used for $\mathcal{M}^3$ can be applied to each of its simple refinements separately. In this way, we can obtain a cut-free Gentzen-type formulation for each of the C-systems we have considered above. However, the rule construction process can be significantly simplified by translating the semantic effect $\mathsf{C}(ax)$ of each extra axiom in a modular way into the corresponding Gentzen-type rules, using the following six facts:

Proposition 36. *Let v be a full $\mathcal{M}$-valuation, where $\mathcal{M}$ is any simple refinement of $\mathcal{M}^3$. Then:*

- *$v(\psi)=t$ iff $\neg\psi\Rightarrow$ is true under v.*
- *$v(\psi)=f$ iff $\psi\Rightarrow$ is true under v.*
- *$v(\psi)=\top$ iff $\Rightarrow\psi$ and $\Rightarrow\neg\psi$ are both true under v.*
- *$v(\psi)\in\{f,\top\}$ iff $\Rightarrow\neg\psi$ is true under v.*
- *$v(\psi)\in\{t,\top\}$ iff $\Rightarrow\psi$ is true under v.*
- *$v(\psi)\in\{t,f\}$ iff $\psi,\neg\psi\Rightarrow$ is true under v.*

The corresponding rules for each $ax\in\mathbf{A_0}$, denoted by $\mathsf{R}(ax)$, are again given in Tables I and II.

Example 37. To see how the Gentzen-type rules from Tables I and II are derived, consider once more the schema $(\mathbf{a}_\vee)$. As explained in Example 25, the validity of this schema is guaranteed by the conditions: for $b\in\{t,f\}$, (i) $t\vee b\subseteq\{t\}$ and (ii) $b\tilde{\vee}t\subseteq\{t\}$. Now the first condition can be reformulated as follows: if $v(\varphi)=t$ and $v(\psi)\in\{t,f\}$, then $v(\varphi\vee\psi)=t$. Using Proposition 36, we can rewrite this as follows: if $\neg\varphi\Rightarrow$ and $\neg\psi,\psi\Rightarrow$ are true, then $\neg(\varphi\vee\psi)\Rightarrow$ is true. By adding context, we obtain the first corresponding rule from Table II. The second one can similarly be derived from condition (ii).

Definition 38. For each $ax\in\mathbf{A_0}$, the set $\mathsf{R}(ax)$ of Gentzen-type rules corresponding to ax is defined as in Tables I and II. For $A\subseteq\mathbf{A_0}$, $\mathbf{G}[A]$ is the Gentzen-type system obtained by adding to $\mathbf{G_K}$ the set of rules $\mathsf{R}(ax)$ for every $ax\in A$.

Theorem 39. *If $A\subseteq\mathbf{A_0}$ is coherent, then:*

1) *$\mathbf{BK}[A]$ is equivalent to $\mathbf{G_K}[A]$.*
2) *$\mathbf{G_K}[A]$ enjoys cut-admissibility.*

Proof: It is easy to see that $\mathbf{G_K}[A]$ is the calculus obtained for $\mathcal{M}^3[A]$ using the algorithm from [5]. Thus the theorem follows from Proposition 27 and the results of [5]. This theorem is also a special case of Theorem 54 in the sequel, for which a direct proof is provided in the appendix. ☐

It is important to note that all the Gentzen-type rules provided in this section have a *uniform* form in the following sense:

1) Each of them introduces exactly one formula in its conclusion, on exactly one of its two sides;
2) The formula which is introduced is either of the form $\diamond(\psi_1,\dots,\psi_n)$ or $\neg\diamond(\psi_1,\dots,\psi_n)$, where $\diamond$ is a primitive n-ary connective of the language;
3) Let $\diamond(\psi_1,\dots,\psi_n)$ be the formula mentioned in the previous item. Then the principal formulas in the

premises of the rule are all taken from the set $\{\psi_1, \dots, \psi_n, \neg\psi_1, \dots, \neg\psi_n\}$;

4) There are no restrictions on the side formulas of the rule (i.e., every context is legitimate).

We call rules of this form *quasi-canonical*, because they provide a natural generalization of the class of canonical rules ([7]) — the type of rules that are used in standard Gentzen-type systems for classical logic. It should be noted that quasi-canonical Gentzen-type systems like those presented here (i.e. Gentzen-type systems, in which all rules are either structural or quasi-canonical) have already been used extensively in the proof theory of non-classical logics. As far as we know, this cannot be said about any Gentzen-type formulation of a C-system that has been suggested before.

IV. Cut-free Calculi for Logics with (l) and (d)

The method for construction of cut-free calculi described in Section III does *not* apply to the C-systems which include one of the axioms $(\mathbf{l}), (\mathbf{d})$. The reason is that it was shown in [4] that such systems cannot have finite-valued characteristic Nmatrices. However, it was also shown there that they do have infinitely-valued characterizations of this type (which still suffice for guaranteeing their decidability). Below we show that these characterizations can be exploited for a modular construction of cut-free sequent calculi too.

A. Non-deterministic Infinite-valued Semantics

The treatment of axioms $(\mathbf{l})$ and $(\mathbf{d})$ is more complicated than that of the axioms from $\mathbf{A_0}$, since the semantic effect of their addition to $\mathbf{BK}$ cannot be formulated as a condition on the three-valued truth-table of some connective, leading to a certain simple refinement of $\mathcal{M}^3$. This is due to the fact that both $(\mathbf{l})$ and $(\mathbf{d})$ involve a conjunction of a formula with its negation. Informally, we need to be able to isolate the case of a conjunction of an "inconsistent" formula ψ with its negation from the cases of conjunction of ψ with other formulas. This requires an infinite number of truth-values, corresponding to the infinitely many formulas of the language. In view of the above, the finite Nmatrix $\mathcal{M}^3$ for **BK** must be replaced by the infinite Nmatrix $\mathcal{M}_0$ defined below. Instead of $t, \top$ and f, it uses three sets of truth-values: $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{\top_i^j \mid i \geq 0, j \geq 0\}$ and $\mathcal{F} = \{f\}$, respectively. Intuitively, the set $\mathcal{T}$ contains infinitely many "copies" of the classical value t, the set $\mathcal{F}$ contains (one "copy" of) f, and the set $\mathcal{I}$ contains infinitely many "copies" of the the inconsistent truth-value $\top$.

Definition 40. The Nmatrix $\mathcal{M}_0 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for $\mathcal{L}_C$, where $\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$ and $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$, is defined as follows:

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{if } a \in \mathcal{F} \\ \{\top_i^{j+1}, t_i^{j+1}\} & \text{if } a = \top_i^j \end{cases}$$

$$\tilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

Proposition 41. *$T \vdash_{\mathcal{M}_0} \psi$ iff $T \vdash_{\mathbf{BK}} \psi$.*

Proof: A modification of the proof of Thm. 7 in [4]. □

We now turn to providing non-deterministic semantics for the extensions of $\mathbf{BK}$ with axioms from $\mathbf{A}$, *including the problematic axioms* $(\mathbf{l})$ *and* $(\mathbf{d})$. The *modularity* of our semantics is preserved — each axiom corresponds to a certain semantic condition on the basic Nmatrix $\mathcal{M}_0$. Moreover, the semantic conditions induced by the axioms from $\mathbf{A_0}$ turn out to be identical to the general formulation $\mathsf{GC}(ax)$, presented in Tables I and II. However, there is a difference in the *meaning* of what is written there: now we take $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{\top_i^j \mid i \geq 0, j \geq 0\}$, and $\mathcal{F} = \{f\}$.

Example 42. Let us take again the axiom $(\mathbf{a}_\vee)$, considered in Example 37 in the context of the three-valued simple refinements of $\mathcal{M}^3$. To guarantee its validity in the context of the infinite-valued refinements of $\mathcal{M}_0$, we must ensure that $(*)$ $v((\circ\varphi \wedge \circ\psi)) \in \mathcal{D}$ implies $v(\circ(\varphi \vee \psi)) \in \mathcal{D}$. In any simple refinement of $\mathcal{M}_0$, $v(\circ(\varphi \vee \psi)) \in \mathcal{D}$ iff $v(\varphi \vee \psi) \in \mathcal{T} \cup \mathcal{F}$. Moreover, $v((\circ\varphi \wedge \circ\psi)) \in \mathcal{D}$ iff $v(\circ\varphi), v(\circ\psi) \in \mathcal{D}$ iff $v(\varphi), v(\psi) \in \mathcal{F} \cup \mathcal{T}$. Since $v(\varphi \vee \psi) \in \mathcal{T} \cup \mathcal{F}$ is already guaranteed when $v(\varphi), v(\psi) \in \mathcal{F}$, the truth table of $\vee$ in $\mathcal{M}_0$ implies that it remains to require the two conditions for $(\mathbf{a}_\vee)$ which are given in Table II: if $a \in \mathcal{T}$ and $b \in \mathcal{T} \cup \mathcal{F}$ then: (i) $a \vee b \subseteq \mathcal{T}$ and (ii) $b \vee a \subseteq \mathcal{T}$. Note the similarity of this derivation of semantic conditions to that in Example 25.

It remains now to define the semantic conditions induced by $(\mathbf{l})$ and $(\mathbf{d})$:

Definition 43.
$\mathsf{GC}(\mathbf{l})$*: For $a = \top_i^j$ and $b \in \{\top_i^{j+1}, t_i^{j+1}\}$, $a \wedge b \subseteq \mathcal{T}$.*
$\mathsf{GC}(\mathbf{d})$*: For $b = \top_i^j$ and $a \in \{\top_i^{j+1}, t_i^{j+1}\}$, $a \wedge b \subseteq \mathcal{T}$.*

Definition 44. For $A \subseteq \mathbf{A}$, the Nmatrix $\mathcal{M}_0[A]$ is the weakest simple refinement of $\mathcal{M}_0$ in which $\mathsf{GC}(ax)$ (from Tables I,II and Definition 43) holds for every $ax \in A$.

Proposition 45. *For $A \subseteq \mathbf{A_0}$, $T \vdash_{\mathcal{M}_0[A]} \psi$ iff $T \vdash_{\mathbf{BK}[A]} \psi$.*

Proof: A modification of proofs from [4] and [2]. □

The notion of a coherent set of axioms from Definition 28 is now modified as follows:

Definition 46. For $A \subseteq \mathbf{A}$, we say that A is coherent if in addition to satisfying all the conditions of Definition 28, it does not contain any of the following pairs of axioms: (i) $(\mathbf{l})$ and $(\mathbf{n}^{\mathbf{r}}_\wedge)$; (ii) $(\mathbf{d})$ and $(\mathbf{n}^{\mathbf{r}}_\wedge)$.

We can now extend Proposition 29 to the context including axioms $(\mathbf{l})$ and $(\mathbf{d})$:

Proposition 47. *For $A \subseteq \mathbf{A}$, $\mathbf{BK}[A]$ is paraconsistent iff A is coherent.*

Remark 48. The list of dependencies among the axioms in $\mathbf{A}$ is similar to that given in Remark 30. It should be also noted that $(\mathbf{k})$ follows both from $(\mathbf{l})$ and from $(\mathbf{d})$ in $\mathbf{BK}$ (see Example 33 for the former case).

B. The Corresponding Gentzen-type Systems

To construct cut-free Gentzen-type systems for logics with $(\mathbf{l})$ and $(\mathbf{d})$, we can no longer rely on the method for construction of analytic calculi given in [5], which was employed in the previous section, as it *does not* apply to logics which have no finite-valued characteristic Nmatrices. However, following the intuitive meaning of $\circ\psi$ given in Remark 13, we can start with the obvious translation of $(\mathbf{l})$ and $(\mathbf{d})$ into Gentzen-type rules which is obtained by substituting in $(\circ \Rightarrow)$ the formulas $\neg(\psi \wedge \neg\psi)$ and $\neg(\neg\psi \wedge \psi)$ (respectively) for $\circ\psi$ (note that by applying the same procedure to $(\Rightarrow \circ)$, we get a rule which is derivable in $\mathbf{BK}$).

Definition 49. The Gentzen-type rules $\mathsf{R}(\mathbf{l})$ and $\mathsf{R}(\mathbf{d})$ are defined as follows:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \neg(\varphi \wedge \neg\varphi) \Rightarrow \Delta}\ \mathsf{R}(\mathbf{l}) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \neg(\neg\varphi \wedge \varphi) \Rightarrow \Delta}\ \mathsf{R}(\mathbf{d})$$

As for the corresponding Gentzen-type rules for every axiom $ax \in \mathbf{A_0}$, luckily we need not start our search from scratch. Although we cannot construct a rule for each line of each truth-table like in the method of [5] (because of both the infinite number of truth-values, and the insufficient expressive power of our language, which does not allow for characterizing each of them), we can still perform the easier task of encoding the semantic effect of each axiom by a Gentzen-type rule. This can be done using the following analogue of Proposition 36:

Proposition 50. *Let v be a full $\mathcal{M}$-valuation, where $\mathcal{M}$ is a simple refinement of $\mathcal{M}_0$. Then:*

- *$v(\psi) \in \mathcal{T}$ iff $\neg\psi \Rightarrow$ is true under v.*
- *$v(\psi) \in \mathcal{F}$ iff $\psi \Rightarrow$ is true under v.*
- *$v(\psi) \in \mathcal{I}$ iff $\Rightarrow \psi$ and $\Rightarrow \neg\psi$ are both true under v.*
- *$v(\psi) \in \mathcal{F} \cup \mathcal{I}$ iff $\Rightarrow \neg\psi$ is true under v.*
- *$v(\psi) \in \mathcal{T} \cup \mathcal{I}$ iff $\Rightarrow \psi$ is true under v.*
- *$v(\psi) \in \mathcal{F} \cup \mathcal{T}$ iff $\psi, \neg\psi \Rightarrow$ is true under v.*

Example 51. Revisiting Example 42, the semantic conditions $\mathsf{GC}(a_\vee)$ from Table II are: for $a \in \mathcal{T}$, $b \in \mathcal{T} \cup \mathcal{F}$: (i) $a \vee b \subseteq \mathcal{T}$ and (ii) $b \vee a \subseteq \mathcal{T}$. The first of then can be reformulated as follows: if $v(\varphi) \in \mathcal{T}$ and $v(\psi) \in \mathcal{T} \cup \mathcal{F}$, then $v(\varphi \vee \psi) \in \mathcal{T}$. Using Proposition 50, we rewrite this as: if $\neg\varphi \Rightarrow$ and $\neg\psi, \psi \Rightarrow$ are true, then $\neg(\varphi \vee \psi) \Rightarrow$ is true. By adding context we obtain the first corresponding rule from Table II. The second rule is obtained similarly from condition (ii). Note the similarity of this construction to that given in Example 37.

We therefore retain our ability to provide cut-free Gentzen-type systems for $\mathbf{BK}[A]$ for $A \subseteq \mathbf{A}$ in a *modular* way. Moreover, as the semantic effects of the axioms in $\mathbf{A_0}$ remain the same, it is not surprising that the Gentzen-type rules corresponding to them are exactly those given in Tables I,II:

Definition 52. For each $ax \in \mathbf{A}$, the set $\mathsf{R}(ax)$ of Gentzen-type rules corresponding to ax is defined as in Tables I,II and Definition 49. For $A \subseteq \mathbf{A}$, $\mathbf{G_K}[A]$ is the Gentzen-type system obtained by adding to $\mathbf{G_K}$ the set of rules $\mathsf{R}(ax)$ for every $ax \in A$.

Example 53. Below is a cut-free proof of $(\mathbf{l})$ in $\mathbf{G_K}[\{(\mathbf{l})\}]$ (note that the rule $\mathsf{R}(\mathbf{l})$ defined above does not even mention the connective $\circ$):

$$\cfrac{\cfrac{\cfrac{\varphi, \neg\varphi \Rightarrow \neg\varphi}{\Rightarrow \neg\varphi, \circ\varphi}\ (\Rightarrow \circ) \quad \cfrac{\varphi, \neg\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \circ\varphi}\ (\Rightarrow \circ)}{\neg(\varphi \wedge \neg\varphi) \Rightarrow \circ\varphi}\ (\mathsf{R}(\mathbf{l}))}{\Rightarrow \neg(\varphi \wedge \neg\varphi) \supset \circ\varphi}\ (\Rightarrow\supset)$$

Now we come to the main theorem of this paper, which applies to *all* extensions of $\mathbf{BK}$ studied here (including those covered by Theorem 32). Its full proof is provided in the appendix. Unlike the proof of Theorem 32, it includes direct proofs of completeness and cut-admissibility, *without relying on the results from previous papers*.

Theorem 54. *For $A \subseteq \mathbf{A}$, $\mathcal{M}_0[A]$ is a characteristic Nmatrix for $\mathbf{G_K}[A]$, and $\mathbf{G_K}[A]$ enjoys cut-admissibility.*

V. $\circ$-FREE C-SYSTEMS

All the C-systems treated above include $\circ$ as a primitive connective. However, many important C-systems, including da Costa's historical C_1, are obtained by defining $\circ\varphi$ using other connectives available in the language. The usual definitions include $\neg(\varphi \wedge \neg\varphi)$ (like in C_1), $\neg(\neg\varphi \wedge \varphi)$, and their disjunction. It is thus important to extend our method also to the $\circ$-free fragments of the C-systems considered in this paper.

The $\circ$-free fragments of all the paraconsistent extensions of $\mathbf{BK}$ by axioms from $\mathbf{A}$ are easily obtained from their corresponding Gentzen-type systems $\mathbf{G_K}[A]$.

Proposition 55. *Let $A \subseteq \mathbf{A}$. If A is coherent, then the system obtained from $\mathbf{G_K}[A]$ by discarding the rules for $\circ$ (i.e. $(\circ \Rightarrow)$, $(\Rightarrow \circ)$, and $\mathsf{R}(\mathbf{i})$ if $(\mathbf{i}) \in A$) is equivalent to the $\circ$-free fragment of $\mathbf{BK}[A]$.*

Proof: Denote by $\mathbf{G}'$ the system obtained from $\mathbf{G_K}[A]$ by discarding the rules for $\circ$. Let $T \cup \{\psi\}$ be a set of formulas over $\mathcal{L}_{cl}$. Clearly, $T \vdash_{\mathbf{G}'} \psi$ implies $T \vdash_{\mathbf{BK[A]}} \psi$. For the converse, suppose that $T \vdash_{\mathbf{BK[A]}} \psi$ for some $T \cup \{\psi\}$ in $\mathcal{L}_{cl}$. Then $\vdash_{\mathbf{G_K}[A]} \Gamma \Rightarrow \psi$ for some finite $\Gamma \subseteq T$. By Theorem 54, $\Gamma \Rightarrow \psi$ has a cut-free derivation P in $\mathbf{G_K}[A]$. Since $\Gamma \Rightarrow \psi$

does not contain $\circ$, P does not contain applications of any rule introducing $\circ$ (since that $\circ$ can only be eliminated using cuts). Hence P is also a derivation in $\mathbf{G}'$, and $T \vdash_{\mathbf{G}'} \psi$. □

Corollary 56. *Let $A \subseteq \mathbf{A} \setminus \{(\mathbf{i})\}$ and $A' = A \cup \{(\mathbf{i})\}$. Then the $\circ$-free fragments of $\mathbf{BK}[A]$ and $\mathbf{BK}[A']$ are identical, that is: for any $T \cup \{\psi\}$ in $\mathcal{L}_{cl}$, $T \vdash_{\mathbf{BK[A]}} \psi$ iff $T \vdash_{\mathbf{BK[A']}} \psi$.*

It follows that the inclusion of axiom $(\mathbf{i})$ in any of the systems studied in this paper does not affect the $\circ$-fragment of that system.

Corollary 57. *The $\circ$-free fragments of $\mathbf{BKcla}$ and $\mathbf{BKcila}$ are identical (and both are equivalent to C_1).*

Corollary 58. The system obtained by discarding $(\circ \Rightarrow)$ and $(\Rightarrow \circ)$ from $\mathbf{G_K}[\{(\mathbf{c}), (\mathbf{l}), (\mathbf{a}_\vee), (\mathbf{a}_\wedge), (\mathbf{a}_\supset)\}]$ is equivalent to C_1.

Remark 59. The above results provide a straightforward way to obtain Hilbert-style axiomatizations for the $\circ$-free fragments of $\mathbf{BK}[A]$ for all $A \subseteq \mathbf{A}$. For this purpose, one can employ some standard method of translating Gentzen-type rules into corresponding axioms. For instance, we can obtain the following axioms, corresponding to the two Gentzen-type rules for $(\mathbf{o}^{\mathbf{1}}_\vee)$ in Table II: $\neg(\varphi \vee \psi) \supset \neg\varphi$ and $\neg(\varphi \vee \psi) \supset (\psi \supset \varphi)$. Note that in this way we obtain a $\circ$-free equivalent for each axiom from $\mathbf{ADC} \setminus \{(\mathbf{i})\}$.

In some cases, there is also an alternative way to obtain Hilbert-style axiomatizations of the $\circ$-free fragments of LFIs considered here (we omit the proof):

Proposition 60. *Let $A \subseteq \mathbf{A}$. For a formula σ, denote by $\mathbf{BK}^\sigma[A]$ the Hilbert-style system obtained from $\mathbf{BK}[A]$ by replacing $\circ\varphi$ by σ in $(\mathbf{b})$ as well as in all the $(\mathbf{a})$- and $(\mathbf{o})$-axioms of A. Then $\mathbf{BK}^\sigma[A]$ is equivalent to the $\circ$-free fragment of $\mathbf{BK}[A]$ whenever:*

- *A contains $(\mathbf{l})$, but not $(\mathbf{d})$, and $\sigma = \neg(\varphi \wedge \neg\varphi)$.*
- *A contains $(\mathbf{d})$, but not $(\mathbf{l})$, and $\sigma = \neg(\neg\varphi \wedge \varphi)$.*
- *A contains $(\mathbf{l})$ and $(\mathbf{d})$, and $\sigma = \neg(\varphi \wedge \neg\varphi) \vee \neg(\neg\varphi \wedge \varphi)$.*

VI. Summary and Further Research

Although analytic calculi for some particular C-systems have been proposed before, to the best of our knowledge, until now there has been no *general* method available for constructing cut-free sequent calculi for C-systems. This paper fills this gap by providing such a method for constructing cut-free calculi in a *modular* way. Our method applies to a large family of C-systems, covering practically all C-systems ever studied in the literature. We believe that these results will help develop efficient tools for automated reasoning with inconsistency, eventually making Logics of Formal (In)consistency a more appealing formalism for reasoning under uncertainty. However, it is clear that for the purposes of building LFI-based theorem provers for real-life applications, the results of this paper need to be extended to the first-order case. To the best of our knowledge, currently there are no known analytic systems for LFIs available on the first-order level. However, [8] provided non-deterministic modular semantics for first-order LFIs, which can possibly be exploited along the lines of the approach presented in this paper.

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APPENDIX

Proof of Theorem 54:

We leave the easy proof of soundness to the reader. Below we prove completeness together with cut-admissibility.

For $A \subseteq \mathbf{A}$, we call a sequent $\Gamma \Rightarrow \Delta$ *saturated with respect to* A if it satisfies the properties (S1)-(S11) below:

- (S1) If $\varphi \wedge \psi \in \Gamma$, then $\varphi, \psi \in \Gamma$. If $\varphi \wedge \psi \in \Delta$, then $\varphi \in \Delta$ or $\psi \in \Delta$. Similarly for $\vee$ and $\supset$.
- (S2) If $\neg\varphi \in \Delta$, then $\varphi \in \Gamma$.
- (S3) If $\circ\varphi \in \Gamma$, then either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$. If $\circ\varphi \in \Delta$, then $\varphi \in \Gamma$ and $\neg\varphi \in \Gamma$.
- (S4) If $(\mathbf{l}) \in A$ and $\neg(\varphi \wedge \neg\varphi) \in \Gamma$, then either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.
- (S5) If $(\mathbf{d}) \in A$ and $\neg(\neg\varphi \wedge \varphi) \in \Gamma$, then either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.
- (S6) If $(\mathbf{c}) \in A$ and $\neg\neg\varphi \in \Gamma$, then $\varphi \in \Gamma$.
- (S7) If $(\mathbf{e}) \in A$ and $\neg\neg\varphi \in \Delta$, then $\varphi \in \Delta$.
- (S8) If $(\mathbf{i}) \in A$, then $\neg \circ \varphi \in \Gamma$ implies $\varphi, \neg\varphi \in \Gamma$.
- (S9) If $(\mathbf{o}^{\mathbf{1}}_{\wedge}) \in A$ and $\neg(\varphi \wedge \psi) \in \Gamma$, then either $\neg\varphi \in \Gamma$ or $\psi \in \Delta$. If $(\mathbf{o}^{\mathbf{2}}_{\wedge}) \in A$ and $\neg(\varphi \wedge \psi) \in \Gamma$, then either $\neg\psi \in \Gamma$ or $\varphi \in \Delta$. Similarly for $(\mathbf{o}^{\mathbf{i}}_{\vee})$ and $(\mathbf{o}^{\mathbf{i}}_{\supset})$, where $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}\}$.
- (S10) If $(\mathbf{n}^{\mathbf{r}}_{\wedge}) \in A$ and $\neg(\varphi \wedge \psi) \in \Delta$, then $\neg\varphi, \neg\psi \in \Delta$. If $(\mathbf{n}^{\mathbf{l}}_{\wedge}) \in A$ and $\neg(\varphi \wedge \psi) \in \Gamma$, then either $\neg\varphi \in \Gamma$ or $\neg\psi \in \Gamma$.
- (S11) Similarly for the rest of the axioms from $\mathbf{A}$.

Now let $A \subseteq \mathbf{A}$, and suppose that $\Gamma_0 \Rightarrow \Delta_0$ has no cut-free proof in $\mathbf{BK}[A]$. It is a standard matter to show that $\Gamma_0 \Rightarrow \Delta_0$ can be extended to a saturated (with respect to A) sequent $\Gamma \Rightarrow \Delta$ such that (i) $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$, and (ii) $\Gamma \Rightarrow \Delta$ has no cut-free proof in $\mathbf{G_K}[A]$. Note that this means that $(*)$ $\Gamma \cap \Delta = \emptyset$ (otherwise $\Gamma \Rightarrow \Delta$ contains the identity axiom, and so has a cut-free proof in $\mathbf{G_K}[A]$).
Let $\lambda i.\alpha_i$ be an enumeration of all the formulas in $\mathcal{L}_C$ that do not begin with $\neg$. Then, for every formula ψ of $\mathcal{L}_C$, there are unique $n(\psi)$ and $k(\psi)$ such that $\psi = \neg_{k(\psi)}\alpha_{n(\psi)}$, where $\neg_k\varphi$ is φ preceded by k negation symbols.

Now we define the refuting valuation v. Below we write $\tilde{\diamond}$ instead of $\tilde{\diamond}_{\mathcal{M}_0[A]}$.

If p is atomic, then

$$v(p) = \begin{cases} f & p \in \Delta \\ t^0_{n(p)} & \neg p \in \Delta \\ \top^0_{n(p)} & otherwise \end{cases}$$

For a formula $\varphi = \diamond(\psi_1, \ldots, \psi_j)$ (where $j \in \{1, 2\}$), define:

$$v(\varphi) = \begin{cases} f & \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_j)) = \{f\} \\ t^{k(\psi)}_{n(\psi)} & t^{k(\psi)}_{n(\psi)} \in \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_j)), \text{ and} \\ & \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_j)) \subseteq \mathcal{T} \text{ or } \neg\psi \in \Delta \\ \top^{k(\psi)}_{n(\psi)} & \text{otherwise} \end{cases}$$

It is easy to verify that the valuation v above is well-defined and that it is a $\mathcal{M}_0[A]$-valuation, that is, for every j-ary connective $\diamond$, $v(\diamond(\psi_1, \ldots, \psi_j)) \in \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_j))$.

It remains to show that v is a refuting valuation for $\Gamma \Rightarrow \Delta$. First, it is easy to prove (by induction on φ) that for every $\varphi \in Frm_{\mathcal{L}}$, $v(\varphi) \in \{f, t^{n(\varphi)}_{k(\varphi)}, \top^{n(\varphi)}_{k(\varphi)}\}$.
Next we prove (by induction on φ) the following four properties:

1) If $\varphi \in \Delta$ then $v(\varphi) = f$.
2) If $\varphi \in \Gamma$ then $v(\varphi) = t^{k(\varphi)}_{n(\varphi)}$ or $v(\varphi) = \top^{k(\varphi)}_{n(\varphi)}$.
3) If $\neg\varphi \in \Delta$ then $v(\varphi) = t^{k(\varphi)}_{n(\varphi)}$.
4) If $\neg\varphi \in \Gamma$ then $v(\varphi) = f$ or $v(\varphi) = \top^{k(\varphi)}_{n(\varphi)}$.

If φ is atomic then 1-4 are immediate from the definition of v, the fact that if $\varphi \in \Gamma$ then $\varphi \notin \Delta$ (by $(*)$ above), and the fact that $k(\varphi) = 0$ in this case.

- Suppose that $\varphi = \neg\psi$.
  1) Suppose $\varphi = \neg\psi \in \Delta$. By the induction hypothesis for ψ, it follows that $v(\psi) = t^{k(\psi)}_{n(\psi)}$. Since $\tilde{\neg}(t^{k(\psi)}_{n(\psi)}) = \{f\}$, by definition of v, $v(\varphi) = f$.
  2) Suppose $\varphi = \neg\psi \in \Gamma$. By the induction hypothesis for ψ, it follows that $v(\psi) = f$ or $v(\psi) = \top^{k(\psi)}_{n(\psi)}$. Either way, $\tilde{\neg}(v(\psi)) \neq \{f\}$. By definition of v, $v(\varphi) \neq f$, and so either $v(\varphi) = t^{k(\varphi)}_{n(\varphi)}$ or $v(\varphi) = \top^{k(\varphi)}_{n(\varphi)}$.
  3) Suppose $\neg\varphi \in \Delta$. By property (S2), $\neg\varphi = \neg\neg\psi \in \Delta$ implies $\neg\psi \in \Gamma$. From the induction hypothesis for ψ, it follows that $v(\psi) = f$ or $v(\psi) = \top^{k(\psi)}_{n(\psi)}$. Now there are two possibilities:
     - $(\mathbf{e}) \in A$. Then by (S7), $\psi \in \Delta$, and so by the induction hypothesis, $v(\psi) = f$. It follows that $t^{k(\varphi)}_{n(\varphi)} \in \tilde{\neg}(v(\psi))$ in this case (because for every $A \subseteq \mathbf{A}$, $\tilde{\neg}(f)$ can be either $\mathcal{T}$ or $\mathcal{D}$).
     - $(\mathbf{e}) \notin A$. Then $v(\psi) \in \{f, \top^{k(\psi)}_{n(\psi)}\}$, and $\tilde{\neg}(\top^{k(\psi)}_{n(\psi)})$ and $\tilde{\neg}(f)$ can be either $\mathcal{D}$, $\mathcal{T}$ or $\{\top^{k(\psi)+1}_{n(\psi)}, t^{k(\psi)+1}_{n(\psi)}\}$, and so $t^{k(\psi)+1}_{n(\psi)} \in \tilde{\neg}(v(\psi))$. Since $k(\psi) + 1 = k(\varphi)$ and $n(\psi) = n(\varphi)$, we have again $t^{k(\varphi)}_{n(\varphi)} \in \tilde{\neg}(v(\psi))$.

     It follows that in both cases $t^{k(\varphi)}_{n(\varphi)} \in \tilde{\neg}(v(\psi))$. Since $\neg\varphi \in \Delta$, $v(\varphi) = t^{k(\varphi)}_{n(\varphi)}$ by definition of v.
  4) Suppose $\neg\varphi \in \Gamma$. Then $\neg\varphi \notin \Delta$ by (*) above, and so $v(\varphi) \in \mathcal{T}$ only if $\tilde{\neg}(v(\psi)) \subseteq \mathcal{T}$. One of the following holds:
     - $(\mathbf{c}) \in A$. Then by (S6) $\neg\varphi = \neg\neg\psi \in \Gamma$ implies $\psi \in \Gamma$. By the induction hypothesis, $v(\psi) = t^{k(\psi)}_{n(\psi)}$ or $v(\psi) = \top^{k(\psi)}_{n(\psi)}$. Thus it is impossible that $\tilde{\neg}(v(\psi)) \subseteq \mathcal{T}$ (since $\tilde{\neg}(a) = \{f\}$ for any $a \in \mathcal{T}$, while always $\tilde{\neg}(a) \cap \mathcal{I} \neq \emptyset$ for any $a \in \mathcal{I}$).
     - $(\mathbf{c}) \notin A$. Then $\tilde{\neg}(v(\psi)) \not\subseteq \mathcal{T}$ even if $v(\psi) = f$ (since in this case $\tilde{\neg}(f) = \mathcal{D}$).

     It follows that in both cases $\tilde{\neg}(v(\psi)) \not\subseteq \mathcal{T}$, and so by the definition of v, $v(\varphi) = f$ or $v(\varphi) = \top^{k(\varphi)}_{n(\varphi)}$.
- Suppose that $\varphi = \circ\psi$.
  1) Suppose $\varphi \in \Delta$. By (S3), this implies $\psi, \neg\psi \in \Gamma$. By

the induction hypothesis, it follows that $v(\psi) = \top^{k(\psi)}_{n(\psi)}$. As $\widetilde{\circ}(v(\psi)) = \{f\}$, $v(\varphi) = f$ by the definition of v.

2) Suppose $\varphi \in \Gamma$. By (S3), either $\psi \in \Delta$ or $\neg\psi \in \Delta$. Hence $v(\psi) \notin \mathcal{I}$ by the induction hypothesis. Then $\widetilde{\circ}(v(\psi)) \neq \{f\}$, and so $v(\varphi) = t^{k(\psi)}_{n(\psi)}$ or $v(\varphi) = \top^{k(\varphi)}_{n(\varphi)}$.
3) Suppose $\neg\varphi \in \Delta$. By(S2), $\varphi = \circ\psi \in \Gamma$. By (S3), either $\psi \in \Delta$ or $\neg\psi \in \Delta$. Hence $v(\psi) \notin \mathcal{I}$ by the induction hypothesis. Thus $\widetilde{\circ}(v(\psi))$ is either $\mathcal{T}$ or $\mathcal{D}$, and so $v(\varphi) = t^{k(\varphi)}_{n(\varphi)}$ by the definition of v.
4) Suppose $\neg\varphi \in \Gamma$. Then $\neg\varphi \notin \Delta$ by $(*)$ above. $\widetilde{\circ}(v(\psi)) \subseteq \mathcal{T}$ can occur only if $(\mathbf{i}) \in A$ and $v(\psi) \in \mathcal{F} \cup \mathcal{T}$. But this is impossible, as by (S8) $\neg\circ\psi \in \Delta$ implies $\psi, \neg\psi \in \Gamma$, and $v(\psi) \in \mathcal{I}$ by the induction hypothesis. Hence $\widetilde{\circ}(v(\psi)) \not\subseteq \mathcal{T}$, and since $\neg\varphi \notin \Delta$, by the definition of v we have $v(\varphi) = f$ or $v(\varphi) = \top^{k(\varphi)}_{n(\varphi)}$.

- Suppose that $\varphi = \psi_1 \wedge \psi_2$.

1) Suppose $\varphi \in \Delta$. By (S1), $\psi_1 \in \Delta$ or $\psi_2 \in \Delta$. By the induction hypothesis, $v(\psi_1) = f$ or $v(\psi_2) = f$. Hence $\tilde{\wedge}(v(\psi_1), v(\psi_2)) = \{f\}$, and by definition of v, $v(\varphi) = f$.
2) Suppose $\varphi \in \Gamma$. By (S1), $\psi_1, \psi_2 \in \Gamma$. By the induction hypothesis, $v(\psi_i) = t^{k(\psi_i)}_{n(\psi_i)}$ or $v(\psi_i) = \top^{k(\psi_i)}_{n(\psi_i)}$ for $i \in \{1, 2\}$. Hence $\widetilde{\wedge}(v(\psi_1), v(\psi_2)) \neq \{f\}$, and by the definition of v, $v(\varphi) = t^{k(\varphi)}_{n(\varphi)}$ or $v(\varphi) = \top^{k(\varphi)}_{n(\varphi)}$.
3) Suppose $\neg\varphi \in \Delta$. By (S2), $\varphi \in \Gamma$. By (S1), $\psi_1, \psi_2 \in \Gamma$. By the induction hypothesis, $v(\psi_1), v(\psi_2) \in \mathcal{D}$. If $t^{k(\varphi)}_{n(\varphi)} \in \tilde{\wedge}(v(\psi_1), v(\psi_2))$, then $v(\varphi) = t^{k(\varphi)}_{n(\varphi)}$ by the definition of v, and we are done. Otherwise, it must be the case that $(\mathbf{n}^{\mathbf{r}}_{\wedge}) \in A$ and either $v(\psi_1) \in \mathcal{I} \cup \mathcal{F}$ or $v(\psi_2) \in \mathcal{I} \cup \mathcal{F}$. By (S10), $\neg\psi_1, \neg\psi_2 \in \Delta$. By the induction hypothesis, $v(\psi_1) = t^{n(\psi_1)}_{k(\psi_1)}$ and $v(\psi_2) = t^{n(\psi_2)}_{k(\psi_2)}$, which is impossible.
4) Suppose $\neg\varphi \in \Gamma$.
   - $(\mathbf{l}) \in A$, $v(\psi_1) = \top^i_j$ and $v(\psi_2) \in \{t^{i+1}_j, \top^{i+1}_j\}$. By definition of v it must be the case that $\psi_2 = \neg\psi_1$. Then by (S4), either $\psi_1 \in \Delta$ or $\psi_2 \in \Delta$. By the induction hypothesis, either $v(\psi_1) = f$ or $v(\psi_2) = f$, which is impossible.
   - $(\mathbf{d}) \in A$, $v(\psi_2) = \top^i_j$ and $v(\psi_1) \in \{t^{i+1}_j, \top^{i+1}_j\}$. Similarly to the previous case, this is impossible.
   - $(\mathbf{n}^{\mathbf{l}}_{\wedge}) \in A$ and $v(\psi_1), v(\psi_2) \in \mathcal{T}$. Then by (S10) either $\neg\psi_1 \in \Gamma$ or $\neg\psi_2 \in \Gamma$. By the induction hypothesis, either $v(\psi_1) \notin \mathcal{T}$ or $v(\psi_2) \notin \mathcal{T}$, which is impossible.
   - $(\mathbf{o}^{\mathbf{1}}_{\wedge}) \in A$, $v(\psi_1) \in \mathcal{T}$ and $v(\psi_2) \in \mathcal{D}$. By (S9), either $\neg\psi_1 \in \Gamma$, or $\psi_2 \in \Delta$. By the induction hypothesis, either $v(\psi_1) \notin \mathcal{T}$, or $v(\psi_2) \notin \mathcal{D}$, which is impossible.
   - $(\mathbf{o}^{\mathbf{2}}_{\wedge}) \in A$, $v(\psi_1) \in \mathcal{D}$ and $v(\psi_2) \in \mathcal{T}$. The proof is symmetric to the previous case.

   Hence $\tilde{\wedge}(v(\psi_1), v(\psi_2)) \not\subseteq \mathcal{T}$, and since in addition $\neg\varphi \notin \Delta$, by the definition of v, either $v(\varphi) = f$ or $v(\varphi) = \top^{k(\varphi)}_{n(\varphi)}$.

We leave the cases of $\vee$ and $\supset$ to the reader. It follows that for every $\psi \in \Gamma$, $v(\psi) \in \mathcal{D}$ and for every $\varphi \in \Delta$, $v(\varphi) = f$. Since $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$, $\Gamma_0 \Rightarrow \Delta_0$ is not valid in $\mathcal{M}_0[A]$.

□