Effective Finite-valued Semantics for Labelled Calculi

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Abstract. We provide a systematic and modular method to define non-deterministic finite-valued semantics for a natural and very general family of *canonical labelled calculi*, of which many previously studied sequent and labelled calculi are particular instances. This semantics is *effective*, in the sense that it naturally leads to a decision procedure for these calculi. It is then applied to provide simple *decidable* semantic criteria for crucial syntactic properties of these calculi, namely (strong) analyticity and cut-admissibility.

1 Introduction

There are two contrary aims in logic: the first is to find calculi that characterize a given semantics, the second is to find semantics for a logic that is only given as a formal calculus. Roughly speaking, the former aim has been reached for all (ordinary) finite-valued logics (including, of course, classical logic), as well as for non-deterministic finite-valued logics ([2, 3]). As for the latter, there is no known systematic method of constructing for a given general calculus, a corresponding "well-behaved" semantics. By "well-behaved" here we mean that it is *effective* in the sense of naturally inducing a decision procedure for its underlying logic. Moreover, it is desirable that such semantics can be applied to provide simple semantic characterization of important syntactic properties of the corresponding calculi, which are hard to establish by other means. Analyticity and cut-admissibility are just a few cases in point.

In [6] and [4] two families of *labelled sequent calculi* have been studied in this context.[3] [6] considers labelled calculi with generalized forms of cuts and axioms

⋆ Supported by The Israel Science Foundation (grant no. 280-10) and by FWF START Y544-N23.

[3] A remark is in order here on the relationship between the labelled calculi studied here and Gabbay's general framework of labelled deductive systems (LDS) from [9]. Both frameworks consider consequence relations between labelled formulas. Methodologically, however, they have different aims: [9] constructs a system for a given logic defined in semantic terms, while we define a a semantics for a given labelled system. Moreover, in LDS *anything* is allowed to serve as labels, while we assume a finite set of labels. In this sense, our labelled calculi are a particular instance of LDS.

and a restricted form of logical rules, and provides some necessary and sufficient conditions for such calculi to have a characteristic finite-valued matrix (see, e.g., [**?**]). In [4] labelled calculi with a less restrictive form of logical rules (but a more restrictive form of cuts and axioms) are considered. The calculi of [4], satisfying a certain coherence condition, have a semantic characterization using a natural generalization of the usual finite-valued matrix called *non-deterministic matrices* ([2, 3]). The semantics provided in [6, 4] for these families of labelled calculi is well-behaved in the above sense, that is the question of whether a sequent s follows in some (non-deterministic) matrix from a finite set of sequents $\mathcal{S}$, can be reduced to considering legal *partial* valuations, defined on the subformulas of $\mathcal{S} \cup \{s\}$. This naturally induces a decision procedure for such logics.

In this paper we show that the class of labelled calculi that have a finite-valued well-behaved semantics is substantially larger than all the families of calculi considered in the literature in this context. We start by defining a very general class of *canonical labelled calculi*, of which the sequent calculi of [2] and the labelled calculi of [6, 4] are particular examples. In addition to the weakening rule, canonical labelled calculi have rules of two forms: primitive rules and introduction rules. The former operate on labels and do not mention any connectives, where the generalized cuts and axioms of [6] are specific instances of such rules. As for the latter, each such rule introduces one logical connective of the language. To provide semantics for these calculi in a systematic and modular way, we generalize the notion of non-deterministic matrices to *partial non-deterministic matrices* (PNmatrices), in which empty sets of options are allowed in the truth-tables of logical connectives. Although applicable to a much wider range of calculi, the semantic framework of finite PNmatrices shares the following attractive property with both usual and non-deterministic matrices: any calculus that has a characteristic PNmatrix is decidable. Moreover, as opposed to the results in [6, 4], *no* conditions are required for a canonical labelled calculi to have a characteristic PNmatrix: *all* such calculi have one, and so *all* of them are decidable. We then apply PNmatrices to provide simple *decidable* characterizations of the crucial syntactic properties of strong analyticity and strong cut-admissibility in canonical labelled calculi.

Due to lack of space, the proofs of all the main propositions appear in appendices A and B.

2 Preliminaries

In what follows $\mathcal{L}$ is a propositional language, and $\pounds$ is a finite non-empty set of labels. We assume that $p_1, p_2, \ldots$ are the atomic formulas of $\mathcal{L}$. We denote by $Frm_{\mathcal{L}}$ the set of all wffs of $\mathcal{L}$. We usually use φ, ψ as metavariables for formulas, Γ, Δ for finite sets of formulas, l for labels, and L for sets of labels.

Definition 1. *A* labelled formula *is an expression of the form $l : \psi$, where $l \in \pounds$ and $\psi \in Frm_{\mathcal{L}}$. A labelled formula $l : \psi$ is* atomic *if ψ is an atomic formula. A* sequent *is a finite set of labelled formulas. An n-*clause *is a sequent consisting of atomic formulas from $\{p_1, \ldots, p_n\}$.*

Notation: Given a labelled formula γ, we denote by $\mathit{frm}[\gamma]$ the (ordinary) formula appearing in γ, and by $\mathit{sub}[\gamma]$ the set of subformulas of the formula $\mathit{frm}[\gamma]$. *frm* and *sub* are extended to sets of labelled formulas and to sets of sets of labelled formulas in the obvious way.

Remark 1. The usual (two-sided) sequent notation $\psi_1, \ldots, \psi_n \Rightarrow \varphi_1, \ldots, \varphi_m$ can be interpreted as $\{f : \psi_1, \ldots, f : \psi_n, t : \varphi_1, \ldots, t : \varphi_m\}$, i.e. a sequent in the sense of Definition 1 for $\pounds = \{t, f\}$.

Notation: Given a set $L \subseteq \pounds$, we write $(L : \psi)$ instead of (the sequent) $\{l : \psi \mid l \in L\}$.

Definition 2. *An $\mathcal{L}$-*substitution *is a function $\sigma : \mathit{Frm}_{\mathcal{L}} \to \mathit{Frm}_{\mathcal{L}}$, which satisfies $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$. A substitution is extended to labelled formulas, sequents, etc. in the obvious way.*

3 Canonical Labelled Systems

In this section we define the family of *canonical labelled systems*. This is a general family of labelled systems, which includes many natural subclasses of previously studied calculi. These include the system **LK** for classical logic, the canonical sequent calculi of [2], the signed calculi of [4] and the labelled calculi of [6].[4]

All canonical labelled systems have in common the *weakening* rule. In addition, they include rules of two types: *primitive rules* and *introduction rules*. Each rule of the latter type introduces exactly one logical connective, while rules of the former type operate on labels and do no mention any logical connectives. Next we provide precise definitions.

Definition 3 (Weakening). *The* weakening *rule allows to infer $s \cup s'$ from s for every two sequents s and s'.*

Definition 4 (Primitive Rules). *A* primitive rule for $\pounds$ *is an expression of the form $\{L_1, \ldots, L_n\}/L$ where $n \geq 0$ and $L_1, \ldots, L_n, L \subseteq \pounds$. An* application of a primitive rule *$\{L_1, \ldots, L_n\}/L$ is is any inference step of the following form:*

$$\frac{(L_1 : \psi) \cup s_1 \quad \ldots \quad (L_n : \psi) \cup s_n}{(L : \psi) \cup s_1 \cup \ldots \cup s_n}$$

where ψ is a formula, and s_i is a sequent for every $1 \leq i \leq n$.

[4] The family of canonical labelled systems also includes the systems dealt with in [10]. [10] extends the results of [2] by considering also "semi-canonical systems", which are obtained from (two-sided) canonical systems by discarding either the cut rule, the identity axioms or both of them. Clearly, these these systems are particular instances of canonical labelled systems, as defined in this paper.

Example 1. Suppose $\pounds = \{a, b, c\}$. consider the primitive rule $\{\{a\}, \{b\}\}/\{b, c\}$. This rule allows to infer $(\{b, c\} : \psi) \cup s_1 \cup s_2$ from $\{a : \psi\} \cup s_1$ and $\{b : \psi\} \cup s_2$ for every two sequents s_1, s_2 and a formula ψ.

Definition 5. *A primitive rule for $\pounds$ of the form $\emptyset/L$ is called a* canonical axiom*. Its applications provide all axioms of the form $(L : \psi)$.*

Example 2. Axioms schemas of two-sided sequent calculi usually have the form $\psi \Rightarrow \psi$. Using the notation from Remark 1, it can be formulated as the canonical axiom $\emptyset/\{t, f\}$.

Definition 6. *A primitive rule for $\pounds$ of the form $\{L_1, \ldots, L_n\}/\emptyset$ is called a* canonical cut*. Its applications allow to infer $s_1 \cup \ldots \cup s_n$ from the sequents $(L_i : \psi) \cup s_i$ for every $1 \leq i \leq n$ (the formula ψ is called the* cut-formula*).*

Example 3. Applications of the cut rule for two-sided sequent calculi are usually presented by the following schema:

$$\frac{\Gamma_1 \Rightarrow \psi, \Delta \quad \Gamma_2, \psi \Rightarrow \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Using the notation from Remark 1, the corresponding canonical cut has the form $\{\{t\}, \{f\}\}/\emptyset$.

Definition 7 (Introduction Rules). *A* canonical introduction rule *for an n-ary connective $\diamond$ of $\mathcal{L}$ and $\pounds$ is an expression of the form $\mathcal{S}/L : \diamond(p_1, \ldots, p_n)$, where $\mathcal{S}$ is a finite set of n-clauses (Definition 1) (called* premises*), and L is a non-empty subset of $\pounds$. An* application *of a canonical introduction rule $\{c_1, \ldots, c_m\}/L : \diamond(p_1, \ldots, p_n)$ is any inference step of the following form:*

$$\frac{\sigma(c_1) \cup s_1 \quad \ldots \quad \sigma(c_m) \cup s_m}{(L : \sigma(\diamond(p_1, \ldots, p_n))) \cup s_1 \cup \ldots \cup s_m}$$

where σ is an $\mathcal{L}$-substitution, and s_i is a sequent for every $1 \leq i \leq m$.

Example 4. The introduction rules for the classical conjunction in **LK** are usually presented as follows:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2 \Rightarrow \Delta_2, \varphi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \psi \wedge \varphi}$$

Using the notation from Remark 1, the canonical representation of the schemas above is:

$$r_1 = \{\{f : p_1, f : p_2\}\}/\{f\} : p_1 \wedge p_2 \qquad r_2 = \{\{t : p_1\}, \{t : p_2\}\}/\{t\} : p_1 \wedge p_2$$

Their applications have the forms:

$$\frac{\{f : \psi, f : \varphi\} \cup s}{\{f : \psi \wedge \varphi\} \cup s} \qquad \frac{\{t : \psi\} \cup s_1 \quad \{t : \varphi\} \cup s_2}{\{t : \psi \wedge \varphi\} \cup s_1 \cup s_2}$$

Definition 8 (Canonical Labelled Systems). *A* canonical labelled system $\mathbf{G}$ *for* $\mathcal{L}$ *and* $\pounds$ *includes the weakening rule, a finite set of primitive rules for* $\pounds$*, and a finite set of introduction rules for connectives of* $\mathcal{L}$ *and* $\pounds$*. We say that a sequent* s follows *in a canonical labelled system* $\mathbf{G}$ *from a set of sequents* $\mathcal{S}$ *(and denote it by* $\mathcal{S} \vdash_{\mathbf{G}} s$*) if there exists a derivation in* $\mathbf{G}$ *of* s *from* $\mathcal{S}$*.*

Notation: Given a canonical labelled system $\mathbf{G}$ for $\mathcal{L}$ and $\pounds$, we denote by $\mathsf{P}_{\mathbf{G}}$ the set of primitive rules of $\mathbf{G}$. In addition, for every connective $\diamond$ of $\mathcal{L}$, we denote by $\mathsf{R}^{\diamond}_{\mathbf{G}}$ the canonical introduction rules for $\diamond$ of $\mathbf{G}$.

Example 5. The standard sequent system **LK** can be represented as a canonical labelled system for the language of classical logic and $\{t, f\}$ (see Examples 2 to 4).

Notation: To improve readability, we usually omit the parentheses from the sets of premises of primitive rules and canonical introduction rules.

Example 6. For $\pounds = \{a, b, c\}$, the canonical labelled system $\mathbf{G}_{abc}$ includes the primitive rules $\emptyset/\{a, b\}, \emptyset/\{b, c\}, \emptyset/\{a, c\}$, and $\{a, b, c\}/\emptyset$. It also has the following canonical introduction rules for a ternary connective $\circ$:

$$\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}/\{a, c\} : \circ(p_1, p_2, p_3)$$

$$\{c : p_2\}, \{a : p_3, b : p_3\}, \{c : p_1\}/\{b, c\} : \circ(p_1, p_2, p_3)$$

Their applications are of the forms:

$$\frac{\{a : \psi_1, c : \psi_2\} \cup s_1 \quad \{a : \psi_3, b : \psi_2\} \cup s_2}{(\{a, c\} : \circ(\psi_1, \psi_2, \psi_3)) \cup s_1 \cup s_2}$$

$$\frac{\{c : \psi_2\} \cup s_1 \quad \{a : \psi_3, b : \psi_3\} \cup s_2 \quad \{c : \psi_1\} \cup s_3}{(\{b, c\} : \circ(\psi_1, \psi_2, \psi_3)) \cup s_1 \cup s_2 \cup s_3}$$

Note that the canonical labelled calculi studied here are substantially more general than the signed calculi of [4] and the labelled calculi of [6], as the primitive rules of both of these calculi include only canonical cuts and axioms. Moreover, in the latter only introduction rules which introduce a singleton are allowed, which is not the case for the calculus in Example 6. In the former, all systems have $\emptyset/\pounds$ as their only axiom, and the set of cuts is always assumed to be $\{\{l_1\}, \{l_2\}/\emptyset \mid l_1 \neq l_2\}$ (again leaving the calculus in Example 6 out of scope).

4 Partial Non-deterministic Matrices

Non-deterministic matrices (Nmatrices) are a natural generalization of the notion of a standard many-valued matrix (see [2]). These are structures, in which the truth-value of a complex formula is chosen non-deterministically out of a *non-empty* set of options (which is determined on the truth-values of its subformulas). In this paper we introduce a further generalization of the concept of an

Nmatrix, in which this set of options is allowed to be *empty*. Intuitively, allowing empty sets of options in a truth-table corresponds to forbidding some combinations of truth-values. As we shall see, this will allow us to characterize a wider class of calculi than that obtained by applying usual Nmatrices. However, as we show in the sequel, the property of *effectiveness* is preserved in PNmatrices, and like finite-valued matrices and Nmatrices, (calculi characterized by) finite-valued PNmatrices are decidable.

4.1 Introducing PNmatrices

Definition 9. *A* partial non-deterministic matrix *(*PNmatrix *for short)* $\mathcal{M}$ *for* $\mathcal{L}$ *and* $\pounds$ *consists of: (i) set* $\mathcal{V}_{\mathcal{M}}$ *of truth-values, (ii) a function* $\mathcal{D}_{\mathcal{M}} : \pounds \to P(\mathcal{V}_{\mathcal{M}})$ *assigning a set of (designated) truth-values to the labels of* $\pounds$*, and (iii) a function* $\diamond_{\mathcal{M}} : {\mathcal{V}_{\mathcal{M}}}^n \to P(\mathcal{V}_{\mathcal{M}})$ *for every* n*-ary connective* $\diamond$ *of* $\mathcal{L}$*. We say that* $\mathcal{M}$ *is* finite *if so is* $\mathcal{V}_{\mathcal{M}}$*.*

Definition 10. *Let* $\mathcal{M}$ *be a PNmatrix for* $\mathcal{L}$ *and* $\pounds$*.*

1. *An* $\mathcal{M}$*-legal* $\mathcal{L}$*-valuation is a function* $v : Frm_{\mathcal{L}} \to \mathcal{V}_{\mathcal{M}}$ *satisfying the condition* $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}}(v(\psi_1), \ldots, v(\psi_n))$ *for every compound formula* $\diamond(\psi_1, \ldots, \psi_n) \in Frm_{\mathcal{L}}$*.*
2. *Let* v *be an* $\mathcal{M}$*-legal* $\mathcal{L}$*-valuation. A sequent* s *is* true *in* v *for* $\mathcal{M}$ *(denoted by* $v \models_{\mathcal{M}} s$*) if* $v(\psi) \in \mathcal{D}_{\mathcal{M}}(l)$ *for some* $l : \psi \in s$*. A set* $\mathcal{S}$ *of sequents is* true *in* v *for* $\mathcal{M}$ *(denoted by* $v \models_{\mathcal{M}} \mathcal{S}$*) if* $v \models_{\mathcal{M}} s$ *for every* $s \in \mathcal{S}$*.*
3. *Given a set* $\mathcal{S} \cup \{s\}$ *of sequents,* $\mathcal{S} \vdash_{\mathcal{M}} s$ *if for every* $\mathcal{M}$*-legal* $\mathcal{L}$*-valuation* v*,* $v \models_{\mathcal{M}} s$ *whenever* $v \models_{\mathcal{M}} \mathcal{S}$*.*

We now define a special subclass of PNmatrices, in which no empty sets of truth-values are allowed in the truth-tables of logical connectives. This corresponds to the case of ordinary Nmatrices [2–4].

Definition 11. *We say that a PNmatrix* $\mathcal{M}$ *for* $\mathcal{L}$ *and* $\pounds$ *is a* proper *if* $\mathcal{V}_{\mathcal{M}}$ *is non-empty and* $\diamond_{\mathcal{M}}(x_1, \ldots, x_n)$ *is non-empty for every* n*-ary connective* $\diamond$ *of* $\mathcal{L}$ *and* $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{M}}$*.*

Remark 2. Nmatrices in their original formulation can be viewed as proper PNmatrices for $\mathcal{L}$ and $\pounds$, where $\pounds$ is a singleton. In this case $\mathcal{D}_{\mathcal{M}}$ is practically a set of designated truth-values. This is useful to define consequence relations between sets of formulas and formulas in the following way: $T \vdash_{\mathcal{M}} \psi$ if whenever the formulas of T are "true in v for $\mathcal{M}$" (that is $v(\varphi) \in \mathcal{D}$ for every $\varphi \in T$), also ψ is "true in v for $\mathcal{M}$" ($v(\psi) \in \mathcal{D}$). However, in this paper we study consequence relations of a different type, namely relations between a set of labelled sequents and a labelled sequent. We need, therefore, a notion of "being true for $\mathcal{M}$" for every $l \in \pounds$. This is achieved by taking $\mathcal{D}_{\mathcal{M}}$ to be a function from $\pounds$ to $P(\mathcal{V}_{\mathcal{M}})$. Finally, note that for simplicity of presentation, unlike in previous works, we allow the set of designated truth-values (for every $l \in \pounds$) to be empty or to include all truth-values in $\mathcal{V}_{\mathcal{M}}$.

Example 7. Let $\pounds = \{a, b\}$ and suppose that $\mathcal{L}$ contains one unary connective $\star$. The PNmatrices $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined as follows: $\mathcal{V}_{\mathcal{M}_1} = \mathcal{V}_{\mathcal{M}_2} = \{t, f\}$, $\mathcal{D}_{\mathcal{M}_1}(a) = \mathcal{D}_{\mathcal{M}_2}(a) = \{t\}$ and $\mathcal{D}_{\mathcal{M}_1}(b) = \mathcal{D}_{\mathcal{M}_2}(b) = \{f\}$. The respective truth-tables for $\star$ are defined as follows:

x	$\star_{\mathcal{M}_1}(x)$
t	$\{f\}$
f	$\{t, f\}$

x	$\star_{\mathcal{M}_2}(x)$
t	$\emptyset$
f	$\{t, f\}$

While both $\mathcal{M}_1$ and $\mathcal{M}_2$ are PNmatrices, only $\mathcal{M}_1$ is proper. Note that in this case we have $\{a : p_1\} \vdash_{\mathcal{M}_2} \emptyset$, simply because there is no $\mathcal{M}_2$-legal $\mathcal{L}$-valuation that assigns t to p_1.

Finally, we extend the notion of *simple refinements* of Nmatrices ([3]) to the context of PNmatrices:

Definition 12. *Let $\mathcal{M}$ and $\mathcal{N}$ be PNmatrices for $\mathcal{L}$ and $\pounds$. We say that $\mathcal{N}$ is a* simple refinement *of $\mathcal{M}$, denoted by $\mathcal{N} \subseteq \mathcal{M}$, if $\mathcal{V}_{\mathcal{N}} \subseteq \mathcal{V}_{\mathcal{M}}$, $\mathcal{D}_{\mathcal{N}}(l) = \mathcal{D}_{\mathcal{M}}(l) \cap \mathcal{V}_{\mathcal{N}}$ for every $l \in \pounds$, and $\diamond_{\mathcal{N}}(x_1, \ldots, x_n) \subseteq \diamond_{\mathcal{M}}(x_1, \ldots, x_n)$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{N}}$.*

Proposition 1. *Let $\mathcal{M}$ and $\mathcal{N}$ be PNmatrices for $\mathcal{L}$ and $\pounds$, such that $\mathcal{N} \subseteq \mathcal{M}$. Then:* (1) *Every $\mathcal{N}$-legal $\mathcal{L}$-valuation is also $\mathcal{M}$-legal; and* (2) $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathcal{N}}$.

4.2 Decidability

For a denotational semantics to be useful, it should be *effective*: the question of whether some conclusion follows from a finite set of assumptions, should be decidable by considering some *computable* set of partial valuations defined on some *finite* set of "relevant" formulas. Usually, the "relevant" formulas are taken as all subformulas occurring in the conclusion and the assumptions. Next, we show that the semantics induced by PNmatrices is effective in this sense.

Definition 13. *Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$, and let $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ closed under subformulas. An $\mathcal{M}$-legal $\mathcal{F}$-valuation is a function $v : \mathcal{F} \to \mathcal{V}_{\mathcal{M}}$ satisfying $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}}(v(\psi_1), \ldots, v(\psi_n))$ for every formula $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$. $\models_{\mathcal{M}}$ is defined for $\mathcal{F}$-valuations exactly as for $\mathcal{L}$-valuations. We say that an $\mathcal{M}$-legal $\mathcal{F}$-valuation is* extendable *in $\mathcal{M}$ if it can be extended to an $\mathcal{M}$-legal $\mathcal{L}$-valuation.*

In proper PNmatrices, all partial valuations are extendable:

Proposition 2. *Let $\mathcal{M}$ be a proper PNmatrix for $\mathcal{L}$ and $\pounds$, and let $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ closed under subformulas. Then any $\mathcal{M}$-legal $\mathcal{F}$-valuation is* extendable *in $\mathcal{M}$.*

Proof. The proof goes exactly like the one of Nmatrices in [1]. Note that the non-emptiness of $\mathcal{V}_{\mathcal{M}}$ is needed in order to extend the empty valuation. Clearly, the different definition of $\mathcal{D}_{\mathcal{M}}$ is immaterial here. □

However, this is not the case for arbitrary PNmatrices:

Example 8. Consider the PNmatrix $\mathcal{M}_2$ from Example 7. Let v be the $\mathcal{M}_2$-legal $\{p_1\}$-valuation defined by $v(p_1) = t$. Obviously, there is no $\mathcal{M}_2$-legal $\mathcal{L}$-valuation that extends v (as there is no way to assign a truth-value to $\star p_1$). Thus v is not extendable in $\mathcal{M}_2$.

Theorem 1. *Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and £ and $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ closed under subformulas. An $\mathcal{M}$-legal $\mathcal{F}$-valuation v is extendable in $\mathcal{M}$ iff v is $\mathcal{N}$-legal for some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$.*

Corollary 1. *Given a finite PNmatrix $\mathcal{M}$ for $\mathcal{L}$ and £, a finite $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ closed under subformulas. and a function $v : \mathcal{F} \to \mathcal{V}_{\mathcal{M}}$, it is decidable whether v is an $\mathcal{M}$-legal $\mathcal{F}$-valuation which is extendable in $\mathcal{M}$.*

Proof. Checking whether v is $\mathcal{M}$-legal is straightforward. To verify that it is extendable in $\mathcal{M}$, we go over all (finite) proper PNmatrices $\mathcal{N}$, such that $\mathcal{N} \subseteq \mathcal{M}$ (there is a finite number of them since $\mathcal{M}$ is finite) and check whether v is $\mathcal{N}$-legal for some such $\mathcal{N}$. We return a positive answer iff we have found some $\mathcal{N} \subseteq \mathcal{M}$ such that v is $\mathcal{N}$-legal. The correctness is guaranteed by Theorem 1. □

Corollary 2. *Given a finite PNmatrix $\mathcal{M}$ for $\mathcal{L}$ and £, a finite set $\mathcal{S}$ of sequents, and a sequent s, it is decidable whether $\mathcal{S} \vdash_{\mathcal{M}} s$ or not.*

In the literature of Nmatrices (see e.g. [1]) effectiveness is usually identified with the property given in Proposition 2.[5] In this case Corollary 1 trivially holds: to check that v is an extendable $\mathcal{M}$-legal $\mathcal{F}$-valuation, it suffices to check that it is $\mathcal{M}$-legal, as extendability is a priori guaranteed. However, the results above show that this property is not a necessary condition for decidability. To guarantee the latter, instead of requiring that *all* partial valuations are extendable, it is sufficient to have an algorithm that establishes which of them are.

4.3 Minimality

In the next section, we show that the framework of PNmatrices provides a semantic way of characterizing canonical labelled systems. A natural question in this context is how one can obtain *minimal* such characterizations. Next we provide lower bounds on the number of truth-values that are needed to characterize $\vdash_{\mathcal{M}}$ of some PNmatrix $\mathcal{M}$ satisfying a separability condition defined below. Moreover, we provide a method to extract from a given (separable) PNmatrix an equivalent PNmatrix with the *minimal* number of truth-values.

Definition 14. *Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and £.*

1. *A truth-value $x \in \mathcal{V}_{\mathcal{M}}$ is called* useful *in $\mathcal{M}$ if $x \in \mathcal{V}_{\mathcal{N}}$ for some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$.*

[5] This property is sometimes called (semantic) *analyticity*. Note that in this paper the term 'analyticity' refers to a *proof-theoretic* property (see Definition 20).

2. *The PNmatrix* $R[\mathcal{M}]$ *is the simple refinement of* $\mathcal{M}$*, defined as follows.* $\mathcal{V}_{R[\mathcal{M}]}$ *consists of all truth-values in* $\mathcal{V}_{\mathcal{M}}$ *which are useful in* $\mathcal{M}$*;* $\mathcal{D}_{R[\mathcal{M}]}(l) = \mathcal{D}_{\mathcal{M}}(l) \cap \mathcal{V}_{R[\mathcal{M}]}$ *for every* $l \in \pounds$*; and* $\diamond_{R[\mathcal{M}]}(x_1, \ldots, x_n) = \diamond_{\mathcal{M}}(x_1, \ldots, x_n) \cap \mathcal{V}_{R[\mathcal{M}]}$ *for every* n*-ary connective* $\diamond$ *of* $\mathcal{L}$ *and* $x_1, \ldots, x_n \in \mathcal{V}_{R[\mathcal{M}]}$.

Proposition 3. *Let* $\mathcal{M}$ *be a PNmatrix for* $\mathcal{L}$ *and* $\pounds$*, and let* v *be an* $\mathcal{M}$*-legal* $\mathcal{L}$*-valuation. Then:* (1) *For every formula* ψ*,* $v(\psi)$ *is useful in* $\mathcal{M}$*; and* (2) *Every* $\mathcal{M}$*-legal* $\mathcal{L}$*-valuation is also* $R[\mathcal{M}]$*-legal.*

Corollary 3. $\vdash_{\mathcal{M}} = \vdash_{R[\mathcal{M}]}$ *for every PNmatrix* $\mathcal{M}$*.*

Proof. One direction follows from Proposition 1, simply because $R[\mathcal{M}]$ is a simple refinement of $\mathcal{M}$ by definition. The converse is easily established using Proposition 3. We leave the details to the reader. □

Definition 15. *Let* $\mathcal{M}$ *be a PNmatrix for* $\mathcal{L}$ *and* $\pounds$*. We say that two truth-values* $x_1, x_2 \in \mathcal{V}_{\mathcal{M}}$ *are* separable *in* $\mathcal{M}$ *for* $l \in \pounds$ *if* $x_1 \in \mathcal{D}_{\mathcal{M}}(l) \Leftrightarrow x_2 \notin \mathcal{D}_{\mathcal{M}}(l)$ *holds.* $\mathcal{M}$ *is called* separable *if every pair of truth-values in* $\mathcal{V}_{\mathcal{M}}$ *are separable in* $\mathcal{M}$ *for some* $l \in \pounds$*.*

We are now ready to obtain a *lower bound* on the number of truth-values needed to characterize $\vdash_{\mathcal{M}}$ for a given separable PNmatrix $\mathcal{M}$:

Theorem 2. *Let* $\mathcal{M}$ *be a separable PNmatrix for* $\mathcal{L}$ *and* $\pounds$*. If* $\vdash_{\mathcal{M}} = \vdash_{\mathcal{N}}$ *for some PNmatrix* $\mathcal{N}$ *for* $\mathcal{L}$ *and* $\pounds$*, then* $\mathcal{N}$ *contains at least* $|\mathcal{V}_{R[\mathcal{M}]}|$ *truth-values.*

Remark 3. As done for usual matrices, it is also possible to define $\vdash_F$, the consequence relation induced by *a family of proper PNmatrices* to be $\bigcap_{\mathcal{M} \in F} \vdash_{\mathcal{M}}$. A PNmatrix can then be thought of as a succinct presentation of a family of proper PNmatrices in the following sense. The consequence relation induced by a PNmatrix $\mathcal{N}$ can be shown to be equivalent to the relation induced by the family of all the proper PNmatrices $\mathcal{M}$, such that $\mathcal{M} \subseteq \mathcal{N}$. Conversely, for every family of proper PNmatrices it is easy to construct an equivalent PNmatrix.

5 Finite PNmatrices for Canonical Labelled Systems

Definition 16. *We say that a PNmatrix* $\mathcal{M}$ *(for* $\mathcal{L}$ *and* $\pounds$*) is* characteristic *for a canonical labelled system* $\mathbf{G}$ *(for* $\mathcal{L}$ *and* $\pounds$*) if* $\vdash_{\mathcal{M}} = \vdash_{\mathbf{G}}$*.*

Next we provide a systematic way to obtain a characteristic PNmatrix $\mathcal{M}_{\mathbf{G}}$ for every canonical labelled system $\mathbf{G}$. The intuitive idea is as follows: the primitive rules of $\mathbf{G}$ determine the set of the truth-values of $\mathcal{M}_{\mathbf{G}}$, while the introduction rules for the logical connectives dictate their corresponding truth-tables. Like in the case of usual Nmatrices, the semantics based on PNmatrices is thus *modular*: each such rule corresponds to a certain semantic condition, and the semantics of a system is obtained by joining the semantic effects of each of its derivation rules.

Definition 17. *Let $r = \{L_1, \ldots, L_n\}/L_0$ be a primitive rule for $\pounds$. Define:*

$$r^* = \{L \subseteq \pounds \mid L_i \cap L = \emptyset \text{ for some } 1 \leq i \leq n \text{ or } L_0 \cap L \neq \emptyset\}$$

Example 9. For a canonical axiom $r = \emptyset/L_0$, $r^* = \{L \subseteq \pounds \mid L_0 \cap L \neq \emptyset\}$. For a canonical cut $r = \{L_1, \ldots, L_n\}/\emptyset$, $r^* = \{L \subseteq \pounds \mid L_i \cap L = \emptyset \text{ for some } 1 \leq i \leq n\}$. In particular, continuing Examples 2 and 3 (for $\pounds = \{t, f\}$), we have $r^* = \{\{t\}, \{f\}, \{t, f\}\}$ for the classical axiom, and $r^* = \{\emptyset, \{t\}, \{f\}\}$ for the classical cut.

Definition 18. *Let $\diamond$ be an n-ary connective, and let $r = \mathcal{S}/L_0 : \diamond(p_1, \ldots, p_n)$ be a canonical introduction rule for $\diamond$ and $\pounds$. For every $L_1, \ldots, L_n \subseteq \pounds$, define:*

$$r^*[L_1, \ldots, L_n] = \begin{cases} \{L \subseteq \pounds \mid L_0 \cap L \neq \emptyset\} & \forall s \in \mathcal{S}((L_1 : p_1) \cup \ldots \cup (L_n : p_n)) \cap s \neq \emptyset \\ P(\pounds) & \text{otherwise} \end{cases}$$

Example 10. Let $\pounds = \{t, f\}$. Recall the usual introduction rules for conjunction from Example 4. By Definition 18:

$$r_1^*[L_1, L_2] = \begin{cases} \{\{f\}, \{t, f\}\} & f \in L_1 \text{ or } f \in L_2 \\ P(\{t, f\}) & \text{otherwise} \end{cases} \qquad r_2^*[L_1, L_2] = \begin{cases} \{\{t\}, \{t, f\}\} & t \in L_1 \cap L_2 \\ P(\{t, f\}) & \text{otherwise} \end{cases}$$

Definition 19 (The PNmatrix $\mathcal{M}_{\mathbf{G}}$). *Let $\mathbf{G}$ be a canonical labelled system for $\mathcal{L}$ and $\pounds$. The PNmatrix $\mathcal{M}_{\mathbf{G}}$ (for $\mathcal{L}$ and $\pounds$) is defined by:*

1. $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}} = \{L \subseteq \pounds \mid L \in r^* \text{ for every } r \in \mathsf{P}_{\mathbf{G}}\}$.
2. *For every $l \in \pounds$,* $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l) = \{L \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}} \mid l \in L\}$.
3. *For every n-ary connective $\diamond$ of $\mathcal{L}$ and $L_1, \ldots, L_n \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$:*

$$\diamond_{\mathcal{M}_{\mathbf{G}}}(L_1, \ldots, L_n) = \{L \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}} \mid L \in r^*[L_1, \ldots, L_n] \text{ for every } r \in \mathsf{R}^{\diamond}_{\mathbf{G}}\}$$

Example 11. Let $\pounds = \{t, f\}$ and consider the calculus $\mathbf{G}_\wedge$ whose primitive rules include only the classical axiom, and the classical cut (see Examples 2 and 3), and whose only introduction rules are the two usual rules for conjunction (see Example 4). By Example 9 and the construction above, $\mathcal{V}_{\mathcal{M}_{\mathbf{G}_\wedge}} = \{\{t\}, \{f\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_\wedge}}(t) = \{t\}$, and $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_\wedge}}(f) = \{f\}$. Using Example 10, we obtain the following interpretation of $\wedge$:

$\wedge_{\mathcal{M}_{\mathbf{G}_\wedge}}$	$\{t\}$	$\{f\}$
$\{t\}$	$\{t\}$	$\{f\}$
$\{f\}$	$\{f\}$	$\{f\}$

We note that the calculus $\mathbf{G}_\wedge$ defined above is an instance of signed canonical calculi of [4], as well as of labelled calculi of [6], and the semantics obtained for it here coincides with the semantics given in [4, 6].

Example 12. Assume that $\mathcal{L}$ contains only a unary connective $\star$, and $\pounds = \{a, b, c\}$. Let us start with the calculus $\mathbf{G}_0$, the primitive rules of which include the canonical axiom $\emptyset/\{a, b, c\}$ and the canonical cuts $\{a\}, \{c\}/\emptyset$ and $\{a\}, \{b\}/\emptyset$, while $\mathbf{G}_0$ has no introduction rules. Here we have $\mathcal{V}_{\mathcal{M}_{\mathbf{G}_0}} = \{\{a\}, \{b\}, \{c\}, \{b, c\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(a) = \{\{a\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(b) = \{\{b\}, \{b, c\}\}$ and $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(c) = \{\{b, c\}\}$. $\star_{\mathcal{M}_{\mathbf{G}_0}}$ is given in the table below (it is completely non-deterministic). One can now obtain a calculus $\mathbf{G}_1$ by adding the rule $\{a : p_1\}/\{b, c\} : \star p_1$. This leads to a refinement of the truth-table, described below. Finally, one can obtain the calculus $\mathbf{G}_2$ by adding $\{b : p_1\}/\{a\} : \star p_1$, resulting in another refinement of truth-table, also described below.

x	$\star_{\mathcal{M}_{\mathbf{G}_0}}(x)$	$\star_{\mathcal{M}_{\mathbf{G}_1}}(x)$	$\star_{\mathcal{M}_{\mathbf{G}_2}}(x)$
$\{a\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{b\},\{b,c\}\}$	$\{\{b\},\{b,c\}\}$
$\{b\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\}\}$
$\{c\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$
$\{b,c\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\}\}$

Theorem 3 (Soundness and completeness). *For every canonical labelled system* $\mathbf{G}$*,* $\mathcal{M}_{\mathbf{G}}$ *is a* characteristic PNmatrix *for* $\mathbf{G}$*.*

Proof. Immediately follows from Corollary 9 in Appendix A. □

Corollary 4 (Decidability). *Given a canonical labelled system* $\mathbf{G}$*, a finite set* $\mathcal{S}$ *of sequents, and a sequent* s*, it is decidable whether* $\mathcal{S} \vdash_{\mathbf{G}} s$ *or not.*

Corollary 5. *The question whether a given canonical labelled system* $\mathbf{G}$ *is consistent (i.e.* $\not\vdash_{\mathbf{G}} \emptyset$*) is decidable.*

$\mathcal{M}_{\mathbf{G}}$ provides a semantic characterization for $\mathbf{G}$, however it may not be a minimal one (in terms of the number of truth-values). For a minimal semantic representation, we should consider the equivalent PNmatrix $R[\mathcal{M}_{\mathbf{G}}]$:

Corollary 6 (Minimality). *For every canonical labelled system* $\mathbf{G}$*,* $R[\mathcal{M}_{\mathbf{G}}]$ *is a minimal (in terms of number of truth-values) characteristic PNmatrix for* $\mathbf{G}$*.*

Proof. The claim follows by Theorem 2 from the fact that $\mathcal{M}_{\mathbf{G}}$ is separable for every system $\mathbf{G}$. □

6 Proof-Theoretic Applications

In this section we apply the semantic framework of PNmatrices to provide *decidable* semantic criteria for syntactic properties of canonical labelled calculi that are usually hard to generally characterize by other means. Namely, we focus on the notions of *analyticity* and *cut-admissibility*, extended to the context of reasoning with assumptions.

6.1 Strong Analyticity

Strong analyticity is a crucial property of a useful (propositional) calculus, as it implies its consistency and decidability. Intuitively, a calculus is *strongly analytic* if whenever a sequent s is provable in it from a set of assumptions $\mathcal{S}$, then s can be proven using only the formulas available within $\mathcal{S}$ and s.

Definition 20. *A canonical labelled system* $\mathbf{G}$ *is* strongly analytic *if whenever* $\mathcal{S} \vdash_{\mathbf{G}} s$*, there exists a derivation in* $\mathbf{G}$ *of* s *from* $\mathcal{S}$ *consisting solely of (sequents consisting of) formulas from* $sub[\mathcal{S} \cup \{s\}]$*.*

Below we provide a *decidable* semantic characterization of strong analyticity of canonical labelled calculi:

Theorem 4 (Characterization of Strong Analyticity). *Let* $\mathbf{G}$ *be a canonical labelled system for* $\mathcal{L}$ *and* $\pounds$. *Suppose that* $\mathbf{G}$ *does not include the (trivial) primitive rule* $\emptyset/\emptyset$. *Then,* $\mathbf{G}$ *is strongly analytic iff* $\mathcal{M}_{\mathbf{G}}$ *is proper.*

Corollary 7. *The question whether a given canonical labelled system is strongly analytic is decidable.*

6.2 Strong Cut-Admissibility

As the property of strong analyticity is sometimes difficult to establish, it is traditional in proof theory to investigate the property of *cut-admissibility*, which means that whenever s is provable in $\mathbf{G}$, it has a cut-free derivation in $\mathbf{G}$. In this paper we investigate a stronger notion of this property, defined as follows for labelled calculi:

Definition 21. *A labelled system* $\mathbf{G}$ *enjoys* strong cut-admissibility *if whenever* $\mathcal{S} \vdash_{\mathbf{G}} s$, *there exists a derivation in* $\mathbf{G}$ *of* s *from* $\mathcal{S}$ *in which only formulas from* $frm[\mathcal{S}]$ *serve as cut-formulas.*

Due to the special form of primitive and introduction rules of canonical calculi (which, except for canonical cuts, enjoy the subformula property), the above property guarantees strong analyticity:

Proposition 4. *Let* $\mathbf{G}$ *be a canonical labelled system. If* $\mathbf{G}$ *enjoys strong cut-admissibility, then* $\mathbf{G}$ *is strongly analytic.*

Although for two-sided canonical sequent calculi the notions of strong analyticity and strong cut-admissibility coincide (see [3]), this is not the case for general labelled calculi, for which the converse of Proposition 4 does not necessarily hold, as shown by the following example:

Example 13. Assume that $\mathcal{L}$ contains only a unary connective $\star$, and $\pounds = \{a, b, c\}$. Let $\mathbf{G}$ be the canonical labelled system $\mathbf{G}$ for $\mathcal{L}$ and $\pounds$, the primitive rules of which include only the canonical cuts $\{a\}, \{b\}/\emptyset$, $\{a\}, \{c\}/\emptyset$, and $\{b\}, \{c\}/\emptyset$, and its only introduction rules are $\{a : p_1\}/\{a, b\} : \star p_1$ and $\{a : p_1\}/\{b, c\} : \star p_1$. To see that this system is strongly analytic, by Theorem 4, it suffices to construct $\mathcal{M}_{\mathbf{G}}$ and check that it is proper. The construction proceeds as follows: $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}} = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l) = \{l\}$ for $l \in \{a, b, c\}$, and the truth-table for $\star$ is the following:

x	$\star_{\mathcal{M}_{\mathbf{G}}}(x)$
$\emptyset$	$\{\emptyset, \{a\}, \{b\}, \{c\}\}$
$\{a\}$	$\{\{b\}\}$
$\{b\}$	$\{\emptyset, \{a\}, \{b\}, \{c\}\}$
$\{c\}$	$\{\emptyset, \{a\}, \{b\}, \{c\}\}$

This is a proper PNmatrix, and so $\mathbf{G}$ is strongly analytic. We also note that it impossible to derive the sequent $\{b : \star p_1\}$ from the singleton set $\{\{a : p_1\}\}$

using only p_1 as a cut-formula. However, by applying the two introduction rules of **G** and then using the cut $\{a\}, \{c\}/\emptyset$ (with $\star p_1$ as cut-formula), we can derive $\{b : \star p_1\}$ from $\{\{a : p_1\}\}$. Thus although this system is strongly analytic, it does not enjoy strong cut-admissibility.

The intuitive explanation is that non-eliminable applications of canonical cuts (like the one in the above example) are not harmful for strong analyticity because they enjoy the subformula property. Thus, the equivalence between strong analyticity and cut-admissibility can be restored if we enforce the following condition:

Definition 22. *A canonical labelled system* **G** *for* $\mathcal{L}$ *and* $\pounds$ *is* cut-saturated *if for every canonical cut* $\{L_1, \dots, L_n\}/\emptyset$ *of* **G** *and* $l \in \pounds$, **G** *contains the primitive rule* $\{L_1, \dots, L_n\}/\{l\}$.

Proposition 5. *For every canonical labelled system* **G**, *there is an equivalent cut-saturated canonical labelled system* $\mathbf{G}'$.

Example 14. Revisiting the system from Example 13, we observe that **G** is not cut-saturated. To obtain a cut-saturated equivalent system $\mathbf{G}'$, we add the following primitive rules to **G**: $r_1 = \{a\}, \{b\}/\{c\}$, $r_2 = \{a\}, \{c\}/\{b\}$, and $r_3 = \{b\}, \{c\}/\{a\}$. Note that the addition of these rules does not affect the set of truth-values, i.e., $\mathcal{V}_{\mathcal{M}_\mathbf{G}} = \mathcal{V}_{\mathcal{M}_{\mathbf{G}'}}$. However, we can now derive $\{b : \star p_1\}$ from $\{\{a : p_1\}\}$ without any cuts by by the two introduction rules and the new rule r_2. Moreover, by Corollary 8 below, $\mathbf{G}'$ *does* enjoy strong cut-admissibility.

We are now ready to provide a decidable semantic characterization of strong cut-admissibility.

Theorem 5. *Let* **G** *be a cut-saturated canonical labelled system for* $\mathcal{L}$ *and* $\pounds$. *Suppose that* **G** *does not include the (trivial) primitive rule* $\emptyset/\emptyset$. *Then,* **G** *enjoys strong cut-admissibility iff* $\mathcal{M}_\mathbf{G}$ *is proper.*

Corollary 8. *Let* **G** *be a cut-saturated canonical labelled system for* $\mathcal{L}$ *and* $\pounds$. *Suppose that* **G** *does not include the (trivial) primitive rule* $\emptyset/\emptyset$. *Then the following statements concerning* **G** *are equivalent: (i)* $\mathcal{M}_\mathbf{G}$ *is proper, (ii)* **G** *is strongly analytic, and (iii)* **G** *enjoys strong cut-admissibility.*

7 Conclusions and Further Research

Establishing proof-theoretical properties of syntactic calculi is in many cases a complex and error-prone task. For instance, proving that a calculus admits cut-elimination is often carried out using heavy syntactic arguments and many case-distinctions, leaving room for mistakes and omissions. This leads to the need of *automatizing* the process of reasoning about calculi[6]. However, a faithful formalization is an elusive goal, as such important properties as cut-admissibility,

[6] In [8], e.g., proofs of cut-admissibility, contraction-admissibility and Graig's interpolation theorem are formalized for some particular sequent calculi using the proof assistant Isabelle.

analyticity and decidability, as well as the dependencies between them are little understood for the general case. We believe that the *abstract* view on labelled calculi taken in this paper is a substantial step towards finding the right level of abstraction for reasoning about these properties. Moreover, the simple and decidable semantic characterizations of these properties for canonical labelled calculi are a key to their faithful axiomatization in this context. To provide these characterizations, we have introduced *PNmatrices*, a generalization of Nmatrices, in which empty entries in logical truth-tables are allowed, while still preserving the *effectiveness* of the semantics. A characteristic PNmatrix $\mathcal{M}_{\mathbf{G}}$ has been constructed for every canonical labelled calculus, which in turn implies its decidability. If in addition $\mathcal{M}_{\mathbf{G}}$ has no empty entries (i.e, is proper) — which is *decidable*, **G** is strongly analytic. For cut-saturated canonical calculi, the latter is also equivalent to strong cut-admissibility.

The results of this paper extend the theory of canonical sequent calculi of [2], as well as of the labelled calculi of [6] and signed calculi of [4], all of which are particular instances of canonical labelled calculi defined here. Moreover, the semantics obtained for these families of calculi in the above mentioned papers, coincide with the PNmatrices semantics obtained for them here. It is particularly interesting to note that [6] provides a list of conditions, under which a labelled calculus has a characteristic finite-valued logic. These conditions include (i) reducibility of cuts (which can be shown to be equivalent to the criterion of coherence of [4]), which entails that $\mathcal{M}_{\mathbf{G}}$ is proper, and (ii) eliminability of compound axioms,[7] which in its turn entails that $\mathcal{M}_{\mathbf{G}}$ is completely deterministic (in other words, it can be identified with an ordinary finite-valued matrix). We conclude that, as shown in this paper, none of the conditions required in any of the mentioned papers [2, 6, 4] from a "well-behaved" calculus are necessary when moving to the more general semantic framework of PNmatrices, where *any* canonical labelled calculus has an effective finite-valued semantics.

An immediate direction for further research is investigating the applications of the theory of canonical labelled calculi developed here. One possibility is exploiting this theory for sequent calculi, whose rules are more complex than the canonical ones, but which can be reformulated in terms of canonical *labelled* calculi. This applies, e.g., to the large family of sequent calculi for paraconsistent logics given in [5]. The rules of these calculi have a particular *uniform form*: (i) each of them introduces exactly one formula in its conclusion; (ii) the formula which is introduced is either of the form $\diamond(\psi_1, \ldots, \psi_n)$ or $\neg \diamond (\psi_1, \ldots, \psi_n)$; (iii) the principal formulas in the premises of the rule are all taken from the set $\{\psi_1, \ldots, \psi_n, \neg\psi_1, \ldots, \neg\psi_n\}$; and (iv) there are no restrictions on the side formulas of the application. For an example, consider the (two-sided) Gentzen-type system $\mathbf{G_K}$ of [5] over the language $\mathcal{L}_C = \{\wedge, \vee, \supset, \circ, \neg\}$, obtained from **LK** by discarding the left rule for negation and adding the following schemas for the unary connective $\circ$:

$$\frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma, \circ\psi \Rightarrow \Delta}\ (\circ \Rightarrow) \qquad \frac{\Gamma, \psi, \neg\psi \Rightarrow \Delta}{\Gamma \Rightarrow \circ\psi, \Delta}\ (\Rightarrow \circ)$$

Clearly, these schemas cannot be formulated as canonical rules in the sense of [2] (since they use $\neg\psi$ as a principal formula). However, we can reformu-

[7] This property intuitively means that compound axioms can be reduced to atomic ones. It is called 'axiom-expansion' in [7].

late $\mathbf{G_K}$ in terms of canonical labelled calculi by using the set of labels $\mathcal{L}_4 = \{t^+, t^-, f^+, f^-\}$, where t and f denote the side on which the formula occurs, and $+$ and $-$ determine whether its occurrence is positive or negative (i.e. preceded with negation). Now each (two-sided) rule of $\mathbf{G_K}$ can be translated into a labelled *canonical* rule over $\mathcal{L}_4$. For instance, $(\circ \Rightarrow)$ and $(\Rightarrow \circ)$ above are translated into $\{t^+ : p_1\}, \{t^- : p_1\}/\{f^+\} : \circ p_1$ and $\{f^+ : p_1, f^- : p_1\}/\{t^+\} : \circ p_1$ respectively. Adding further rules, it can be shown that for each (non-canonical) two-sided calculus $\mathbf{G}$ from [5], an equivalent[8] labelled *canonical* calculus $\mathbf{G}'$ can be constructed, which automatically implies the decidability of the calculi from [5]. Characterizing the non-canonical sequent calculi which have useful translations into labelled canonical calculi is a question for further research. Another direction is generalizing the results of this paper to more complex classes of labelled calculi, e.g., like those defined in [11] for inquisitive logic. Extending the results to the first-order case is another future goal.

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[8] Here we mean that $\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathbf{G}'} \mathsf{M}_4(\Gamma \Rightarrow \Delta)$, where M_4 is a mapping from two-sided sequents to labelled sequents over $\mathcal{L}_4$ defined as follows: $\mathsf{M}_4(\Gamma \Rightarrow \Delta) = \{f^+ : \psi \mid \psi \in \Gamma^+\} \cup \{f^- : \psi \mid \psi \in \Gamma^-\} \cup \{t^+ : \psi \mid \psi \in \Delta^+\} \cup \{t^- : \psi \mid \psi \in \Delta^-\}$, where for each set of $\mathcal{L}_C$-formulas T, T^+ denotes the positive formulas of T, while T^- – the negative ones.

A Soundness and Completeness Proofs

A.1 Soundness

Definition 23. *Let* $\mathbf{G}$ *be a canonical labelled system for* $\mathcal{L}$ *and* $\pounds$. *For every set* $\mathcal{S} \cup \{s\}$ *of sequents and set* $\mathcal{F}$ *of formulas, we write* $\mathcal{S} \vdash^{\mathcal{F}}_{\mathbf{G}} s$ *iff there if there exists a derivation in* $\mathbf{G}$ *of* s *from* $\mathcal{S}$ *consisting only of (sequents consisting of) formulas from* $\mathcal{F}$.

Theorem 6. *Let* $\mathbf{G}$ *be a canonical labelled system for* $\mathcal{L}$ *and* $\pounds$. *Let* $\mathcal{S} \cup \{s\}$ *be a set of sequents, and let* $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ *such that* $sub[\mathcal{S} \cup \{s\}] \subseteq \mathcal{F}$. *Suppose that* $\mathcal{S} \vdash^{\mathcal{F}}_{\mathbf{G}} s$. *Then,* $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$ *implies* $v \models_{\mathcal{M}_{\mathbf{G}}} s$ *for every* $\mathcal{M}_{\mathbf{G}}$*-legal* $\mathcal{F}$*-valuation* v.

Proof. Assume there exists a derivation P in $\mathbf{G}$ of a sequent s from a set $\mathcal{S}$ of sequents consisting only of (sequents consisting of) formulas from $\mathcal{F}$. Let v be an $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{F}$-valuation. Suppose that $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$. Using induction on the length of P, we show that $v \models_{\mathcal{M}_{\mathbf{G}}} s$ for every sequent s occurring in P. This trivially holds for the sequents of $\mathcal{S}$. We show that this property is also preserved by applications of the rules of $\mathbf{G}$. This obviously holds for the weakening rule. We show it holds for primitive rules and for canonical rules as well:

- Suppose $(L : \psi) \cup s_1 \cup \ldots \cup s_n$ is derived from the sequents $(L_1 : \psi) \cup s_1, \ldots, (L_n : \psi) \cup s_n$ using the primitive rule $r = \{L_1, \ldots, L_n\}/L$. Assume that $v \models_{\mathcal{M}_{\mathbf{G}}} (L_i : \psi) \cup s_i$ for every $1 \leq i \leq n$. We show that $v \models_{\mathcal{M}_{\mathbf{G}}} (L : \psi) \cup s_1 \cup \ldots \cup s_n$. By definition it suffices to show that there exists some $l : \varphi \in (L : \psi) \cup s_1 \cup \ldots \cup s_n$ such that $v(\varphi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. If there exists some $l : \varphi \in s_1 \cup \ldots \cup s_n$ such that $v(\varphi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$, then we are done. Assume otherwise. Then our assumption entails that $v \models_{\mathcal{M}_{\mathbf{G}}} (L_i : \psi)$ for every $1 \leq i \leq n$, and so for every $1 \leq i \leq n$, there exists some $l \in L_i$ such that $v(\psi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. The definition of $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}$ entails that for every $1 \leq i \leq n$, there exists some $l \in L_i$ such that $l \in v(\psi)$. In other words, for every $1 \leq i \leq n$, $L_i \cap v(\psi) \neq \emptyset$. Since v is $\mathcal{M}_{\mathbf{G}}$-legal, $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. In particular, $v(\psi) \in r^*$, and so $v(\psi) \cap L \neq \emptyset$. Hence there exists some $l_0 \in L$, such that $l_0 \in v(\psi)$. It follows that $v(\psi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l_0)$. Thus, in this case $v \models_{\mathcal{M}_{\mathbf{G}}} (L : \psi)$.
- Suppose $(L_0 : \sigma(\diamond(p_1, \ldots, p_n))) \cup s_1 \cup \ldots \cup s_m$ is derived from the sequents $\sigma(c_1) \cup s_1, \ldots, \sigma(c_m) \cup s_m$ using the canonical introduction rule $r = \{c_1, \ldots, c_m\}/L_0 : \diamond(p_1, \ldots, p_n)$. Assume that $v \models_{\mathcal{M}_{\mathbf{G}}} \sigma(c_i) \cup s_i$ for every $1 \leq i \leq m$. We show that $v \models_{\mathcal{M}_{\mathbf{G}}} (L_0 : \sigma(\diamond(p_1, \ldots, p_n))) \cup s_1 \cup \ldots \cup s_m$. By definition it suffices to show that there exists some $l : \varphi \in (L_0 : \sigma(\diamond(p_1, \ldots, p_n))) \cup s_1 \cup \ldots \cup s_m$ such that $v(\varphi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. If there exists some $l : \varphi \in s_1 \cup \ldots \cup s_m$ such that $v(\varphi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$, then we are done. Assume otherwise. Then our assumption entails that $v \models_{\mathcal{M}_{\mathbf{G}}} \sigma(c_i)$ for every $1 \leq i \leq m$. Thus for every $1 \leq i \leq m$ there exists some $l : p \in c_i$, such that $v(\sigma(p)) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. The definition of $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}$ entails that for every $1 \leq i \leq m$ there exists some $l : p \in c_i$, such that $l \in v(\sigma(p))$. Let $L_i = v(\sigma(p_i))$ for every $1 \leq i \leq n$. It follows that $((L_1 : p_1) \cup \ldots \cup (L_n : p_n)) \cap c_i \neq \emptyset$ for every

$1 \leq i \leq m$. Thus $r^*[L_1, \ldots, L_n] = \{L \subseteq \pounds \mid L \cap L_0 \neq \emptyset\}$. Since v is $\mathcal{M}_{\mathbf{G}}$-legal and $\sigma(\diamond(p_1, \ldots, p_n)) \in \mathcal{F}$, $v(\sigma(\diamond(p_1, \ldots, p_n))) \in r^*[v(\sigma(p_1), \ldots, v(\sigma(p_n))]$. Hence, $v(\sigma(\diamond(p_1, \ldots, p_n))) \cap L_0 \neq \emptyset$. Thus there exists some $l \in L_0$, such that $l \in v(\sigma(\diamond(p_1, \ldots, p_n)))$. It follows that $v(\sigma(\diamond(p_1, \ldots, p_n))) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. Thus, in this case $v \models_{\mathcal{M}_{\mathbf{G}}} (L_0 : \sigma(\diamond(p_1, \ldots, p_n)))$. □

A.2 Completeness

Definition 24. *Let* $\mathbf{G}$ *be a canonical labelled system for* $\mathcal{L}$ *and* $\pounds$*. For every set* $\mathcal{S} \cup \{s\}$ *of sequents and sets* $\mathcal{F}, \mathcal{C}$ *of formulas, we write* $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s$ *iff there if there exists a derivation* P *in* $\mathbf{G}$ *of* s *from* $\mathcal{S}$ *such that:*

1. *P consists only of (sequents consisting of) formulas from* $\mathcal{F}$.
2. *Only formulas from* $\mathcal{C}$ *serve as cut-formulas in* P.

Notation: We denote by $\mathbf{G_{cf}}$ the canonical labelled system obtained from a canonical labelled system $\mathbf{G}$ by discarding all the canonical cut rules of $\mathbf{G}$.

Theorem 7. *Let* $\mathbf{G}$ *be a canonical labelled system for* $\mathcal{L}$ *and* $\pounds$*. Let* $\mathcal{S} \cup \{s\}$ *be a set of sequents,* $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ *such that* $sub[\mathcal{S} \cup \{s\}] \subseteq \mathcal{F}$*, and* $\mathcal{C} \subseteq Frm_{\mathcal{L}}$*. Suppose that* $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$ *implies* $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$ *for every* $\mathcal{M}_{\mathbf{G_{cf}}}$*-legal* $\mathcal{F}$*-valuation* v *such that* $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ *for every* $\psi \in \mathcal{C} \cap \mathcal{F}$*. Then* $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s$*.*

Proof. Assume that $\mathcal{S} \nvdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s$. We construct an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation v such that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in \mathcal{C} \cap \mathcal{F}$, and $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$ but $v \not\models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$. Call a set Ω of labelled formulas *maximal* if it satisfies the following conditions:

1. $frm[\Omega] \subseteq \mathcal{F}$.
2. $\mathcal{S} \nvdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s$ for every sequent $s \subseteq \Omega$.
3. For every labelled formula $l : \psi \notin \Omega$ for $\psi \in \mathcal{F}$, there exists a sequent $s \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s \cup \{l : \psi\}$.

Let Ω be a maximal set extending s. An existence of such a set is ensured by the next lemma.

Lemma: Let s' be a set of labelled formulas, such that $frm[s'] \subseteq \mathcal{F}$. If $\mathcal{S} \nvdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s'$, then there exists a maximal set Ω such that $s' \subseteq \Omega$.
Proof: Let $\gamma_1, \gamma_2, \ldots$ be an enumeration of all labelled formulas, such that $frm[\gamma_i] \in \mathcal{F}$ and $\gamma_i \notin s'$ for every $i \geq 1$. We recursively define a sequence of sets of labelled formulas, $\{\Omega_k\}_{k=0}^{k=\infty}$. Let $\Omega_0 = s$. For $k \geq 1$, let $\Omega_k = \Omega_{k-1}$ iff there exists a sequent $s \subseteq \Omega_{k-1}$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s \cup \{\gamma_k\}$. Otherwise, let $\Omega_k = \Omega_{k-1} \cup \{\gamma_k\}$. Finally, let $\Omega = \bigcup_{k \geq 0} \Omega_k$. It is easy to verify that Ω has all required properties.

Next, let v be a function from $\mathcal{F}$ to $P(\pounds)$ defined by $v(\psi) = \{l \in \pounds \mid l : \psi \notin \Omega\}$ for every $\psi \in \mathcal{F}$. We claim that:

(A) For every sequent c, such that $frm[c] \subseteq \mathcal{F}$: there exists a sequent $s' \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} c \cup s'$ iff $(v(\psi) : \psi) \cap c \neq \emptyset$.
(B) v is an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation.
(C) $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in \mathcal{C} \cap \mathcal{F}$.
(D) $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$.
(E) $v \not\models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$.

Proof of (A): Let c be a sequent such that $frm[c] \subseteq \mathcal{F}$. Suppose that there exists a sequent $s' \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} c \cup s'$. The maximality of Ω entails that $c \not\subseteq \Omega$. Thus there exists a signed formula $l : \psi \in c$ such that $l : \psi \notin \Omega$. The construction of v entails that $l \in v(\psi)$, and so $(v(\psi) : \psi) \cap c \neq \emptyset$. For the converse, assume that $(v(\psi) : \psi) \cap c \neq \emptyset$. Hence there exists some $l \in v(\psi)$ such that $l : \psi \in c$. By definition, $l : \psi \notin \Omega$. The maximality of Ω entails that there exists a sequent $s' \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} s' \cup \{l : \psi\}$. Using weakening, we obtain $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} c \cup s'$.

Proof of (B): We first show that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}}$ for every $\psi \in \mathcal{F}$. Thus we prove that for every formula $\psi \in \mathcal{F}$, $v(\psi) \in r^*$ for every rule $r \in \mathsf{P}_{\mathbf{G_{cf}}}$. Let $\psi \in \mathcal{F}$, and let $r = \{L_1, \ldots, L_n\}/L$ be a primitive rule of $\mathbf{G_{cf}}$. To see that $v(\psi) \in r^*$, we show that if $L_i \cap v(\psi) \neq \emptyset$ for every $1 \leq i \leq n$, then $L \cap v(\psi) \neq \emptyset$. Suppose that $L_i \cap v(\psi) \neq \emptyset$ for every $1 \leq i \leq n$. (A) entails that for every $1 \leq i \leq n$, there exists some sequent $s_i \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} (L_i : \psi) \cup s_i$. Using the rule r (which is not a canonical cut), we obtain $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} (L : \psi) \cup s_1 \cup \ldots \cup s_n$. (A) again entails that $L \cap v(\psi) \neq \emptyset$.

Next, we show that $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}_{\mathbf{G_{cf}}}}(v(\psi_1), \ldots, v(\psi_n))$ for every formula $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$. Thus we prove that for every formula $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$, $v(\diamond(\psi_1, \ldots, \psi_n)) \in r^*[v(\psi_1), \ldots, v(\psi_n)]$ for every rule $r \in \mathsf{R}^{\diamond}_{\mathbf{G}}$. Let $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$, and let $r = \mathcal{S}/L : \diamond(p_1, \ldots, p_n)$ be a rule in $\mathsf{R}^{\diamond}_{\mathbf{G}}$. To see that $v(\diamond(\psi_1, \ldots, \psi_n)) \in r^*[v(\psi_1), \ldots, v(\psi_n)]$, we show that if $((v(\psi_1) : p_1) \cup \ldots \cup (v(\psi_n) : p_n)) \cap c \neq \emptyset$ for every $c \in \mathcal{S}$, then $v(\diamond(\psi_1, \ldots, \psi_n)) \cap L \neq \emptyset$. Suppose that $((v(\psi_1) : p_1) \cup \ldots \cup (v(\psi_n) : p_n)) \cap c \neq \emptyset$ for every $c \in \mathcal{S}$. Let σ be an $\mathcal{L}$-substitution assigning ψ_i to p_i for every $1 \leq i \leq n$. Thus $((v(\psi_1) : \psi_1) \cup \ldots \cup (v(\psi_n) : \psi_n)) \cap \sigma(c) \neq \emptyset$ for every $c \in \mathcal{S}$. Hence for every $c \in \mathcal{S}$, there exists some $1 \leq i \leq n$, such that $(v(\psi_i) : \psi_i) \cap \sigma(c) \neq \emptyset$. (A) entails that for every $c \in \mathcal{S}$, there exists some sequent $s_c \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} \sigma(c) \cup s_c$. By applying r, we obtain $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} (L : \diamond(\psi_1, \ldots, \psi_n)) \cup \bigcup_{c \in \mathcal{S}} s_c$. Since $\bigcup_{c \in \mathcal{S}} s_c \subseteq \Omega$, (A) entails that $v(\diamond(\psi_1, \ldots, \psi_n)) \cap L \neq \emptyset$.

Proof of (C): Let $\psi \in \mathcal{C} \cap \mathcal{F}$. We show that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. Following (B), it suffices to show that $v(\psi) \in r^*$ for every canonical cut r of $\mathbf{G}$. Let $r = \{L_1, \ldots, L_n\}/\emptyset$ be a canonical cut of $\mathbf{G}$. To have $v(\psi) \in r^*$, it suffices to prove that $L_i \cap v(\psi) = \emptyset$ for some $1 \leq i \leq n$. Suppose otherwise. (A) entails that for every $1 \leq i \leq n$, there exists some sequent $s_i \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} (L_i : \psi) \cup s_i$. Using the canonical cut r (with the cut-formula $\psi \in \mathcal{C}$), we obtain $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F},\mathcal{C} \rangle} s_1 \cup \ldots \cup s_n$. This contradicts the fact that $s_1 \cup \ldots \cup s_n \subseteq \Omega$.

Proof of (D): Let $s' \in \mathcal{S}$. Clearly, $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s'$. By (A), there exists some $l \in v(\psi)$ such that $l : \psi \in s'$. Since $l \in v(\psi)$, we have $v(\psi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G_{cf}}}}(l)$. Hence, $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s'$. Consequently, $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$.

Proof of (E): Since $s \subseteq \Omega$, $l \notin v(\psi)$ for every $l : \psi \in s$. It follows that for every $l : \psi \in s$, $v(\psi) \notin \mathcal{D}_{\mathcal{M}_{\mathbf{G_{cf}}}}(l)$. Therefore, $v \not\models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$.

Finally, properties (B)-(E) show that v is an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation with all required properties. □

A.3 Corollaries

Proposition 6. *Let* $\mathbf{G}$ *be a canonical labelled system for* $\mathcal{L}$ *and* $\pounds$*. Suppose that* $\mathbf{G}'$ *is obtained from* $\mathbf{G}$ *by omitting some primitive rules. Let* $\mathcal{F}$ *be a set of formulas closed under subformulas. Then, every* $\mathcal{M}_{\mathbf{G}'}$*-legal* $\mathcal{F}$*-valuation* v *such that* $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ *for every* $\psi \in \mathcal{F}$*, is also an* $\mathcal{M}_{\mathbf{G}}$*-legal* $\mathcal{F}$*-valuation. In addition, for every sequent* s *such that* $frm[s] \subseteq \mathcal{F}$*,* $v \models_{\mathcal{M}_{\mathbf{G}}} s$ *iff* $v \models_{\mathcal{M}_{\mathbf{G}'}} s$*.*

Proof. Let v be an $\mathcal{M}_{\mathbf{G}'}$-legal $\mathcal{F}$-valuation. We first note that $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}} \subseteq \mathcal{V}_{\mathcal{M}_{\mathbf{G}'}}$ (this follows by definition from the fact that $\mathsf{P}_{\mathbf{G}'} \subseteq \mathsf{P}_{\mathbf{G}}$). Now let $\diamond$ be an n-ary connective and $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$. Since v is $\mathcal{M}_{\mathbf{G}'}$-legal, $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}_{\mathbf{G}'}}(v(\psi_1), \ldots, v(\psi_n))$. We show that $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}_{\mathbf{G}}}(v(\psi_1), \ldots, v(\psi_n))$. Let $r \in \mathsf{R}^{\diamond}_{\mathbf{G}}$. Since $\mathsf{R}^{\diamond}_{\mathbf{G}'}$, $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}_{\mathbf{G}'}}(v(\psi_1), \ldots, v(\psi_n))$ implies that $v(\diamond(\psi_1, \ldots, \psi_n)) \in r^*[v(\psi_1), \ldots, v(\psi_n)]$. Since $v(\diamond(\psi_1, \ldots, \psi_n)) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$, we obtain $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}_{\mathbf{G}}}(v(\psi_1), \ldots, v(\psi_n))$. It follows that v is $\mathcal{M}_{\mathbf{G}}$-legal. The second part is left to the reader. □

Corollary 9. *For every canonical labelled system* $\mathbf{G}$*,* $\vdash_{\mathcal{M}_{\mathbf{G}}} = \vdash_{\mathbf{G}}$*.*

Proof. One direction follows directly from Theorem 6 (by choosing $\mathcal{F} = Frm_{\mathcal{L}}$). For the converse, suppose that $\mathcal{S} \vdash_{\mathcal{M}_{\mathbf{G}}} s$. We prove that $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$ implies $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$ for every $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{L}$-valuation v such that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in Frm_{\mathcal{L}}$. Theorem 7 then implies that $\mathcal{S} \vdash_{\mathbf{G}} s$ (choose $\mathcal{F} = \mathcal{C} = Frm_{\mathcal{L}}$). Let v be an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{L}$-valuation such that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in Frm_{\mathcal{L}}$, and $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$. Since $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in Frm_{\mathcal{L}}$, Proposition 6 entails that v is also an $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{L}$-valuation. Similarly, $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$. Our assumption entails that $v \models_{\mathcal{M}_{\mathbf{G}}} s$. This implies that $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$. By Theorem 7, $\mathcal{S} \vdash_{\mathbf{G}} s$. □

Corollary 10. *Let* $\mathbf{G}$ *be a canonical labelled system for* $\mathcal{L}$ *and* $\pounds$*, and let* $\mathcal{S} \cup \{s\}$ *be a set of sequents. There exists a derivation of* s *from* $\mathcal{S}$ *in* $\mathbf{G}$ *in consisting solely of (sequents consisting of) formulas from* $sub[\mathcal{S} \cup \{s\}]$ *iff* $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$ *implies* $v \models_{\mathcal{M}_{\mathbf{G}}} s$ *for every* $\mathcal{M}_{\mathbf{G}}$*-legal* $sub[\mathcal{S} \cup \{s\}]$*-valuation* v*.*

Proof. One direction follows directly from Theorem 6 (by choosing $\mathcal{F} = sub[\mathcal{S} \cup \{s\}]$). For the converse, suppose that for every $\mathcal{M}_{\mathbf{G}}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation v, $v \models_{\mathcal{M}_{\mathbf{G}}} s$ whenever $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$. We prove that $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$ implies $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$ for every $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation v such that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$

for every $\psi \in sub[\mathcal{S} \cup \{s\}]$. Theorem 7 then implies that there exists a derivation of s from $\mathcal{S}$ in $\mathbf{G}$ consisting solely of (sequents consisting of) formulas from $sub[\mathcal{S} \cup \{s\}]$ (choose $\mathcal{F} = \mathcal{C} = sub[\mathcal{S} \cup \{s\}]$). Let v be an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation such that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in sub[\mathcal{S} \cup \{s\}]$, and $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$. Since $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in sub[\mathcal{S} \cup \{s\}]$, Proposition 6 entails that v is also an $\mathcal{M}_{\mathbf{G}}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation. Similarly, $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$. Our assumption entails that $v \models_{\mathcal{M}_{\mathbf{G}}} s$. This implies that $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$. The claim then follows from Theorem 7. □

B Proofs of Selected Propositions

Proof of Proposition 1:
(1) is easy and left to the reader. For (2), let $\mathcal{S} \vdash_{\mathcal{M}} s$. Let v be an $\mathcal{N}$-legal $\mathcal{L}$-valuation, such that $v \models_{\mathcal{N}} \mathcal{S}$. For every $s' \in \mathcal{S}$, $v(\psi) \in \mathcal{D}_{\mathcal{N}}(l)$ for some $l : \psi \in s'$. Since $\mathcal{D}_{\mathcal{N}}(l) \subseteq \mathcal{D}_{\mathcal{M}}(l)$ for every $l \in \pounds$, we have that for every $s' \in \mathcal{S}$, $v(\psi) \in \mathcal{D}_{\mathcal{M}}(l)$ for some $l : \psi \in s'$. Thus $v \models_{\mathcal{M}} \mathcal{S}$, and since v is $\mathcal{M}$-legal (using (1)) $v \models_{\mathcal{M}} s$. Then $v(\psi) \in \mathcal{D}_{\mathcal{M}}(l)$ for some $l : \psi \in s$. Since $\mathcal{D}_{\mathcal{N}} = \mathcal{D}_{\mathcal{M}} \cap \mathcal{V}_{\mathcal{N}}$ and $v(\psi) \in \mathcal{V}_{\mathcal{N}}$ (since v is $\mathcal{N}$-legal), $v(\psi) \in \mathcal{D}_{\mathcal{N}}(l)$ and so $v \models_{\mathcal{N}} s$. □

Proof of Theorem 1:

($\Rightarrow$) Let v' be an $\mathcal{M}$-legal $\mathcal{L}$-valuation that extends v. Define the PNmatrix $\mathcal{N}$ as follows: $\mathcal{V}_{\mathcal{N}} = \mathsf{Image}(v')$; for every $l \in \pounds$, $\mathcal{D}_{\mathcal{N}}(l) = \mathcal{D}_{\mathcal{M}}(l) \cap \mathcal{V}_{\mathcal{N}}$; and for every n-ary connective $\diamond$ of $\mathcal{L}$, and $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{N}}$, $\diamond_{\mathcal{N}}(x_1, \ldots, x_n)$ is $\{v'(\diamond(\psi_1, \ldots, \psi_n)) \mid \psi_1, \ldots, \psi_n \in \mathit{Frm}_{\mathcal{L}}$ and $v'(\psi_i) = x_i$ for every $1 \leq i \leq n\}$. Note that $\diamond_{\mathcal{N}}(x_1, \ldots, x_n) \subseteq \mathcal{V}_{\mathcal{N}}$ for every n-ary connective $\diamond$ of $\mathcal{L}$, and $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{N}}$. First we verify that $\mathcal{N}$ is proper. Clearly, $\mathcal{V}_{\mathcal{N}}$ is non-empty. Let $\diamond$ be an n-ary connective of $\mathcal{L}$, and let $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{N}}$. Since $\mathcal{V}_{\mathcal{N}} = \mathsf{Image}(v')$, there are some $\psi_1, \ldots, \psi_n \in \mathit{Frm}_{\mathcal{L}}$, such that $v'(\psi_i) = x_i$ for every $1 \leq i \leq n$. Since v' is an $\mathcal{L}$-valuation, $v'(\diamond(\psi_1, \ldots, \psi_n))$ is defined, and so $\diamond_{\mathcal{N}}(x_1, \ldots, x_n)$ is non-empty. To see that $\mathcal{N} \subseteq \mathcal{M}$ it suffices to show that for every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{N}}$, we have $\diamond_{\mathcal{N}}(x_1, \ldots, x_n) \subseteq \diamond_{\mathcal{M}}(x_1, \ldots, x_n)$. Let $\diamond$ be an n-ary connective of $\mathcal{L}$, $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{N}}$, and let $y \in \diamond_{\mathcal{N}}(x_1, \ldots, x_n)$. Then there are some $\psi_1, \ldots, \psi_n \in \mathit{Frm}_{\mathcal{L}}$, such that $y = v'(\diamond(\psi_1, \ldots, \psi_n))$ and $v'(\psi_i) = x_i$ for all $1 \leq i \leq n$. Since v' is $\mathcal{M}$-legal, $y \in \diamond_{\mathcal{M}}(v'(\psi_1), \ldots, v'(\psi_n)) = \diamond_{\mathcal{M}}(x_1, \ldots, x_n)$. Finally, we show that v is $\mathcal{N}$-legal. Let $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$. By definition $v'(\diamond(\psi_1, \ldots, \psi_n)) = v(\diamond(\psi_1, \ldots, \psi_n))$, and since $\mathcal{F}$ is closed under subformulas we also have $v'(\psi_i) = v(\psi_i)$ for every $1 \leq i \leq n$. The construction of $\diamond_{\mathcal{N}}$ then ensures that $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{N}}(v(\psi_1), \ldots, v(\psi_n))$.

($\Leftarrow$) Suppose that there is some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$, such that v is $\mathcal{N}$-legal. By Proposition 2, there exists an $\mathcal{N}$-legal $\mathcal{L}$-valuation v' that extends v. By Proposition 1, v' is $\mathcal{M}$-legal. Thus v is extendable in $\mathcal{M}$. □

Proof of Corollary 2:
Using Corollary 1, it is possible to enumerate all functions $v : sub[\mathcal{S} \cup \{s\}] \to \mathcal{V}_{\mathcal{M}}$, and check if one of them is an $\mathcal{M}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation extendable in $\mathcal{M}$, such that $v \models_{\mathcal{M}} \mathcal{S}$ but $v \not\models_{\mathcal{M}} s$. We claim that

$\mathcal{S} \vdash_{\mathcal{M}} s$ iff such a function is not found. To see this, note that if $\mathcal{S} \nvdash_{\mathcal{M}} s$, then by definition there exists an $\mathcal{M}$-legal $\mathcal{L}$-valuation v' such that $v' \models_{\mathcal{M}} \mathcal{S}$ but $v' \not\models_{\mathcal{M}} s$. Its restriction v to $sub[\mathcal{S} \cup \{s\}]$ is an $\mathcal{M}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation extendable in $\mathcal{M}$, such that $v \models_{\mathcal{M}} \mathcal{S}$ but $v \not\models_{\mathcal{M}} s$. On the other hand, if there exists an $\mathcal{M}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation v extendable in $\mathcal{M}$, such that $v \models_{\mathcal{M}} \mathcal{S}$ but $v \not\models_{\mathcal{M}} s$, then for any of its $\mathcal{M}$-legal extensions v', we have $v' \models_{\mathcal{M}} \mathcal{S}$ but $v' \not\models_{\mathcal{M}} s$. Consequently, $\mathcal{S} \nvdash_{\mathcal{M}} s$ in this case. □

Proof of Proposition 3:
(2) easily follows from (1). For (1), note that v is trivially extendable in $\mathcal{M}$, and so Theorem 1 entails that there is some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$, such that v is $\mathcal{N}$-legal. Clearly, $v(\psi) \in \mathcal{V}_{\mathcal{N}}$ for every formula ψ. Thus $v(\psi)$ is useful in $\mathcal{M}$ for every formula ψ. □

Proof of Theorem 2:
Let $\mathcal{N}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$ with $|\mathcal{V}_{\mathcal{N}}| < |\mathcal{V}_{R[\mathcal{M}]}|$. We show that $\vdash_{\mathcal{N}} \neq \vdash_{\mathcal{M}}$. For every $y \in \mathcal{V}_{\mathcal{N}}$, define $V_y = \{x \in \mathcal{V}_{\mathcal{M}} \mid \forall l \in \pounds . y \in \mathcal{D}_{\mathcal{N}}(l) \Leftrightarrow x \in \mathcal{D}_{\mathcal{M}}(l)\}$. Since $\mathcal{M}$ is separable, V_y is either singleton or empty for every $y \in \mathcal{V}_{\mathcal{N}}$. Since $|\mathcal{V}_{\mathcal{N}}| < |\mathcal{V}_{R[\mathcal{M}]}|$ (and $\mathcal{V}_{R[\mathcal{M}]} \subseteq \mathcal{V}_{\mathcal{M}}$), there exists some $x_0 \in \mathcal{V}_{R[\mathcal{M}]}$, such that $x_0 \notin V_y$ for every $y \in \mathcal{V}_{\mathcal{N}}$. Let $L = \{l \in \pounds \mid x_0 \in \mathcal{D}_{\mathcal{M}}(l)\}$. Let $\mathcal{S}$ be the set of all 1-clauses of the form $\{l : p_1\}$ for $l \in L$, and s be the 1-clause $(\pounds \setminus L : p_1)$. We claim that $\mathcal{S} \vdash_{\mathcal{N}} s$. Suppose otherwise. Then there exists an $\mathcal{N}$-legal $\mathcal{L}$-valuation v such that $v \models_{\mathcal{N}} \mathcal{S}$, but $v \not\models_{\mathcal{N}} s$. Thus $v(p_1) \in \mathcal{D}_{\mathcal{N}}(l)$ for every $l \in L$, and $v(p_1) \notin \mathcal{D}_{\mathcal{N}}(l)$ for every $l \notin L$. But, it then follows that $x_0 \in V_{v(p_1)}$, and this contradicts our assumption concerning x_0.

On the other hand, it is easy to see that $\mathcal{S} \nvdash_{\mathcal{M}} s$. Indeed, consider an $R[\mathcal{M}]$-legal $\{p_1\}$-valuation v that assigns x_0 to p_1. Since x_0 is useful in $\mathcal{M}$, there exists some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$ such that $x_0 \in \mathcal{V}_{\mathcal{N}}$. v is trivially an $\mathcal{N}$-legal $\{p_1\}$-valuation, and so by Theorem 1, v is extendable in $\mathcal{M}$. Let v' be an $\mathcal{M}$-legal $\mathcal{L}$-valuation which extends v. Clearly, $v' \models_{\mathcal{M}} \mathcal{S}$, but $v' \not\models_{\mathcal{M}} s$. □

Proof of Theorem 4:
Suppose that $\mathcal{M}_{\mathbf{G}}$ is proper. Assume that there does not exist a derivation of a sequent s from a set $\mathcal{S}$ of sequents in $\mathbf{G}$ that consists solely of formulas from $sub[\mathcal{S} \cup \{s\}]$. We show that $\mathcal{S} \nvdash_{\mathbf{G}} s$. By Corollary 10 (in Appendix A), there exists some $\mathcal{M}_{\mathbf{G}}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation v, such that $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$ but $v \not\models_{\mathcal{M}_{\mathbf{G}}} s$. By Proposition 2, v is extendable to a full $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{L}$-valuation v'. Clearly, $v' \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$ but $v' \not\models_{\mathcal{M}_{\mathbf{G}}} s$, and so $\mathcal{S} \nvdash_{\mathcal{M}_{\mathbf{G}}} s$. By the soundness of $\mathcal{M}_{\mathbf{G}}$ for $\mathbf{G}$, $\mathcal{S} \nvdash_{\mathbf{G}} s$.

For the converse, suppose that $\mathcal{M}_{\mathbf{G}}$ is not proper. If $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ is empty, then $\vdash_{\mathcal{M}_{\mathbf{G}}} \emptyset$, and so (by Theorem 3) $\vdash_{\mathbf{G}} \emptyset$. Clearly, without using a rule of the form $\emptyset/\emptyset$, there is no derivation in $\mathbf{G}$ that does not contain any formula. It follows that $\mathbf{G}$ is not strongly analytic in this case. Otherwise $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ is non-empty. Thus $\diamond_{\mathcal{M}_{\mathbf{G}}}(L_1, \ldots, L_n) = \emptyset$ for some n-ary connective $\diamond$ of $\mathcal{L}$ and $L_1, \ldots, L_n \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. For every $1 \leq i \leq n$, let $\mathcal{S}_i$ be the set of all clauses of the form $\{l : p_i\}$ where $l \in L_i$, and let $s_i = \{l : p_i \mid l \notin L_i\}$. We claim that $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n \vdash_{\mathbf{G}} s_1 \cup \ldots \cup s_n$, but there does not exist a derivation of $s_1 \cup \ldots \cup s_n$ from $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$ in $\mathbf{G}$ that consists solely of formulas from $\{p_1, \ldots, p_n\}$. For the latter, note that for the $\mathcal{M}_\mathbf{G}$-legal $\{p_1, \ldots, p_n\}$-valuation v assigning L_i to p_i for every $1 \leq i \leq n$, we have $v \models_{\mathcal{M}_\mathbf{G}} \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$ but $v \not\models_{\mathcal{M}_\mathbf{G}} s_1 \cup \ldots \cup s_n$. Thus the claim follows by Corollary 10. For the former, note that every $\mathcal{M}_\mathbf{G}$-legal $\mathcal{L}$-valuation v' for which $v' \models_{\mathcal{M}_\mathbf{G}} \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$ but $v' \not\models_{\mathcal{M}_\mathbf{G}} s_1 \cup \ldots \cup s_n$, we must have $v'(p_i) = L_i$ for every $1 \leq i \leq n$. But then $v'(\diamond(p_1, \ldots, p_n))$ should be an element of the empty set. Since such an $\mathcal{L}$-valuation does not exist, Theorem 3 entails that $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n \vdash_\mathbf{G} s_1 \cup \ldots \cup s_n$. □

Proof of Theorem 5:
Suppose that $\mathcal{M}_\mathbf{G}$ is not proper. By Theorem 4, $\mathbf{G}$ is not strongly analytic. By Proposition 4, it follows that $\mathbf{G}$ does not enjoy strong cut-admissibility.
For the converse, we need the following lemmas:

Lemma 1. *Let* $\mathbf{G}$ *be a canonical labelled system for* $\mathcal{L}$ *and* $\pounds$. *For every* n*-ary connective* $\diamond$ *of* $\mathcal{L}$, *and every* $L_1, \ldots, L_n, L'_1, \ldots, L'_n \in \mathcal{V}_{\mathcal{M}_\mathbf{G}}$ *such that* $L_i \subseteq L'_i$ *for every* $1 \leq i \leq n$, *we have* $\diamond_{\mathcal{M}_\mathbf{G}}(L'_1, \ldots, L'_n) \subseteq \diamond_{\mathcal{M}_\mathbf{G}}(L_1, \ldots, L_n)$.

Lemma 2. *Let* $\mathbf{G}$ *be a cut-saturated canonical labelled system for* $\mathcal{L}$ *and* $\pounds$. $\mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}} = \mathcal{V}_{\mathcal{M}_\mathbf{G}} \cup \{\pounds\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G_{cf}}}}(l) = \mathcal{D}_{\mathcal{M}_\mathbf{G}}(l) \cup \{\pounds\}$ *for every* $l \in \pounds$, *and* $\diamond_{\mathcal{M}_{\mathbf{G_{cf}}}}(L_1, \ldots, L_n) = \diamond_{\mathcal{M}_\mathbf{G}}(L_1, \ldots, L_n) \cup \{\pounds\}$ *for every* n*-ary connective* $\diamond$ *of* $\mathcal{L}$ *and every* $L_1, \ldots, L_n \in \mathcal{V}_{\mathcal{M}_\mathbf{G}}$.[9]

Now suppose that $\mathcal{M}_\mathbf{G}$ is proper. Assume that there does not exist a derivation of a sequent s from a set $\mathcal{S}$ of sequents in $\mathbf{G}$ in which only formulas from $frm[\mathcal{S}]$ serve as cut-formulas. We show that $\mathcal{S} \not\vdash_\mathbf{G} s$. By choosing $\mathcal{F} = Frm_\mathcal{L}$ and $\mathcal{C} = frm[\mathcal{S}]$ in Theorem 7 (see Appendix A), we obtain that there exists some $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{L}$-valuation v' assigning only values from $\mathcal{V}_{\mathcal{M}_\mathbf{G}}$ to the formulas in $frm[\mathcal{S}]$, such that $v' \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$ but $v' \not\models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$. By Lemma 2, v' is a function from $Frm_\mathcal{L}$ to $\mathcal{V}_{\mathcal{M}_\mathbf{G}} \cup \{\pounds\}$. We construct a function $v : Frm_\mathcal{L} \to \mathcal{V}_{\mathcal{M}_\mathbf{G}}$, such that v is an $\mathcal{M}_\mathbf{G}$-legal $\mathcal{L}$-valuation; $v(\psi) \subseteq v'(\psi)$ for every $\psi \in Frm_\mathcal{L}$; and $v(\psi) = v'(\psi)$ whenever $v'(\psi) \in \mathcal{V}_{\mathcal{M}_\mathbf{G}}$. In particular, it is easy to verify that we will have $v \models_{\mathcal{M}_\mathbf{G}} \mathcal{S}$, and $v \not\models_{\mathcal{M}_\mathbf{G}} s$, and consequently $\mathcal{S} \not\vdash_\mathbf{G} s$. The construction of v is done by recursion on the build-up of formulas. First, for atomic formulas, if $v'(p) \in \mathcal{V}_{\mathcal{M}_\mathbf{G}}$, we choose $v(p) = v'(p)$. Otherwise, $v'(p) = \pounds$, and we (arbitrarily) choose $v(p)$ to be an element of $\mathcal{V}_{\mathcal{M}_\mathbf{G}}$ ($\mathcal{V}_{\mathcal{M}_\mathbf{G}}$ is non-empty since $\mathcal{M}_\mathbf{G}$ is proper). Now, let $\diamond$ be an n-ary connective of $\mathcal{L}$, and suppose $v(\psi_i)$ was defined for every $1 \leq i \leq n$. We choose $v(\diamond(\psi_1, \ldots, \psi_n))$ to be equal to $v'(\diamond(\psi_1, \ldots, \psi_n))$ if the latter is in $\mathcal{V}_{\mathcal{M}_\mathbf{G}}$. Otherwise, $v'(\diamond(\psi_1, \ldots, \psi_n)) = \pounds$, and we choose $v(\diamond(\psi_1, \ldots, \psi_n))$ to be some element of $\diamond_{\mathcal{M}_\mathbf{G}}(v(\psi_1), \ldots, v(\psi_n))$ (such an element exists since $\mathcal{M}_\mathbf{G}$ is proper). Obviously, $v(\psi) \subseteq v'(\psi)$ for every $\psi \in Frm_\mathcal{L}$, and $v(\psi) = v'(\psi)$ whenever $v'(\psi) \in \mathcal{V}_{\mathcal{M}_\mathbf{G}}$. To see that v is a valuation in $\mathcal{M}_\mathbf{G}$, suppose (for contradiction) that $v(\diamond(\psi_1, \ldots, \psi_n)) \notin \diamond_{\mathcal{M}_\mathbf{G}} (v(\psi_1), \ldots, v(\psi_n))$ for some

[9] Recall that we denote by $\mathbf{G_{cf}}$ the canonical labelled system obtained from $\mathbf{G}$ by discarding all the canonical cuts.

formula $\diamond(\psi_1,\dots,\psi_n)$. By Lemma 2, $v(\diamond(\psi_1,\dots,\psi_n)) \notin \diamond_{\mathcal{M}_{\mathbf{G_{cf}}}}(v(\psi_1),\dots,v(\psi_n))$. Now, since $v(\psi) \subseteq v'(\psi)$ for every $\psi \in \mathit{Frm}_{\mathcal{L}}$, Lemma 1 entails that $v(\diamond(\psi_1,\dots,\psi_n)) \notin \diamond_{\mathcal{M}_{\mathbf{G_{cf}}}}(v'(\psi_1),\dots,v'(\psi_n))$. Consequently, since v' is a valuation in $\mathcal{M}_{\mathbf{G_{cf}}}$, $v'(\diamond(\psi_1,\dots,\psi_n)) \neq v(\diamond(\psi_1,\dots,\psi_n))$. Now, if $v'(\diamond(\psi_1,\dots,\psi_n)) = \pounds$, then $v(\diamond(\psi_1,\dots,\psi_n)) \in \diamond_{\mathcal{M}_{\mathbf{G}}}(v(\psi_1),\dots,v(\psi_n))$ by definition. Otherwise, $v(\diamond(\psi_1,\dots,\psi_n)) = v'(\diamond(\psi_1,\dots,\psi_n))$, reaching a contradiction. $\square$