Generalized Non-deterministic Matrices and (n,k)-ary Quantifiers

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Abstract. An (n, k)-ary quantifier is a generalized logical connective, binding k variables and connecting n formulas. Canonical Gentzen-type systems with (n, k)-ary quantifiers are systems which in addition to the standard axioms and structural rules have only logical rules in which exactly one occurrence of an (n, k)-ary quantifier is introduced. The semantics of such systems for the case of $k \in \{0, 1\}$ are provided in [16] using two-valued non-deterministic matrices (2Nmatrices). A constructive syntactic coherence criterion for the existence of a 2Nmatrix for which a canonical system is strongly sound and complete, is formulated there. In this paper we extend these results from the case of $k \in \{0, 1\}$ to the general case of $k \geq 0$. We show that the interpretation of quantifiers in the framework of Nmatrices is not sufficient for the case of $k > 1$ and introduce *generalized Nmatrices* which allow for a more complex treatment of quantifiers. Then we show that (i) a canonical calculus G is coherent iff there is a 2GNmatrix, for which G is strongly sound and complete, and (ii) any coherent canonical calculus admits cut-elimination.

1 Introduction

Propositional canonical Gentzen-type systems, introduced in [2, 3], are systems which in addition to the standard axioms and structural rules have only logical rules in which exactly one occurrence of a connective is introduced and no other connective is mentioned. Intuitively, the term "canonical systems" refers to systems in which the introduction rules of a logical connective determine the semantic meaning of that connective[1]. A natural constructive *coherence* criterion can be defined for the non-triviality of such systems, and it can be shown that a canonical system admits cut-elimination iff it is coherent. The semantics of such systems are provided by two-valued non-deterministic matrices (2Nmatrices), which form a natural generalization of the classical matrix. A characteristic 2Nmatrix can be constructed for every coherent canonical propositional system.

In [16] the notion of a canonical system is extended to languages with (n, k)-ary quantifiers. An (n, k)*-ary quantifier* (for $n > 0$, $k \geq 0$) is a generalized logical connective, which binds k variables and connects n formulas. Any n-ary propositional connective can be thought of as an $(n, 0)$-ary quantifier: for instance,

[1] This is according to a long tradition in the philosophy of logic, established by Gentzen in his classical paper *"Investigations Into Logical Deduction"* ([11]).

the standard $\wedge$ connective is an $(2,0)$-ary quantifier, as it binds no variables and connects two formulas: $\wedge(\psi_1, \psi_2)$. The standard first-order quantifiers $\exists$ and $\forall$ are $(1,1)$-quantifiers, while the simplest Henkin quantifier $\mathcal{Q}^H$ ([13]) is a (4,1)-quantifier, as it binds 4 variables and connects one formula[2]:

$$\mathcal{Q}^H x_1 x_2 y_1 y_2 \psi(x_1, x_2, y_1, y_2) := \begin{matrix} \forall x_1 \; \exists y_1 \\ \forall x_2 \; \exists y_2 \end{matrix} \psi(x_1, x_2, y_1, y_2)$$

Non-deterministic matrices (Nmatrices) are a natural generalization of the standard multi-valued matrix introduced in [2,3] and extended in [4,16]. In these structures the truth-value assigned to a complex formula is chosen non-deterministically out of a given non-empty set of options. [16] use two-valued Nmatrices (2Nmatrices) extended to languages with (n,k)-ary quantifiers to provide non-deterministic semantics for canonical systems for the case of $k \in \{0,1\}$. It is shown that there is a strong connection between the coherence of a canonical calculus G and the existence of a 2Nmatrix, for which G is strongly sound and complete.

In this paper we extend these results from the case of $k \in \{0,1\}$ to the general case of $k \geq 0$. We show that the interpretation of quantifiers used in [16] is not sufficient for the case of $k > 1$ and conclude that a more general interpretation of quantifiers is needed. Then we introduce *generalized Nmatrices* (GNmatrices), a generalization of Nmatrices, in which the approach to quantifiers used in Church's type theory ([10]) is adapted. Then it is shown that the following statements concerning a canonical calculus G with (n,k)-ary quantifiers for $k \geq 0$ and $n > 0$ are equivalent: (i) G is coherent, and (ii) there exists a 2GNmatrix, for which G is strongly sound and complete. Finally, we show that any coherent canonical calculus with (n,k)-ary quantifiers admits cut-elimination.

2 Preliminaries

In what follows, L is a language with (n,k)-ary quantifiers, that is with quantifiers $\mathcal{Q}_1, ..., \mathcal{Q}_m$ with arities (n_1, k_1), ..., (n_m, k_m) respectively. For any $n > 0$ and $k \geq 0$, if a quantifier $\mathcal{Q}$ in a language L is of arity (n,k), then $\mathcal{Q}x_1...x_k(\psi_1, ..., \psi_n)$ is an L-formula whenever $x_1, ..., x_k$ are distinct variables and $\psi_1, ..., \psi_n$ are formulas of L. Denote by Frm_L (Frm_L^{cl}) the set of L-formulas (closed L-formulas). Denote by Trm_L (Trm_L^{cl}) the set of L-terms (closed L-terms). $Var = \{v_1, v_2, ...,\}$ is the set of variables of L. We use the metavariables x, y, z to range over elements of Var. Given an L-formula A, $Fv[A]$ is the set of variables occurring free in A. $\equiv_\alpha$ is the α-equivalence relation between formulas, i.e identity up to the renaming of bound variables. We use [ ] for application of functions in the meta-language, leaving the use of ( ) to the object language. We write $\mathcal{Q}\overrightarrow{x}A$ instead of $\mathcal{Q}x_1...x_kA$, and $\psi\{\overrightarrow{\mathbf{t}}/\overrightarrow{z}\}$ instead of $\psi\{\mathbf{t}_1/z_1, ..., \mathbf{t}_k/z_k\}$.

[2] In this way of recording combinations of quantifiers, dependency relations between variables are expressed as follows: an existentially quantified variable depends on those universally quantified variables which are on the left of it in the same row.

In the following two subsections, we briefly reproduce the relevant definitions from [16] of canonical systems with (n, k)-ary quantifiers and of the semantic framework of Nmatrices.

2.1 Canonical Systems with (n, k)-ary quantifiers

We use a simplified representation language from [16] for a schematic representation of canonical rules.

Definition 1 *For $k \geq 0$, $n \geq 1$ and a set of constants Con, $L_k^n(Con)$ is the language with n k-ary predicate symbols $p_1, ..., p_n$ and the set of constants Con (and no quantifiers or connectives). The set of variables of $L_k^n(Con)$ is $Var = \{v_1, v_2, ..., \}$.*

Note that $L_k^n(Con)$ and L share the same set of variables. Henceforth we also assume[3] that for every (n, k)-ary quantifier $\mathcal{Q}$ of L, $L_k^n(Con)$ is a subset of L.

Definition 2 *Let Con be some set of constants. A* canonical quantificational rule of arity (n, k) *is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m}/C$, where $m \geq 0$, C is either $\Rightarrow \mathcal{Q}v_1...v_k(p_1(v_1, ..., v_k), ..., p_n(v_1, ..., v_k))$ or $\mathcal{Q}v_1...v_k(p_1(v_1, ..., v_k), ..., p_n(v_1, ..., v_k)) \Rightarrow$ for some (n, k)-ary quantifier $\mathcal{Q}$ of L and for every $1 \leq i \leq m$: $\Pi_i \Rightarrow \Sigma_i$ is a clause[4] over $L_k^n(Con)$.*

Henceforth, in cases where the set of constants Con is clear from the context (it is the set of all constants occurring in a canonical rule), we will write L_k^n instead of $L_k^n(Con)$.
A canonical rule is a schematic representation of the actual rule, while for a specific application of the rule we need to instantiate the schematic variables by the terms and formulas of L. This is done using a mapping function:

Definition 3 *Let $R = \Theta/C$ be an (n, k)-ary canonical rule, where C is of one of the forms $(\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v})) \Rightarrow)$ or $(\Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v})))$. Let Γ be a set of L-formulas and $z_1, ..., z_k$ - distinct variables of L. An $\langle R, \Gamma, z_1, ..., z_k\rangle$-*mapping *is any function χ from the predicate symbols, terms and formulas of L_k^n to formulas and terms of L, satisfying the following conditions:*

- *For every $1 \leq i \leq n$, $\chi[p_i]$ is an L-formula. $\chi[y]$ is a variable of L, and $\chi[x] \neq \chi[y]$ for every two variables $x \neq y$. $\chi[c]$ is an L-term, such that $\chi[x]$ does not occur in $\chi[c]$ for any variable x occurring in Θ.*
- *For every $1 \leq i \leq n$, whenever $p_i(\boldsymbol{t}_1, ..., \boldsymbol{t}_k)$ occurs in Θ, for every $1 \leq j \leq k$: $\chi[\boldsymbol{t}_j]$ is a term free for z_j in $\chi[p_i]$, and if $\boldsymbol{t}_j$ is a variable, then $\chi[\boldsymbol{t}_j]$ does not occur free in $\Gamma \cup \{\mathcal{Q}z_1...z_k(\chi[p_1], ..., \chi[p_n])\}$.*
- *$\chi[p_i(\boldsymbol{t}_1, ..., \boldsymbol{t}_k)] = \chi[p_i]\{\chi[\boldsymbol{t}_1]/z_1, ..., \chi[\boldsymbol{t}_k]/z_k\}$.*

χ is extended to sets of L_k^n-formulas as follows: $\chi[\Delta] = \{\chi[\psi] \mid \psi \in \Delta\}$.

[3] This assumption is not necessary, but it makes the presentation easier, as will be explained in the sequel.

[4] By a clause we mean a sequent containing only atomic formulas.

Definition 4 *An* application *of a canonical rule of arity* (n, k) $R = \{\Pi_i \Rightarrow \Sigma_i\}_{1 \le i \le m} / \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v})) \Rightarrow$ *is any inference step of the form:*

$$\frac{\{\Gamma, \chi[\Pi_i] \Rightarrow \Delta, \chi[\Sigma_i]\}_{1 \le i \le m}}{\Gamma, \mathcal{Q}z_1...z_k\ (\chi[p_1], ..., \chi[p_n]) \Rightarrow \Delta}$$

where $z_1, ..., z_k$ *are variables,* Γ, Δ *are any sets of L-formulas and* χ *is some* $\langle R, \Gamma \cup \Delta, z_1, ..., z_k\rangle$*-mapping.*
An application of a canonical quantificational rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \le i \le m} / \Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))$ *is defined similarly.*

For example, the two standard introduction rules for the $(1,1)$-ary quantifier $\forall$ can be formulated as follows: $\{p(c) \Rightarrow\}/\forall v_1\, p(v_1) \Rightarrow$ and $\{\Rightarrow p(v_1)\}/ \Rightarrow \forall v_1\, p(v_1)$. Applications of these rules have the forms:

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta}\ (\forall \Rightarrow) \quad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}\ (\Rightarrow \forall)$$

where z is free for w in A, z is not free in $\Gamma \cup \Delta \cup \{\forall w A\}$, and $\mathbf{t}$ is any term free for w in A.
Notation: (Following [3, 16]). Let $-t = f, -f = t$. Let $ite(t, A, B) = A$ and $ite(f, A, B) = B$. Let Φ, A^s (where Φ may be empty) denote $ite(s, \Phi \cup \{A\}, \Phi)$. For instance, the sequents $A \Rightarrow$ and $\Rightarrow A$ are denoted by $A^{-a} \Rightarrow A^a$ for $a = f$ and $a = t$ respectively. With this notation, an (n, k)-ary canonical rule has the form $\{\Sigma_j \Rightarrow \Pi_j\}_{1 \le j \le m} / \mathcal{Q}\overrightarrow{z}(p_1(\overrightarrow{z}), ..., p_n(\overrightarrow{z}))^{-s} \Rightarrow \mathcal{Q}\overrightarrow{z}(p_1(\overrightarrow{z}), ..., p_n(\overrightarrow{z}))^s$ for some $s \in \{t, f\}$. For further abbreviation, we denote such rule by $\{\Sigma_j \Rightarrow \Pi_j\}_{1 \le j \le m} / \mathcal{Q}(s)$.

Definition 5 *A Gentzen-type calculus G is* canonical *if in addition to the* α*-axiom* $A \Rightarrow A'$ *for* $A \equiv_\alpha A'$ *and the standard structural rules, G has only canonical rules.*

Definition 6 *Two* (n, k)*-ary canonical introduction rules* Θ_1/C_1 *and* Θ_2/C_2 *for* $\mathcal{Q}$ *are* dual *if for some* $s \in \{t, f\}$*:* $C_1 = A^{-s} \Rightarrow A^s$ *and* $C_2 = A^s \Rightarrow A^{-s}$*, where* $A = \mathcal{Q}v_1...v_k(p_1(v_1, ..., v_k), ..., p_n(v_1, ..., v_k))$*.*

Definition 7 *For two sets of clauses* Θ_1, Θ_2 *over* L^n_k*,* $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ *is a set* $\Theta_1 \cup \Theta'_2$*, where* Θ'_2 *is obtained from* Θ_2 *by a fresh renaming of constants and variables which occur in* Θ_1*.*

Definition 8 (Coherence)[5] *A canonical calculus G is* coherent *if for every two dual canonical rules* $\Theta_1/ \Rightarrow A$ *and* $\Theta_2/A \Rightarrow$*, the set of clauses* $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ *is classically inconsistent.*

[5] The coherence criterion for the propositional case was first introduced in [2, 3] and then extended to the first-order case in [16]. A strongly related coherence criterion was also used in [14], where linear logic is used to reason about various sequent systems. Also, the coherence criterion defined in this paper can be shown to be equivalent in the context of canonical calculi to the reductivity condition of [9] (defined for Gentzen-type systems with (n, k)-ary quantifiers which are more general than the canonical calculi), as will be explained in the sequel.

Proposition 9 (Decidability of coherence) *([16]) The coherence of a canonical calculus G is decidable.*

2.2 Non-deterministic matrices

Non-deterministic matrices[6] (Nmatrices), were first introduced in [2, 3] and extended to the first-order case in [4, 17]. These structures are a generalization of the standard concept of a many-valued matrix, in which the truth-value of a formula is chosen non-deterministically from a given non-empty set of truth-values. For interpretation of quantifiers, generalized *distribution quantifiers*[7] are used.

Definition 10 *([16])* **(Non-deterministic matrix)** *A non-deterministic matrix (Nmatrix) for L is a tuple $\mathcal{M} =< \mathcal{V}, \mathcal{D}, \mathcal{O} >$, where: (i) $\mathcal{V}$ is a non-empty set of truth values, (ii) $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$, and (iii) $\mathcal{O}$ is a set of interpretation functions: for every (n, k)-ary quantifier $\mathcal{Q}$ of L, $\mathcal{O}$ includes the corresponding distribution function $\tilde{\mathcal{Q}}_{\mathcal{M}} : P^+(\mathcal{V}^n) \to P^+(\mathcal{V})$. A 2Nmatrix is any Nmatrix with $\mathcal{V} = \{t, f\}$ and $\mathcal{D} = \{t\}$.*

The notion of an L-structure is defined standardly (see, e.g. [16, 4]). In order to interpret quantifiers, the substitutional approach is used, which assumes that every element of the domain has a term referring to it. Thus given a structure $S = \langle D, I\rangle$, the language L is extended with *individual constants*: $\{\overline{a} \mid a \in D\}$. Call the extended language $L(D)$. The interpretation function I is extended as follows: $I[\overline{a}] = a$.
An L-substitution σ is any function from variables to $Trm^{cl}_{L(D)}$. For an L-substitution σ and a term $\mathbf{t}$ (a formula ψ), the closed term $\sigma[\mathbf{t}]$ (the sentence $\sigma[\psi]$) is obtained from $\mathbf{t}$ (ψ) by substituting every variable x for $\sigma[x]$.

Definition 11 (Congruence of terms and formulas)[8] *Let S be an L-structure for an Nmatrix $\mathcal{M}$. The relation $\sim^S$ between terms of $L(D)$ is defined inductively as follows: (i) $x \sim^S x$, (ii) For closed terms $\boldsymbol{t}, \boldsymbol{t}'$ of $L(D)$: $\boldsymbol{t} \sim^S \boldsymbol{t}'$ when $I[\boldsymbol{t}] = I[\boldsymbol{t}']$, (iii) If $\boldsymbol{t}_1 \sim^S \boldsymbol{t}'_1, ..., \boldsymbol{t}_n \sim^S \boldsymbol{t}'_n$, then $f(\boldsymbol{t}_1, ..., \boldsymbol{t}_n) \sim^S f(\boldsymbol{t}'_1, ..., \boldsymbol{t}'_n)$.*
The relation $\sim^S$ between formulas of $L(D)$ is defined as follows:

- *If $\boldsymbol{t}_1 \sim^S \boldsymbol{t}'_1, \boldsymbol{t}_2 \sim^S \boldsymbol{t}'_2, ..., \boldsymbol{t}_n \sim^S \boldsymbol{t}'_n$, then $p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n) \sim^S p(\boldsymbol{t}'_1, ..., \boldsymbol{t}'_n)$.*
- *If $\psi_1\{\overrightarrow{z}/\overrightarrow{x}\} \sim^S \varphi_1\{\overrightarrow{z}/\overrightarrow{y}\}, ..., \psi_n\{\overrightarrow{z}/\overrightarrow{x}\} \sim^S \varphi_n\{\overrightarrow{z}/\overrightarrow{y}\}$, where $\overrightarrow{x} = x_1...x_k$ and $\overrightarrow{y} = y_1...y_k$ are distinct variables and $\overrightarrow{z} = z_1...z_k$ are new distinct variables, then $\mathcal{Q}\overrightarrow{x}(\psi_1, ..., \psi_n) \sim^S \mathcal{Q}\overrightarrow{y}(\varphi_1, ..., \varphi_n)$ for any (n, k)-ary quantifier $\mathcal{Q}$ of L.*

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[6] For the connection between Nmatrices and other abstract semantics, see e.g. [7].

[7] Distribution quantifiers were introduced in [6], with the intention to generalize Mostowski's proposal.

[8] The motivation for this definition is purely technical and is related to extending the language with the set of individual constants $\{\overline{a} \mid a \in D\}$. Suppose we have a closed term $\mathbf{t}$, such that $I[\mathbf{t}] = a \in D$. But a also has an individual constant $\overline{a}$ referring to it. We would like to be able to substitute $\mathbf{t}$ for $\overline{a}$ in every context.

The following is a straightforward generalization of Lemma 3.6 from [16].

Lemma 12 *Let S be an L-structure for an Nmatrix $\mathcal{M}$. Let ψ, ψ' be formulas of $L(D)$. Let $\boldsymbol{t}_1, ..., \boldsymbol{t}_n, \boldsymbol{t}'_1, ..., \boldsymbol{t}'_n$ be closed terms of $L(D)$, such that $\boldsymbol{t}_i \sim^S \boldsymbol{t}'_i$ for every $1 \leq i \leq n$. Then (1) If $\psi \equiv_\alpha \psi'$, then $\psi \sim^S \psi'$, and (2) If $\psi \sim^S \psi'$, then $\psi\{\overrightarrow{\boldsymbol{t}}/\overrightarrow{x}\} \sim^S \psi'\{\overrightarrow{\boldsymbol{t}'}/\overrightarrow{x}\}$.*

Definition 13 **(Legal valuation)** *Let $S = \langle D, I\rangle$ be an L-structure for a Nmatrix $\mathcal{M}$. An S-valuation $v : Frm^{cl}_{L(D)} \to \mathcal{V}$ is* legal in $\mathcal{M}$ *if it satisfies the following conditions:*

- *$v[\psi] = v[\psi']$ for every two sentences ψ, ψ' of $L(D)$, such that $\psi \sim^S \psi'$.*
- *$v[p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)] = I[p][I[\boldsymbol{t}_1], ..., I[\boldsymbol{t}_n]]$.*
- *$v[\mathcal{Q}x_1, ..., x_k(\psi_1, ..., \psi_n)$ is in the set $\tilde{\mathcal{Q}}_{\mathcal{M}}[\{\langle v[\psi_1\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}], ..., v[\psi_n\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}]\rangle \mid a_1, ..., a_k \in D\}]$ for every (n, k)-ary quantifier $\mathcal{Q}$ of L.*

Definition 14 *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$.*

1. *An $\mathcal{M}$-legal S-valuation v is* a model of a sentence ψ in $\mathcal{M}$, *denoted by $S, v \models_{\mathcal{M}} \psi$, if $v[\psi] \in \mathcal{D}$.*
2. *Let v be an $\mathcal{M}$-legal S-valuation. A sequent $\Gamma \Rightarrow \Delta$ is* $\mathcal{M}$-valid in $\langle S, v\rangle$ *if for every S-substitution σ: if $S, v \models_{\mathcal{M}} \sigma[\psi]$ for every $\psi \in \Gamma$, then there is some $\varphi \in \Delta$, such that $S, v \models_{\mathcal{M}} \sigma[\varphi]$. A sequent $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid if for every L-structure S and every $\mathcal{M}$-legal S-valuation v, $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid in $\langle S, v\rangle$.*
3. *The consequence relation $\vdash_{\mathcal{M}}$ between sets of L-formulas is defined as follows: $\Gamma \vdash_{\mathcal{M}} \Delta$ if $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid.*

Definition 15 *A system G is* sound *for an Nmatrix $\mathcal{M}$ if $\vdash_G \subseteq \vdash_{\mathcal{M}}$. A system G is* complete *for an Nmatrix $\mathcal{M}$ if $\vdash_{\mathcal{M}} \subseteq \vdash_G$. An Nmatrix $\mathcal{M}$ is* characteristic *for G if G is sound and complete for $\mathcal{M}$.*
A system G is strongly sound *for $\mathcal{M}$ if for every set of sequents $\mathcal{S}$: if $\Gamma \Rightarrow \Delta$ is derivable in G from $\mathcal{S}$, then for every L-structure S and every S-substitution v, whenever $\mathcal{S}$ is $\mathcal{M}$-valid in $\langle S, v\rangle$, $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid in $\langle S, v\rangle$.*

Note that strong soundness implies (weak) soundness.

In addition to L-structures for languages with (n, k)-ary quantifiers, we will also use L^n_k-structures for the simplified languages L^n_k, using which the canonical rules are formulated. To make the distinction clearer, we shall use the metavariable S for the former and $\mathcal{N}$ for the latter. Since the formulas of L^n_k are always atomic, the specific 2Nmatrix for which $\mathcal{N}$ is defined is immaterial, and can be omitted. Henceforth we may speak simply of validity of sets of sequents over L^n_k.

Definition 16 *([16]) Let $\mathcal{N} = \langle D, I\rangle$ be an L^n_k-structure. $Dist_{\mathcal{N}}$ (the distribution of $\mathcal{N}$) is the set $\{\langle I[p_1][a_1, ..., a_k], ..., I[p_n][a_1, ..., a_k]\rangle \mid a_1, ..., a_k \in D\}$.*

3 Generalizing the framework of Nmatrices

It is shown in [16] for the case of $k \in \{0,1\}$ that a canonical calculus has a strongly characteristic 2Nmatrix iff it is coherent. Moreover, if a 2Nmatrix $\mathcal{M}$ is suitable for a calculus G, then G is strongly sound for $\mathcal{M}$:

Definition 17 *([16]) Let G be a canonical calculus over L. A 2Nmatrix $\mathcal{M}$ is* suitable *for G if for every (n,k)-ary canonical rule $\Theta/\mathcal{Q}(s)$ of G, it holds that for every L_k^n-structure $\mathcal{N}$ in which Θ is valid: $\tilde{\mathcal{Q}}_{\mathcal{M}}[Dist_{\mathcal{N}}] = \{s\}$.*

We will now show that the above property does not hold for the case of $k > 1$. We first prove that the suitability of $\mathcal{M}$ for G is not only a sufficient, but also a *necessary* condition for the strong soundness of G for $\mathcal{M}$ for any $k \geq 0$. Then we will construct a coherent calculus with a (2,1)-ary quantifier, for which there is no suitable 2Nmatrix. This immediately implies that G has no strongly characteristic 2Nmatrix.

Proposition 18 *If a canonical calculus G is strongly sound for a 2Nmatrix $\mathcal{M}$, then $\mathcal{M}$ is suitable for G.*

Proof: Let G be a canonical calculus which is strongly sound for $\mathcal{M}$ and suppose for contradiction that $\mathcal{M}$ is not suitable for G. Then there is some (n,k)-ary canonical rule $R = \Theta/\mathcal{Q}(s)$ of G, such that there is some L_k^n-structure $\mathcal{N}$ in which Θ is valid, but $\tilde{\mathcal{Q}}_{\mathcal{M}}[Dist_{\mathcal{N}}] \neq \{s\}$. Suppose that $s = t$. Then $R = \Theta/\Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))$ and $(*)$ $f \in \tilde{\mathcal{Q}}_{\mathcal{M}}[Dist_{\mathcal{N}}]$. Let S be any extension of $\mathcal{N}$ to L (recall that we assume for simplicity that L_k^n is a subset of L). It is easy to see that Θ is also $\mathcal{M}$-valid in $\langle S, v\rangle$ for every S-valuation v (note that Θ only contains atomic formulas). Obviously, $\Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))$ is derivable from Θ in G. Now since G is strongly sound for $\mathcal{M}$, $(**)$ $\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))$ should also be $\mathcal{M}$-valid in $\langle S, v\rangle$ for every S-valuation v. Let v_0 be any $\mathcal{M}$-legal S-valuation, such that $v_0[\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))] = f$ (the existence of such a valuation follows from $(*)$ and the fact that $\{\langle v_0[p_1\{\overrightarrow{a}/\overrightarrow{v}\}], ..., v[p_n\{\overrightarrow{a}/\overrightarrow{v}\}]\rangle \mid a_1, ..., a_k \in D\} = Dist_{\mathcal{N}}$). Obviously $\Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))$ is not $\mathcal{M}$-valid in $\langle S, v_0\rangle$, in contradiction to $(**)$. □

Next, consider the calculus G, which consists of the following two dual introduction rules $\Theta_1/\Rightarrow \mathcal{Q}v_1v_2p(v_1,v_2)$ and $\Theta_2/\mathcal{Q}v_1v_2p(v_1,v_2) \Rightarrow$, where $\Theta_1 = \{p(c,v_1) \Rightarrow\}$ and $\Theta_2 = \{\Rightarrow p(v_1,c)\}$. The set of clauses $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \{p(c,v_1) \Rightarrow, \Rightarrow p(v_2,d)\}$ is classically inconsistent, and so G is coherent. Suppose by contradiction that there is a 2Nmatrix $\mathcal{M}$ suitable for G. Consider the L_k^n-structures $\mathcal{N}_1 = \langle D_1, I_1\rangle$ and $\mathcal{N}_2 = \langle D_2, I_2\rangle$, defined as follows. $D_1 = D_2 = \{a_1, a_2\}$, $I_1[p][a_1,a_1] = I_1[a_1,a_2] = f$, $I_1[p][a_2,a_1] = I_1[p][a_2,a_2] = t$, $I_1[c] = a_1$. $I_2[p][a_1,a_1] = I_2[a_1,a_2] = t$, $I_2[p][a_2,a_1] = I_2[p][a_2,a_2] = f$, $I_2[c] = a_1$. Obviously, Θ_1 is valid in $\mathcal{N}_1$, and so by suitability of $\mathcal{M}$, $\tilde{\mathcal{Q}}_{\mathcal{M}}[Dist_{\mathcal{N}_1}] = t$. Θ_2 is valid in $\mathcal{N}_2$, and so $\tilde{\mathcal{Q}}_{\mathcal{M}}[Dist_{\mathcal{N}_2}] = f$. But this is impossible, since $Dist_{\mathcal{N}_1} = Dist_{\mathcal{N}_2} = \{t, f\}$. Thus G has no suitable 2Nmatrix, although it is coherent. By Prop. 18 above, G has no strongly characteristic 2Nmatrix.

We conclude that the interpretation of (n,k)-ary quantifiers using distributions is not sufficient for the case of $k > 1$. Using them, we cannot capture any kind of dependencies between elements of the domain. For instance, there is no way we can express the fact that there exists an element b in the domain, such that for *every* element a, $p(a,b)$ holds. It is clear that a more general interpretation of a quantifier is needed.

We will generalize the interpretation of quantifiers as follows. Given an L-structure $S = \langle D, I\rangle$, an interpretation of an (n,k)-ary quantifier $\mathcal{Q}$ in S is an operation $\tilde{\mathcal{Q}}_S : (D^k \to \mathcal{V}^n) \to P^+(\mathcal{V})$, which for every function (from k-ary vectors of the domain elements to n-ary vectors of truth-values) returns a non-empty set of truth-values.

Definition 19 *A* generalized non-deterministic matrix *(henceforth GNmatrix) for L is a tuple $\mathcal{M} =< \mathcal{V}, \mathcal{D}, \mathcal{O} >$, where:*

- *$\mathcal{V}$ is a non-empty set of truth values.*
- *$\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.*
- *For every (n,k)-ary quantifier $\mathcal{Q}$ of L, $\mathcal{O}$[9] includes a corresponding operation $\tilde{\mathcal{Q}}_S : (D^k \to \mathcal{V}^n) \to P^+(\mathcal{V})$ for every L-structure $S = \langle D, I\rangle$. A 2GNmatrix is any GNmatrix with $\mathcal{V} = \{t, f\}$ and $\mathcal{D} = \{t\}$.*

Examples:

1. Given an L-structure $S = \langle D, I\rangle$, the standard $(1,1)$-ary quantifier $\forall$ is interpreted as follows for any $g \in D \to \{t, f\}$: $\tilde{\forall}_S[g] = \{t\}$ if for every $a \in D$, $g[a] = t$, and $\tilde{\forall}_S[g] = \{f\}$ otherwise. The standard $(1,1)$-ary quantifier $\exists$ is interpreted as follows for any $g \in D \to \{t, f\}$: $\tilde{\exists}_S[g] = \{t\}$ if there exists some $a \in D$, such that $g[a] = t$, and $\tilde{\exists}_S[g] = \{f\}$ otherwise.
2. Given an L-structure $S = \langle D, I\rangle$, the $(1,2)$-ary bounded universal[10] quantifier $\overline{\forall}$ is interpreted as follows: for any $g \in D \to \{t, f\}^2$, $\tilde{\overline{\forall}}_S[g] = \{t\}$ if for every $a \in D$, $g[a] \neq \langle t, t\rangle$, and $\tilde{\overline{\forall}}_S[g] = \{f\}$ otherwise. The $(1,2)$-ary bounded existential[11] quantifier $\overline{\exists}$ is interpreted as follows: for any $g \in D \to \{t, f\}^2$, $\tilde{\overline{\forall}}_S[g] = \{t\}$ if there exists some $a \in D$, such that $g[a] = \langle t, t\rangle$, and $\tilde{\overline{\forall}}_S[g] = \{f\}$ otherwise.
3. Consider the $(2,2)$-ary quantifier $\mathcal{Q}$, with the intended meaning of $Qxy(\psi_1, \psi_2)$ as $\exists y \forall x(\psi_1(x,y) \wedge \neg\psi_2(x,y))$. Its interpretation for every L-structure $S = \langle D, I\rangle$, every $g \in D^2 \to \{t, f\}^2$ is as follows: $\tilde{Q}_S[g] = t$ iff there exists some $a \in D$, such that for every $b \in D$: $g[a,b] = \langle t, f\rangle$.
4. Consider the $(4,1)$-ary Henkin quantifier[12] $\mathcal{Q}_H$ discussed in section 1. Its interpretation for for every L-structure $S = \langle D, I\rangle$ and every $g \in D^4 \to \{t, f\}$

[9] In the current definition, $\mathcal{O}$ is not a class and the tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is not well-defined. We can overcome this technical problem by assuming that the domains of all the structures are prefixes of the set of natural numbers. A more general solution to this problem is a question for further research.

[10] The intended meaning of $\overline{\forall}x(p_1(x), p_2(x))$ is $\forall x(p_1(x) \to p_2(x))$.

[11] The intended meaning of $\overline{\exists}x(p_1(x), p_2(x))$ is $\exists x(p_1(x) \wedge p_2(x))$.

[12] We note that the current framework of canonical systems is not adequate to handle such quantifiers.

is as follows: $\tilde{\mathcal{Q}}^H_S[g] = \{t\}$ if for every $a \in D$ there exists some $b \in D$ and for every $c \in D$ there exists some $d \in D$, such that $g[a,b,c,d] = t$. $\tilde{\mathcal{Q}}^H_S[g] = \{f\}$ otherwise.

Definition 20 (Legal valuation) *Let $S = \langle D, I\rangle$ be an L-structure for a GN-matrix $\mathcal{M}$. An S-valuation $v : Frm^{\mathsf{cl}}_{L(D)} \to \mathcal{V}$ is* legal *in $\mathcal{M}$ if it satisfies the following conditions: $v[\psi] = v[\psi']$ for every two sentences ψ, ψ' of $L(D)$, such that $\psi \sim^S \psi'$, $v[p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)] = I[p][I[\boldsymbol{t}_1], ..., I[\boldsymbol{t}_n]]$, and $v[\mathcal{Q}x_1, ..., x_k(\psi_1, ..., \psi_n)$ is in the set $\tilde{\mathcal{Q}}_S[\lambda a_1, ..., a_k \in D.\langle v[\psi_1\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}], ..., v[\psi_n\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}]\rangle]$ for every (n,k)-ary quantifier $\mathcal{Q}$ of L.*

The semantic notions from Defn. 14 and 15 are defined similarly for the case of GNmatrices.
Next we generalize the notion of a *distribution* of L^n_k-structures (see Defn. 16).

Definition 21 *Let $\mathcal{N} = \langle D, I\rangle$ be a structure for L^n_k. The* functional distribution *of $\mathcal{N}$ is a function $FDist_{\mathcal{N}} \in D^k \to \{t, f\}^n$, such that: $FDist_{\mathcal{N}} = \lambda a_1, ..., a_k \in D.\langle I[p_1][a_1, ..., a_k], ..., I[p_n][a_1, ..., a_k]\rangle$.*

4 Semantics for canonical calculi

In this section we show that a canonical calculus G with (n,k)-ary quantifiers is coherent iff it has a strongly characteristic 2GNmatrix.
First we construct a strongly characteristic 2GNmatrix for every coherent canonical calculus.

Definition 22 *Let G be a coherent canonical calculus. For every L-structure $S = \langle D, I\rangle$, the GNmatrix $\mathcal{M}_G$ contains the operation $\tilde{\mathcal{Q}}_S$ defined as follows. For every (n,k)-ary quantifier $\mathcal{Q}$ of L, every $r \in \{t,f\}$ and every $g \in D^k \to \{t,f\}^n$:*

$$\tilde{\mathcal{Q}}_S[g] = \begin{cases} \{r\} & \Theta/\mathcal{Q}(r) \in G \text{ and there is an } L^n_k - \text{structure } \mathcal{N} = \langle D_{\mathcal{N}}, I_{\mathcal{N}}\rangle \\ & \text{such that } D_{\mathcal{N}} = D,\ FDist_{\mathcal{N}} = g \text{ and } \Theta \text{ is valid in } \mathcal{N}. \\ \{t,f\} & \text{otherwise} \end{cases}$$

It should be noted that as opposed to the definition of the Nmatrix $\mathcal{M}_G$ in [16] (see Defn. 4.2 there), the above definition is not constructive. This is because the question whether Θ is valid in some L^n_k-structure with a given functional distribution is not generally decidable. Next, let us show that $\mathcal{M}_G$ is well-defined. Assume by contradiction that there are two dual rules $\Theta_1/\Rightarrow A$ and $\Theta_2/A \Rightarrow$, such that there exist two L^n_k-structures $\mathcal{N}_1 = \langle D, I_1\rangle$ and $\mathcal{N}_2 = \langle D, I_2\rangle$, which satisfy: $FDist_{\mathcal{N}_1} = FDist_{\mathcal{N}_2}$ and Θ_i is valid in $\mathcal{N}_i$ for $i \in \{1,2\}$. But then $\mathcal{N}_1$ and $\mathcal{N}_2$ only differ in their interpretations of constants from Θ_1 and Θ_2. Then we can easily construct an L^n_k-structure $\mathcal{N}_3 = \langle D, I_3\rangle$, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is valid in $\mathcal{N}_3$ (the renaming is essential since it may be the case that the same constant occurs both in Θ_1 and Θ_2). And so $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent, in contradiction to the coherence of G.

Theorem 23 *Any coherent canonical calculus G is strongly sound for $\mathcal{M}_G$.*

Proof: Let $S = \langle D, I\rangle$ be some L-structure and v - an $\mathcal{M}$-legal S-valuation. Let $\mathcal{S}$ be any set of sequents closed under substitution. We will show that if the sequents of $\mathcal{S}$ are $\mathcal{M}$-valid in $\langle S, v\rangle$, then any sequent provable from $\mathcal{S}$ in G is $\mathcal{M}$-valid in $\langle S, v\rangle$. Obviously, the axioms of G are $\mathcal{M}$-valid, and the structural rules, including cut, are strongly sound. It remains to show that for every application of a canonical rule R of G: if the premises of R are $\mathcal{M}$-valid in $\langle S, v\rangle$, then its conclusion is $\mathcal{M}$-valid in $\langle S, v\rangle$. Let R be an (n, k)-ary rule of G of the form: $R = \Theta_R / \Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}), ..., p_n(\overrightarrow{v}))$, where $\Theta_R = \{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}$. An application of R is of the form:

$$\frac{\{\Gamma, \chi[\Sigma_j] \Rightarrow \chi[\Pi_j], \Delta\}_{1\leq j\leq m}}{\Gamma \Rightarrow \Delta, \mathcal{Q}\overrightarrow{z}(\chi[p_1], ..., \chi[p_n])}$$

where χ is some $\langle R, \Gamma \cup \Delta, \overrightarrow{z}\rangle$-mapping. Let $\{\Gamma, \chi[\Sigma_j] \Rightarrow \chi[\Pi_j], \Delta\}_{1\leq j\leq m}$ be $\mathcal{M}$-valid in $\langle S, v\rangle$. Let σ be an S-substitution, such that $S, v \models_{\mathcal{M}} \sigma[\Gamma]$ and for every $\psi \in \Delta$: $S, v \not\models_{\mathcal{M}} \sigma[\psi]$. Denote by $\widetilde{\psi}$ the L-formula obtained from a formula ψ by substituting every free occurrence of $w \in Fv[\psi] - \{z\}$ for $\sigma[w]$. Construct the L^n_k-structure $\mathcal{N} = \langle D_{\mathcal{N}}, I_{\mathcal{N}}\rangle$ as follows: $D_{\mathcal{N}} = D$, for every $a_1, ..., a_k \in D$: $I_{\mathcal{N}}[p_i][a_1, ..., a_k] = v[\widetilde{\chi[p_i]}\{\overrightarrow{a}/\overrightarrow{z}\}]$, and for every constant c occurring in Θ_R, $I_{\mathcal{N}}[c] = I[\sigma[\chi[c]]]$. It is not difficult to show that Θ_R is valid in $\mathcal{N}$. Thus by definition of $\mathcal{M}_G$, $\tilde{\mathcal{Q}}_S[FDist_{\mathcal{N}}] = \{t\}$. Finally, by definition of $\mathcal{N}$, $FDist_{\mathcal{N}} = \lambda a_1, ..., a_k \in D.\{\langle v[\widetilde{\chi[p_1]}\{\overrightarrow{a}/\overrightarrow{z}\}], ..., v[\widetilde{\chi[p_n]}\{\overrightarrow{a}/\overrightarrow{z}\}]\rangle\}$. Since v is $\mathcal{M}$-legal, $v[\sigma[\mathcal{Q}\overrightarrow{z}(\chi[p_1], ..., \chi[p_n])]] = v[\mathcal{Q}\overrightarrow{z}(\widetilde{\chi[p_1]}, ..., \widetilde{\chi[p_n]})] \in FDist_{\mathcal{N}} = \{t\}$. And so $\Gamma \Rightarrow \Delta, \mathcal{Q}\overrightarrow{z}(\chi[p_1], ..., \chi[p_n])$ is $\mathcal{M}$-valid in $\langle S, v\rangle$. □

Example 1. The canonical calculus G_1 consists of (1,1)-ary rule $\Rightarrow p(v_1)/ \Rightarrow \forall v_1 p(v_1)$. Clearly, G_1 is coherent. For every L-structure $S = \langle D, I\rangle$, $\mathcal{M}_{G_1}$ contains the operation $\tilde{\forall}_S$ defined as follows for every $g \in D \to \mathcal{V}$:

$$\tilde{\forall}_S[g] = \begin{cases} \{t\} & \textit{if for all } a \in D: \ g[a] = t \\ \{t, f\} & \textit{otherwise} \end{cases}$$

Example 2. The canonical calculus G_2 consists of the following rules: (i) $\{p_1(v_1) \Rightarrow p_2(v_1)\}/ \Rightarrow \overline{\forall} v_1\ (p_1(v_1), p_2(v_1))$, (ii) $\{p_2(c) \Rightarrow\ ,\ \Rightarrow p_1(c)\}/\overline{\forall} v_1(p_1(v_1), p_2(v_1)) \Rightarrow$ and (iii) $\{\Rightarrow\ p_1(c)\ ,\ \Rightarrow\ p_2(c)\}/ \Rightarrow\ \overline{\exists} v_1(p_1(v_1), p_2(v_1))$. G' is obviously coherent. The operations $\tilde{\overline{\forall}}_S$ and $\tilde{\overline{\exists}}_S$ in $\mathcal{M}_{G_2}$ are defined as follows for every $g \in D \to \{t, f\}^2$:

$$\tilde{\overline{\forall}}_S[g] = \begin{cases} \{t\} & \textit{if there are no such } a, b \in D, \textit{ that } g[a, b] = \langle t, f\rangle \\ \{f\} & \textit{otherwise} \end{cases}$$

$$\tilde{\overline{\exists}}_S[g] = \begin{cases} \{t\} & \textit{if there are } a, b \in D, s.t.\ g[a, b] = \langle t, t\rangle \\ \{t, f\} & \textit{otherwise} \end{cases}$$

The rule (i) dictates the condition that $\overline{\forall}_S[g] = \{t\}$ for the case that there are no $a, b \in D$, s.t. $g[a,b] = \langle t, f\rangle$. The rule (ii) dictates the condition that $\overline{\forall}_S[g] = \{f\}$ for the case that there are such $a, b \in D$. Since G_2 is coherent, the dictated conditions are non-contradictory. The rule (iii) dictates the condition that $\overline{\exists}_S[g] = \{t\}$ in the case that there are $a, b \in D$, s.t. $g[a,b] = \langle t, t\rangle$. There is no rule which dictates conditions for the case of $\langle t, t\rangle \notin H$, and so the interpretation in this case is non-deterministic.

Example 3. Consider the canonical calculus G_3 consisting of the following $(2,2)$-ary rule: $\{p_1(v_1, v_2) \Rightarrow\ ;\ \Rightarrow p_2(c, v_1)\}/ \Rightarrow \mathcal{Q}v_1v_2(p_1(v_1,v_2), p_2(v_1,v_2))$. G_3 is (trivially) coherent. For a tuple $v = \langle a_1, ..., a_n\rangle$, denote by $(v)_i$ the i-th element of v. For every L-structure $S = \langle D, I\rangle$, $\mathcal{M}_{G_3}$ contains the operation $\tilde{\mathcal{Q}}_S$ defined as follows for every $g \in D^2 \to \{t,f\}^2$:

$$\tilde{\mathcal{Q}}_S[g] = \begin{cases} \{t\} & \textit{if there is some } a \in D, \textit{ s.t. for every } b, c \in D \\ & (g[b,c])_1 = f \textit{ and } (g[a,b])_2 = t \\ \{t,f\} & \textit{otherwise} \end{cases}$$

Next we show that for every canonical calculus G: (i) $\mathcal{M}_G$ is a characteristic 2Nmatrix for G, and (ii) G admits cut-elimination. For this we first prove the following proposition.

Proposition 24 *Let G be a coherent calculus. Let $\Gamma \Rightarrow \Delta$ be a sequent which satisfies the free-variable condition*[13]*. If $\Gamma \Rightarrow \Delta$ has no cut-free proof in G, then $\Gamma \not\vdash_{\mathcal{M}_G} \Delta$.*

Proof: Let $\Gamma \Rightarrow \Delta$ be a sequent which satisfies the free-variable condition. Suppose that $\Gamma \Rightarrow \Delta$ has no cut-free proof from in G. To show that $\Gamma \Rightarrow \Delta$ is not $\mathcal{M}_G$-valid, we will construct an L-structure S, an S-substitution σ^* and an $\mathcal{M}_G$-legal valuation v, such that $v[\sigma^*[\psi]] = t$ for every $\psi \in \Gamma$, while $v[\sigma^*[\varphi]] = f$ for every $\varphi \in \Delta$.
It is easy to see that we can limit ourselves to the language L^*, which is a subset of L, consisting of all the constants and predicate and function symbols, occurring in $\Gamma \Rightarrow \Delta$.
Let $\mathbf{T}$ be the set of all the terms in L^* which do not contain variables occurring bound in $\Gamma \Rightarrow \Delta$. It is a standard matter to show that Γ, Δ can be extended to two (possibly infinite) sets Γ', Δ' (where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$), satisfying the following properties:

1. For every finite $\Gamma_1 \subseteq \Gamma'$ and $\Delta_1 \subseteq \Delta'$, $\Gamma_1 \Rightarrow \Delta_1$ has no cut-free proof in G.
2. There are no $\psi \in \Gamma'$ and $\varphi \in \Delta'$, such that $\psi \equiv_\alpha \varphi$.
3. If $\{\Pi_j \Rightarrow \Sigma_j\}_{1 \le j \le m}/\mathcal{Q}(r)$ is an (n,k)-ary rule of G and $\mathcal{Q}z_1...z_k\ (A_1, ..., A_n) \in ite(r, \Delta', \Gamma')$, then there is some $1 \le j \le m$ satisfying the following condition. Let $\mathbf{t}_1, ..., \mathbf{t}_m$ be the L_k^n-terms occurring in $\Pi_j \cup \Sigma_j$, where $\mathbf{t}_{j_1}, ..., \mathbf{t}_{j_l}$ are constants and $\mathbf{t}_{j_{l+1}}, ..., \mathbf{t}_{j_m}$ are variables. Then for every $\mathbf{s}_1, ..., \mathbf{s}_l \in \mathbf{T}$ there are

[13] A sequent $\mathcal{S}$ satisfies the free-variable condition if the set of variables occurring free in $\mathcal{S}$ and the set of variables occurring bound in $\mathcal{S}$ are disjoint.

some[14] $\mathbf{s}_{l+1}, ..., \mathbf{s}_m \in \mathbf{T}$, such that whenever $p_i(\mathbf{t}_{n_1}, ..., \mathbf{t}_{n_k}) \in ite(r, \Pi_j, \Sigma_j)$ for some $1 \le n_1, ..., n_k \le m$: $A_i\{\mathbf{s}_{n_1}/z_1, ..., \mathbf{s}_{n_k}/z_k\} \in ite(r, \Gamma', \Delta')$.

Let $S = \langle D, I\rangle$ be the L^*-structure defined as follows: $D = \mathbf{T}$, $I[c] = c$ for every constant c of L^*; $I[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$ for every n-ary function symbol f; $I[p][\mathbf{t}_1, ..., \mathbf{t}_n] = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$ for every n-ary predicate symbol p. It is easy to show by induction on $\mathbf{t}$ that: $(*)$ For every $\mathbf{t} \in \mathbf{T}$: $I[\sigma^*[\mathbf{t}]] = \mathbf{t}$.
Let σ^* be any S-substitution satisfying $\sigma^*[x] = \overline{x}$ for every $x \in \mathbf{T}$. (Note that every $x \in \mathbf{T}$ is also a member of the domain and thus has an individual constant referring to it in $L^*(D)$).
For an $L(D)$-formula ψ (an $L(D)$-term $\mathbf{t}$), we will denote by $\widehat{\psi}$ $(\widehat{\mathbf{t}})$ the L-formula (L-term) obtained from ψ $(\mathbf{t})$ by replacing every individual constant of the form $\overline{\mathbf{s}}$ for some $\mathbf{s} \in \mathbf{T}$ by the term $\mathbf{s}$. Then the following property can be proved by an induction on ψ: $(**)$ For every $\psi \in \Gamma' \cup \Delta'$: $\widehat{\sigma^*[\psi]} = \psi$.
Define the S-valuation v as follows: (i) $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$, (ii) If there is some $C \in \Gamma' \cup \Delta'$, s.t. $C \equiv_\alpha \widehat{\mathcal{Q}\overrightarrow{z}(\psi_1, ..., \psi_n)}$, then $v[\mathcal{Q}\overrightarrow{z}(\psi_1, ..., \psi_n)] = t$ iff $C \in \Gamma'$. Otherwise $v[\mathcal{Q}\overrightarrow{z}(\psi_1, ..., \psi_n)] = t$ iff
$\tilde{\mathcal{Q}}_S[\lambda a_1 ... a_k \in D.\{\langle v[\psi_1\{\overrightarrow{a}/\overrightarrow{z}\}], ..., v[\psi_n\{\overrightarrow{a}/\overrightarrow{z}\}]\rangle\}] = \{t\}$.
It is not difficult to show that v is legal in $\mathcal{M}_G$.
Next we show that for every $\psi \in \Gamma' \cup \Delta'$: $v[\sigma^*[\psi]] = t$ iff $\psi \in \Gamma'$. If $\psi = p(\mathbf{t}_1, ..., \mathbf{t}_n)$, then $v[\sigma^*[\psi]] = I[p][I[\sigma^*[\mathbf{t}_1]], ..., I[\sigma^*[\mathbf{t}_n]]]$. Note[15] that for every $1 \le i \le n$, $\mathbf{t}_i \in \mathbf{T}$. By $(*)$, $I[\sigma^*[\mathbf{t}_i]] = \mathbf{t}_i$, and by the definition of I, $v[\sigma^*[\psi]] = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$. Otherwise $\psi = Q\overrightarrow{z}(\psi_1, ..., \psi_n)$. If $\psi \in \Gamma'$, then by $(**)$: $\widehat{\sigma^*[\psi]} = \psi \in \Gamma'$ and so $v[\sigma^*[\psi]] = t$. If $\psi \in \Delta'$ then by property 2 of $\Gamma' \cup \Delta'$ it cannot be the case that there is some $C \in \Gamma'$, such that $C \equiv_\alpha \widehat{\sigma^*[\psi]} = \psi$ and so $v[\sigma^*[\psi]] = f$.
We have constructed an L-structure S, an S-substitution σ^* and an $\mathcal{M}_G$-legal valuation v, such that $v[\sigma^*[\psi]] = t$ for every $\psi \in \Gamma'$, while $v[\sigma^*[\varphi]] = f$ for every $\varphi \in \Delta'$. Since $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, $\Gamma \Rightarrow \Delta$ is not $\mathcal{M}_G$-valid. □

Theorem 25 *Let G be a canonical calculus. Then the following statements concerning G are equivalent:*

1. *G is coherent.*
2. *There exists a 2GNmatrix $\mathcal{M}$, such that G is strongly sound and complete for $\mathcal{M}$.*

Proof: (1) ⇒ (2):
Suppose that G is coherent. By theorem 23, G is strongly sound for $\mathcal{M}_G$. For completeness, let $\Gamma \Rightarrow \Delta$ be a sequent which has no proof in G. If it does not satisfy the free-variable condition, obtain a sequent $\Gamma' \Rightarrow \Delta'$ which does satisfy this condition by renaming the bound variables. (Otherwise, set $\Gamma' = \Gamma$ and

[14] Note that in contrast to $\mathbf{t}_1, ..., \mathbf{t}_m$, $\mathbf{s}_1, ..., \mathbf{s}_m$ are L-terms and not L_k^n-terms.
[15] This is obvious if $\mathbf{t}_i$ does not occur in $\Gamma \Rightarrow \Delta$. If it occurs in $\Gamma \Rightarrow \Delta$, then since $\Gamma \Rightarrow \Delta$ satisfies the free-variable condition, $\mathbf{t}_i$ does not contain variables bound in this set and so $\mathbf{t}_i \in \mathbf{T}$ by definition of $\mathbf{T}$.

$\Delta' = \Delta$). Then also $\Gamma' \Rightarrow \Delta'$ has no proof in G (otherwise we could obtain a proof of $\Gamma \Rightarrow \Delta$ from a proof of $\Gamma' \Rightarrow \Delta'$ by using cuts on logical axioms). By proposition 24, $\Gamma' \not\vdash_{\mathcal{M}_G} \Delta'$. That is, there is an L-structure S, an S-substitution σ and an $\mathcal{M}_G$-legal valuation v, such that $v[\sigma[\psi]] = t$ for every $\psi \in \Gamma'$, while $v[\sigma[\varphi]] = f$ for every $\varphi \in \Delta'$. By lemma 12-1, v respects the $\equiv_\alpha$-relation, and so $v[\sigma[\psi]] = t$ for every $\psi \in \Gamma$, while $v[\sigma[\varphi]] = f$ for every $\varphi \in \Delta$. Hence, $\Gamma \not\vdash_{\mathcal{M}_G} \Delta$, and G is complete (and strongly sound) for $\mathcal{M}_G$.
(2) $\Rightarrow$ (1):
Suppose that G is strongly sound and complete for some 2GNmatrix $\mathcal{M}$. Assume by contradiction that G is not coherent. Then there exist two dual (n, k)-ary rules $R_1 = \Theta_1 / \Rightarrow A$ and $R_2 = \Theta_2 / A \Rightarrow$ in G, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent. Recall that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \Theta_1 \cup \Theta_2'$, where Θ_2' is obtained from Θ_2 by renaming constants and variables that occur also in Θ_1 (see defn. 7). For simplicity[16] we assume that the fresh constants used for renaming are all in L. Since $\Theta_1 \cup \Theta_2'$ is classically consistent, there exists an L_k^n-structure $\mathcal{N} = \langle D, I \rangle$, in which both Θ_1 and Θ_2' are valid. Recall that we also assume that L_k^n is a subset of L[17] and so $\dfrac{\Theta_1}{\Rightarrow A}$ and $\dfrac{\Theta_2'}{A \Rightarrow}$ are applications of R_1 and R_2 respectively. Let S be any extension of $\mathcal{N}$ to L and v - any $\mathcal{M}$-legal S-valuation. It is easy to see that Θ_1 and Θ_2' are $\mathcal{M}$-valid in $\langle S, v \rangle$ (since they only contain atomic formulas). Since G is strongly sound for $\mathcal{M}$, both $\Rightarrow A$ and $A \Rightarrow$ should also be $\mathcal{M}$-valid in $\langle S, v \rangle$, which is of course impossible. □

Corollary 26 *The existence of a strongly characteristic 2GNmatrix for a canonical calculus G is decidable.*

Proof: By Theorem 25, the question whether G has a strongly characteristic 2Nmatrix is equivalent to the question whether G is coherent, and this, by Proposition 9, is decidable.

Corollary 27 *If G is a coherent canonical calculus then it admits cut-elimination.*

As was shown in [16], the opposite does not hold: a canonical calculus which is not coherent can still admit cut-elimination.
Remark: The above results are related to the results in [9], where a general class of sequent calculi with (n, k)-ary quantifiers, called *standard* calculi is defined. Standard calculi may include any set of structural rules, and so canonical calculi are a particular instance of standard calculi which include all of the standard structural rules. [9] formulate syntactic sufficient and (under some limitations) necessary conditions for modular cut-elimination, a particular version of cut-elimination with non-logical axioms consisting only of atomic formulas. The reductivity condition of [9] can be shown to be equivalent to our coherence criterion in the context of canonical systems. Thus from the results of [9] it

[16] This assumption is not necessary and is used only for simplification of presentation, since we can instantiate the constants by any L-terms.
[17] This assumption is again not essential for the proof, but it simplifies the presentation.

follows that coherence is a necessary condition for modular cut-elimination in canonical calculi.

5 Summary and further research

In this paper we have extended the results of [16] for canonical systems with (n, k)-ary quantifiers from the case of $k \in \{0, 1\}$ to the general case of $k \geq 0$ (while preserving the decidability of coherence). We have demonstrated that the framework of Nmatrices is not sufficient to provide semantics for canonical systems for the case of $k > 1$, and generalized the framework of Nmatrices by introducing more general interpretations of quantifiers. Then we have shown that a canonical calculus G is coherent iff there is a 2GNmatrix $\mathcal{M}$ for which G is strongly sound and complete. Furthermore, any coherent calculus admits cut-elimination. However, the opposite direction does not hold: we have seen that coherence is not a necessary condition for (standard) cut-elimination in canonical calculi. From the results of [9] it follows that coherence *is* a necessary condition for modular cut-elimination. Whether it is possible to extend these results to more general forms of cut-elimination, is a question for further research.
Other research directions include extending the results of this paper to more general systems, such as the standard calculi of [9], which use non-standard sets of structural rules, and treating more complex quantifier extensions, such as Henkin quantifiers. Although the syntactic formulation of canonical systems given in this paper is not expressible enough to deal with Henkin quantifiers, the proposed semantic framework of GNmatrices provides an adequate interpretation of such quantifiers. This might be a promising starting point for a proof-theoretical investigation of canonical systems with Henkin quantifiers.
Yet another research direction is gaining an insight into the connection between non-determinism and axiom expansion in canonical systems. In [5] it is shown (on the propositional level) that any many-sided calculus which satisfies: (i) a condition similar to coherence and (ii) axiom expansion[18] (i.e axioms can be reduced to atomic axioms), has a deterministic characteristic matrix. We conjecture that there is a direct connection between axiom expansion in a coherent canonical system, and the degree of non-determinism in its characteristic 2Nmatrix.

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[18] This is also closely connected to the criterion of rigidity of [8] for propositional simple calculi.

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