Non-deterministic semantics for first-order paraconsistent logics

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Abstract

Using non-deterministic structures called Nmatrices, we provide simple modular non-deterministic semantics for a large family of first-order paraconsistent logics with a formal consistency operator, also known as LFIs. This includes da-Costa's well known predicate calculus C_1^*. We show how consistency propagation in quantified formulas is captured in the semantic framework of Nmatrices, and analyze the semantic effects of different styles of propagation considered in the literature of LFIs. Then we demonstrate how the tool of Nmatrices can be applied to prove a non-trivial property of first-order LFIs discussed in this paper.

Introduction

Contradictory data is one of the most complex problems in reasoning under uncertainty. Delgrande & Mylopoulos describe the problem as follows:

> "It is a fact of life that large knowledge bases are inherently inconsistent, in the same way large programs are inherently buggy. Moreover, within a conventional logic, the inconsistency of a knowledge base has the catastrophic consequence that *everything* is derivable from the knowledge base."

Thus to handle inconsistent knowledge bases one needs a logic that allows contradictory yet non-trivial theories. Logics of this sort are called *paraconsistent*.

There are several approaches to the problem of designing a useful paraconsistent logic (see, e.g. (Batens & Mortensen 2000; Carnielli & Marcos 2002; Bremer 2005)). One of the best known is da Costa's approach ((da Costa. 1974; Carnielli 2006; Carnielli & Marcos 2002; 1999)), which has led to the family of *Logics of Formal Inconsistency* (LFIs). This family is based on two main ideas. First of all, propositions are divided into two sorts: the "normal" (or "consistent") and the "abnormal" (or "inconsistent") ones. The second idea is to express the meta-theoretical notions of consistency/inconsistency at the object language level, by adding to the language a new connective $\circ$, with the intended meaning of $\circ\varphi$ being "φ is consistent". This operator allows for controlling explosiveness by logically separating the notions of contradictoriness and inconsistency. In this way one can limit the applicability of the rule $\varphi, \neg\varphi \vdash \psi$ (which amounts to "a single contradiction entails everything" and leads to trivialization in case of contradictions in classical logic) to the case when φ is consistent (i.e., $\varphi, \neg\varphi, \circ\varphi \vdash \psi$).

Although the syntactic formulations of LFIs are relatively simple, the known semantic interpretations are more complicated: the vast majority of LFIs cannot be characterized by means of finite multi-valued matrices. The first systems of da-Costa have been introduced only proof-theoretically, and only some years later bivalued semantics have been proposed for their interpretation. Alternative semantics like *possible translations semantics* (Carnielli 2000; Carnielli & Marcos 1999) and *society semantics* ((Carnielli & Lima-Marques 1999)) were also proposed at a later stage.

An alternative framework for providing semantics for paraconsistent logics is introduced in (Avron & Lev 2001) and used in (Avron 2005; 2006; Avron & Zamansky 2005)). This framework uses a generalization of the standard multi-valued matrices, called *non-deterministic matrices* (Nmatrices). Nmatrices are multi-valued structures, in which the value assigned by a valuation to a complex formula can be chosen *non-deterministically* out of a certain nonempty set of options. The framework of Nmatrices has a number of advantages compared to the other approaches mentioned above. First of all, the semantics provided by Nmatrices is *modular*: the main effect of each of the rules of a proof system is reducing the degree of non-determinism of operations, by forbidding some options. The semantics of a proof system is obtained by combining the semantic constraints imposed by its rules in a rather straightforward way. Another attractive property of semantics based on Nmatrices is *effectiveness*, i.e. the possibility to extend partial valuations closed under subformulas to full valuations (see (Avron 2006) for further details). Effectiveness is a crucial condition for the usefulness of semantics, in particular for constructing counterexamples. Furthermore, the use of finite Nmatrices in the propositional case has the benefits of preserving *decidability* and *compactness*.

In this paper we extend the scope of the *modular* framework based on Nmatrices from propositional to the first-order case. We provide semantics for a useful family of first-order LFIs, including da-Costa's well-known predicate calculus C_1^*. The semantics we provide is three-valued for most of the systems, and infinite-valued for the rest of them. We also demonstrate how consistency propagation in quantified formulas is captured in the extended framework, by analyzing the semantic effects of various consistency propagation schemata. Finally, the semantic tool of Nmatrices is applied to prove a proof-theoretical property of some of the discussed LFIs, namely that two sentences are logically indistinguishable (i.e., intersubstitutable in any context) iff they are identical up to the names of their bound variables and deletion/addition of void quantifiers. This is closely connected to an important result of Mortensen ((Mortensen 1980; Carnielli & Marcos 2002)) on algebraization of C_1.

Preliminaries

In what follows L is a first-order language including the connectives $\wedge, \vee, \rightarrow, \neg, \circ$. The set Frm_L of its formulas is defined standardly. Denote by Frm_L^{cl} the set of closed formulas (sentences) of L and by Trm_L^{cl} - its set of closed terms.
Denote by $Fv[\psi]$ ($Fv[\mathbf{t}]$) the set of variables occurring free in a formula ψ (a term **t**). Denote by $\psi\{\mathbf{t}/x\}$ the formula obtained from ψ by substituting the term **t** for every free occurrence of x in ψ. A term **t** is *free for x in ψ* if either ψ is atomic, $\psi = \diamond(\psi_1, ..., \psi_n)$ and **t** is free for x in $\psi_1, ..., \psi_n$, or $\psi = Qy\varphi$ and either $x = y$ or $y \notin Fv[\mathbf{t}]$ and **t** is free for x in φ.
Given a set A, denote by $P^+(A)$ the set of all non-empty subsets of A.

A taxonomy of LFIs

Let $\mathbf{HCL}^+$ be some standard propositional Hilbert-type system with MP as the only rule of inference which is sound and complete for the positive fragment of classical propositional logic.

Definition 1 (The system B) $\mathbf{B}$[1] *is the system obtained from* $\mathbf{HCL}^+$ *by the addition of the following schemata:*

$(\mathbf{t})$ $\neg\varphi \vee \varphi$
$(\mathbf{p})$ $\circ\,\varphi \rightarrow ((\varphi \wedge \neg\varphi) \rightarrow \psi)$

Definition 2 (The set of schemata SC) *Let SC be the set consisting of the following schemata:*

(c) $\neg\neg\varphi \rightarrow \varphi$
(e) $\varphi \rightarrow \neg\neg\varphi$
(i) $\neg\circ\varphi \rightarrow (\varphi \wedge \neg\varphi)$
(l) $\neg(\varphi \wedge \neg\varphi) \rightarrow \circ\varphi$
(d) $\neg(\neg\varphi \wedge \varphi) \rightarrow \circ\varphi$
(b) $\neg(\varphi \wedge \neg\varphi) \vee (\neg\varphi \wedge \varphi)) \rightarrow \circ\varphi$
for $\diamond \in \{\vee, \wedge, \rightarrow\}$*:*
($\mathbf{a}_\diamond$) $\circ\varphi \wedge \circ\psi \rightarrow \circ(\varphi \diamond \psi)$
($\mathbf{o}_\diamond$) $\circ\varphi \vee \circ\psi \rightarrow \circ(\varphi \diamond \psi)$

Note that **(d)** and **(l)** are not equivalent in this context, as will be explained in the sequel.

Definition 3 $(\mathbf{B}[\mathbf{X}])$ *For $X \subseteq SC$, $\mathbf{B}[X]$ is the system obtained from $\mathbf{B}$ by adding the schemata from X.*
We write $\Gamma \vdash_{\mathbf{B}[X]} \psi$ if ψ is provable from Γ in $\mathbf{B}[X]$.

We usually denote a system $\mathbf{B}[X]$ by $\mathbf{B}s$, where s is a string consisting of the names of the schemata from X. For instance, $\mathbf{Bia}$ denotes the system $\mathbf{B}[\{(\mathbf{i}), (\mathbf{a})\}]$.
Remark: The difference between **Bcial** and da-Costa's original formulation of C_1 (see (da Costa. 1974)) is that the connective $\circ$ in C_1 was not taken as primitive, but $\circ\psi$ was instead denoted ψ° and was taken as an abbreviation of $\neg(\psi \wedge \neg\psi)$ (see (Carnielli 2006; Avron 2006) for further details).

Non-deterministic matrices

Our main semantic tool in what follows will be the following generalization of the concept of a multi-valued matrix given in (Avron & Lev 2001; Avron 2005; Avron & Zamansky 2005).

Definition 4 (Non-deterministic matrix) *A non-deterministic matrix (Nmatrix) for L is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where: $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$ and $\mathcal{O}$ includes the following interpretation functions:*

- $\tilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$ *for every n-ary connective $\diamond$.*
- $\tilde{Q}_{\mathcal{M}} : P^+(\mathcal{V}) \rightarrow P^+(\mathcal{V})$ *for every quantifier Q.*

Definition 5 (L-structure) *Let $\mathcal{M}$ be an Nmatrix. An L-structure for $\mathcal{M}$ is a pair $S = \langle D, I \rangle$ where D is a (non-empty) domain and I is a function interpreting constants, predicate symbols and function symbols of L, satisfying the following conditions: $I[c] \in D$, $I[p^n] : D^n \rightarrow \mathcal{V}$ is an n-ary predicate, and $I[f^n] : D^n \rightarrow D$ is an n-ary function.*
I is extended to interpret closed terms of L as follows:

$$I[f(t_1, ..., t_n)] = I[f][I[t_1], ..., I[t_n]]$$

Henceforth we will be interested in a special kind of L-structures, called *Henkin L-structures* (see e.g.(Leblanc 2001)), satisfying that for every element of the domain there is a closed L-term referring to it. The results presented below can be easily generalized to general L-structures by adding a set of *individual constants* $\{\overline{a} \mid a \in D\}$ to the language (see (Avron & Zamansky 2005) for further details).

In first-order logic we expect certain classes of formulas to be always assigned the same truth values. Examples of such classes are formulas which are identical up to the names of their bound variables, or formulas obtained by deletion/addition of void quantifiers, i.e. quantifiers that do not bind any variables, and are in fact redundant. In order to guarantee that the same truth values are chosen for such formulas, we define the relation $\sim_\alpha^v$ on formulas:

Definition 6 $(\sim_\alpha^{\mathbf{v}})$ *Let ψ, ψ' be L-formulas. $\psi \sim_\alpha^v \psi'$ if ψ' can be obtained from ψ by renaming of bound variables and deletions/additions of void quantifiers.*

It is easy to see that $\sim_\alpha^v$ is an equivalence relation.

Definition 7 $(S$**-valuation)** *Let $S = \langle D, I \rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. An S-valuation $v : Frm_L^{\mathsf{cl}} \rightarrow \mathcal{V}$ is* legal *in $\mathcal{M}$ if it satisfies the following conditions:*

- *v respects the relation $\sim_\alpha^v$, i.e. $v[\psi] = v[\psi']$ for every two L-sentences ψ, ψ', such that $\psi \sim_v^\alpha \psi'$.*

[1] The system **B** is called mbC in (Carnielli 2006).

- $v[p(t_1,...,t_n)] = I[p][I[t_1],...,I[t_n]]$.
- $v[\diamond(\psi_1,...,\psi_n)] \in \tilde{\diamond}_{\mathcal{M}}[v[\psi_1],...,v[\psi_n]]$.
- $v[Qx\psi] \in \tilde{Q}_{\mathcal{M}}[\{v[\psi\{t/x\}] \mid t \in Trm^{\mathsf{cl}}_L\}]$.

Definition 8 (Model, $\mathcal{M}$-validity, $\vdash_{\mathcal{M}}$) *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$, ψ a sentence of $L(D)$ and Γ, Δ sets of formulas of $L(D)$.*

1. *An $\mathcal{M}$-legal S-valuation v is* a model of a sentence ψ (a set of sentence Γ) in $\mathcal{M}$, *denoted by $S, v \models_{\mathcal{M}} \psi$ ($S, v \models_{\mathcal{M}} \Gamma$), if $v[\psi] \in \mathcal{D}$ ($v[\psi] \in \mathcal{D}$ for every $\psi \in \Gamma$).*
2. *The consequence relation $\vdash_{\mathcal{M}}$ between sets of L-sentences is defined as follows: $\Gamma \vdash_{\mathcal{M}} \Delta$ if for every L-structure S and every $\mathcal{M}$-legal S-valuation v: $S, v \models_{\mathcal{M}} \Gamma$ implies that there is some $\psi \in \Delta$, such that $S, v \models_{\mathcal{M}} \psi$.*
3. *An Nmatrix $\mathcal{M}$ is* sound *for a system C if $\vdash_C \subseteq \vdash_{\mathcal{M}}$. An Nmatrix $\mathcal{M}$ is* complete *for a system C if $\vdash_{\mathcal{M}} \subseteq \vdash_C$.*

The following definition and proposition are a straightforward extension of defn. 3 and proposition 1 from (Avron 2006):

Definition 9 (Refinement) *Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1\rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2\rangle$ be Nmatrices for L. $\mathcal{M}_2$ is a* refinement *of $\mathcal{M}_1$ if $\mathcal{V}_2 \subseteq \mathcal{V}_1$, $\mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{V}_2$, $\tilde{\diamond}_{\mathcal{M}_2}[a_1,...,a_n] \subseteq \tilde{\diamond}_{\mathcal{M}_1}[a_1,...a_n]$ for every n-ary connective $\diamond$ of L and every $a_1,...,a_n \in \mathcal{V}_2$ and $\tilde{Q}_{\mathcal{M}_2}[H] \subseteq \tilde{Q}_{\mathcal{M}_1}[H]$ for every quantifier Q of L and every $H \subseteq \mathcal{V}_2$.*

Proposition 1 *If $\mathcal{M}_2$ is a refinement of $\mathcal{M}_1$, then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$.*

Now we come to the semantics of (Avron 2006) for the propositional systems presented above. Recall that the idea of LFIs is to divide the propositions into “consistent” and “inconsistent” and to limit the applicability of the inference $\psi, \neg\psi \vdash \varphi$ (which amounts to “a single contradiction entails everything”) to the case where ψ is “consistent” (expressing the assumption of its consistency within the language using the operator $\circ$). This is reflected in the use of sets of “consistent” and “inconsistent” truth values: $\mathcal{T}$ is the set of designated “consistent” truth values, $\mathcal{F}$ - the set of non-designated “consistent” truth values and $\mathcal{I}$ - the set of “inconsistent” truth values (in this paper we only consider the cases when the “inconsistent” truth values are all designated). The intuitive meaning of this is that if a proposition ψ is assigned a truth value from $\mathcal{T}$ (meaning that ψ is “consistently true” and so its negation is “false”), then $\neg\psi$ is assigned a truth value from $\mathcal{F}$. If ψ is assigned a truth value from $\mathcal{I}$, then $\neg\psi$ is also assigned a truth value from $\mathcal{I}$ (meaning that ψ is “inconsistently true”, and so both ψ and $\neg\psi$ can be “true” at the same time).

Definition 10 (Basic B-Nmatrix, B-Nmatrix) *A basic* **B**-*Nmatrix is an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$, such that $\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$, where $\mathcal{T}, \mathcal{I}, \mathcal{F}$ are non-empty disjoint sets, $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$, and $\mathcal{O}$ is defined by:*

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \textit{either } a \in \mathcal{D} \textit{ or } b \in \mathcal{D} \\ \mathcal{F} & \textit{otherwise} \end{cases}$$

$$a \tilde{\to} b = \begin{cases} \mathcal{D} & \textit{either } a \in \mathcal{F} \textit{ or } b \in \mathcal{D} \\ \mathcal{F} & \textit{otherwise} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \textit{either } a \in \mathcal{F} \textit{ or } b \in \mathcal{F} \\ \mathcal{D} & \textit{otherwise} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & a \in \mathcal{T} \\ \mathcal{T} & a \in \mathcal{F} \\ \mathcal{D} & a \in \mathcal{I} \end{cases} \qquad \tilde{\circ} a = \begin{cases} \mathcal{V} & a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & a \in \mathcal{I} \end{cases}$$

A **B**-*Nmatrix is an Nmatrix which is a refinement of some basic* **B**-*Nmatrix.*
$\mathcal{M}_B$ is the 3-valued basic **B**-*Nmatrix with $\mathcal{T} = \{t\}, \mathcal{F} = \{f\}$ and $\mathcal{I} = \{I\}$.*

It is shown in (Avron 2006) that **B** is sound for any basic **B**-Nmatrix and $\mathcal{M}_B$ is a characteristic Nmatrix for **B**. As for the rest of the systems defined above, each of the schemata **(i)**, **(c)**, **(e)**, **(a$_\diamond$)**, **(o$_\diamond$)** imposes some semantic condition on the basic Nmatrix.

Definition 11 (Refining conditions for SC) *The refining conditions for the schemata from SC are:*

$Cond(\mathbf{i}) : a \in \mathcal{T} \cup \mathcal{F} \Rightarrow \tilde{\circ}[a] \subseteq \mathcal{T}$.
$Cond(\mathbf{c}) : a \in \mathcal{F} \Rightarrow \tilde{\neg}[a] \subseteq \mathcal{T}$.
$Cond(\mathbf{e}) : a \in \mathcal{I} \Rightarrow \tilde{\neg}[a] \subseteq \mathcal{I}$.
$Cond(\mathbf{a}_\diamond) : a \in \mathcal{T} \cup \mathcal{F}$ *and* $b \in \mathcal{T} \cup \mathcal{F} \Rightarrow \tilde{\diamond}[a,b] \subseteq \mathcal{T} \cup \mathcal{F}$.
$Cond(\mathbf{o}_\diamond) : a \in \mathcal{T} \cup \mathcal{F}$ *or* $b \in \mathcal{T} \cup \mathcal{F} \Rightarrow \tilde{\diamond}[a,b] \subseteq \mathcal{T} \cup \mathcal{F}$.

Definition 12 (The set S)
Let $S = \{\mathbf{i}, \mathbf{c}, \mathbf{e}, \mathbf{ce}, \mathbf{cie}, \mathbf{ia}, \mathbf{cia}, \mathbf{iae}, \mathbf{ciae}, \mathbf{io}, \mathbf{ioe}, \mathbf{cioe}\}$.

Remark: Note that systems which include the schema **(a$_\diamond$)**, or the schema **(o$_\diamond$)**, but not the schema **(i)** are not considered here for the sake of a simpler presentation: non-deterministic semantics for such systems require five truth values rather than three. A completely modular five-valued semantics is provided for such systems in (Avron 2005).

Definition 13 ($\mathcal{M}_{Bs}$) *For $s \in S$, $\mathcal{M}_{Bs}$ is the unique* **B**-*Nmatrix in which $\mathcal{V} = \{t\}$, $\mathcal{F} = \{f\}$ and $\mathcal{I} = \{I\}$ and which satisfies $Cond(x)$ for every x occurring in s.*

Theorem 1 *(Avron 2006) For every $s \in S$, $\mathcal{M}_{Bs}$ is a characteristic Nmatrix for* **B***s.*

Next (Avron 2006) defines a infinite-valued non-deterministic semantics for systems containing the schemata **(l)**, **(d)** and **(d)** and shows that no finite-valued semantics exists for them:

Definition 14 ($\mathcal{M}_{\mathbf{Bl}}$, $\mathcal{M}_{\mathbf{Bd}}$, $\mathcal{M}_{\mathbf{Bb}}$) *Let $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}, \mathcal{F} = \{f\}$. The Nmatrix $\mathcal{M}_{Bl} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ for L is defined as follows: $\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$, $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$ and $\mathcal{O}$ is defined by:*

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \textit{if either } a \in \mathcal{D} \textit{ or } b \in \mathcal{D} \\ \mathcal{F} & \textit{if } a, b \in \mathcal{F} \end{cases}$$

$$a \tilde{\to} b = \begin{cases} \mathcal{D} & \textit{if either } a \in \mathcal{F} \textit{ or } b \in \mathcal{D} \\ \mathcal{F} & \textit{if } a \in \mathcal{D} \textit{ and } b \in \mathcal{F} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \textit{if either } a \in \mathcal{F} \textit{ or } b \in \mathcal{F} \\ \mathcal{T} & \textit{if } a = I_i^j \textit{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \textit{otherwise} \end{cases}$$

$$\tilde{\neg}a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases}$$

$$\tilde{\circ}a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{T} \cup \mathcal{F} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

The Nmatrix $\mathcal{M}_{Bd}$ is defined like $\mathcal{M}_{Bl}$, except that $\tilde{\wedge}$ is defined as follows:

$$a\tilde{\wedge}b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } b = I_i^j \text{ and } a \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

The Nmatrix $\mathcal{M}_{Bb}$ is defined like $\mathcal{M}_{Bl}$, except that $\tilde{\wedge}$ is defined as follows:

$$a\tilde{\wedge}b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & (\text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\}) \\ & or(b = I_i^j \text{ and } a \in \{I_i^{j+1}, t_i^{j+1}\}) \\ \mathcal{D} & \text{otherwise} \end{cases}$$

Definition 15 **($\mathcal{M}_{\mathbf{Bsz}}$)** *For $z \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$ and $s \in S$, $\mathcal{M}_{Bsz}$ is the weakest refinement of $\mathcal{M}_{Bz}$ which satisfies $Cond(x)$ for every x occurring in s. In other words, $\mathcal{M}_{Bsz}$ is obtained from $\mathcal{M}_{Bz}$ by the following modifications:*

1. *If **i** occurs in s, then for every $a \in \mathcal{F} \cup \mathcal{T}$, $\tilde{\circ}[a] = \mathcal{T}$.*
2. *If **c** occurs in s, then $\tilde{\neg}[f] = \mathcal{T}$.*
3. *If **e** occurs in s, then $\tilde{\neg}[I_i^j] = \{I_i^{j+1}\}$.*
4. *If $\mathbf{a}_\wedge$ occurs in s, then: for $a \in \mathcal{T}$ and $b \in \mathcal{T}$: $\tilde{\wedge}[a, b] = \mathcal{T}$. If $\mathbf{a}_\vee$ occurs in s, then: for $a \in \mathcal{T}$ and $b \notin \mathcal{I}$, or $b \in \mathcal{T}$ and $a \notin \mathcal{I}$: $\tilde{\vee}[a, b] = \mathcal{T}$. If $\mathbf{a}_\rightarrow$ occurs in s, then: for $a \in \mathcal{F}$ and $b \notin \mathcal{I}$, or $b \in \mathcal{T}$ and $a \notin \mathcal{I}$: $\tilde{\rightarrow}[a, b] = \mathcal{T}$. Similarly for **($\mathbf{o}_\diamond$)**.*

Theorem 2 *((Avron 2006)) For $z \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$ and $s \in S$, $\mathcal{M}_{Bsz}$ is a characteristic Nmatrix for $\mathbf{B}sz$.*

Semantics for first-order LFIs

Systems with 3-valued semantics

Definition 16 **(The system $\mathbf{B}^f$)** $\mathbf{B^f}$ *is the first-order system obtained from **B** by the addition of the following schemata:*

$(\mathbf{v}_\alpha)$ $\psi \rightarrow \psi'$ for $\psi \sim_\alpha^v \psi'$

$(\forall_\mathbf{f})$ $\forall x\psi \rightarrow \psi\{t/x\}$

$(\exists_\mathbf{t})$ $\psi\{t/x\} \rightarrow \exists x\psi$

and the following inference rules:

$$(\forall_\mathbf{t})\ \frac{(\varphi \rightarrow \psi)}{(\varphi \rightarrow \forall x\psi)}$$

$$(\exists_\mathbf{f})\ \frac{(\psi \rightarrow \varphi)}{(\exists x\psi \rightarrow \varphi)}$$

where t is free for x in ψ and $x \notin Fv[\varphi]$.

It is important to note that unlike in classical predicate logic, the **($\mathbf{v}_\alpha$)**-schema is not derivable from the rest of the schemata of $\mathbf{B^f}$ and so has to be added explicitly. For instance, $\neg\forall xp(x) \rightarrow \neg\forall yp(y)$ and $\neg\forall xp(c) \rightarrow \neg p(c)$ are not derivable without it.

Definition 17 **(Basic $\mathbf{B}^f$-Nmatrix, $\mathbf{B}^f$-Nmatrix, $\mathcal{M}_\mathbf{B}^\mathbf{f}$)** *A basic $\mathbf{B}^f$-Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for L is defined similarly to a basic **B**-Nmatrix (defn. 10), with the addition of the following interpretations of quantifiers for every $H \subseteq P^+(\mathcal{V})$:*

$$\tilde{\forall}[H] = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$\tilde{\exists}[H] = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

A $\mathbf{B}^f$-Nmatrix is any refinement of a basic $\mathbf{B}^f$-Nmatrix.

The Nmatrix $\mathcal{M}_B^f$ for L is the 3-valued basic $\mathbf{B}^f$-Nmatrix with $\mathcal{T} = \{t\}$, $\mathcal{F} = \{f\}$ and $\mathcal{I} = \{I\}$.

Theorem 3 *Let Γ be a set of L-sentences and μ - an L-sentence. Then $\Gamma \vdash_{\mathbf{B^f}} \mu$ iff $\Gamma \vdash_{\mathcal{M}_B^f} \mu$.*

Proof: The proof of soundness is not hard and is left for the reader.
For completeness, suppose that $\Gamma \not\vdash_{\mathbf{B^f}} \mu$. We will construct an L-structure S and an $\mathcal{M}$-legal S-valuation v, refuting $\Gamma \vdash_{\mathcal{M}_B^f} \mu$, that is $S, v \models_{\mathcal{M}_B^f} \Gamma$, but $S, v \not\models_{\mathcal{M}_B^f} \mu$.
It is easy to see that we can restrict ourselves to L_r, the subset of L consisting of all the constants, function and predicate symbols occurring in $\Gamma \cup \{\mu\}$. Let L' be the language obtained from L_r by adding a countably infinite set of new constants. It is a standard matter to show that Γ can be extended to a maximal set Γ^*, such that $\Gamma^* \not\vdash_{\mathbf{B^f}} \mu$ over L', satisfying:

1. $\Gamma \subseteq \Gamma^*$.
2. For every sentence $\exists x\psi \in \Gamma^*$ there is a constant c_ψ such that $\psi\{c_\psi/x\} \in \Gamma^*$.
3. For every sentence $\forall x\psi \notin \Gamma^*$ there is a constant c_ψ such that $\psi\{c_\psi/x\} \notin \Gamma^*$.

Let $\psi, \varphi, \forall xA$ be L'-sentences. It is easy to show that Γ^* satisfies the following properties:

1. $\psi \notin \Gamma^*$ iff $\psi \rightarrow \mu \in \Gamma^*$.
2. If $\psi \notin \Gamma^*$, then $\psi \rightarrow \varphi \in \Gamma^*$.
3. $\psi \vee \varphi \in \Gamma^*$ iff either $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$.
4. $\psi \wedge \varphi \in \Gamma^*$ iff both $\varphi \in \Gamma^*$ and $\psi \in \Gamma^*$.
5. $\varphi \rightarrow \psi \in \Gamma^*$ iff either $\varphi \notin \Gamma^*$ or $\psi \in \Gamma^*$.
6. Either $\psi \in \Gamma^*$ or $\neg\psi \in \Gamma^*$.
7. If ψ and $\neg\psi$ are both in Γ^*, then $\circ\psi \notin \Gamma^*$.
8. If $\psi \in \Gamma^*$, then for every L'-sentence ψ' such that $\psi' \sim_\alpha^v \psi$: $\psi' \in \Gamma^*$.
9. If $\forall xA \in \Gamma^*$, then for every closed L'-term **t**: $A\{\mathbf{t}/x\} \in \Gamma^*$. If $\forall xA \notin \Gamma^*$, then there is some constant c_A of L', such that $A\{c_A/x\} \notin \Gamma^*$.

10. If $\exists x A \in \Gamma^*$, then there is some constant c_A of L, such that $A\{c_A/x\} \in \Gamma^*$. If $\exists x A \notin \Gamma^*$, then for every closed term $\mathbf{t}$ of L': $A\{\mathbf{t}/x\} \notin \Gamma^*$.

The L'-structure $S = \langle D, I \rangle$ is defined as follows:

- D is the set of all the closed terms of L'.
- For every constant c of L': $I[c] = c$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$:

$$I[p][\mathbf{t}_1, ..., \mathbf{t}_n = \begin{cases} f & p(\mathbf{t}_1, ..., \mathbf{t}_n) \notin \Gamma^* \\ t & \neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \notin \Gamma^* \\ I & p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^* \\ & \text{and } \neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^* \end{cases}$$

Next we define the refuting S-valuation v as follows:

$$v[\psi] = \begin{cases} f & \psi \notin \Gamma^* \\ t & \neg\psi \notin \Gamma^* \\ I & \psi \in \Gamma^* \text{ and } \neg\psi \in \Gamma^* \end{cases}$$

Note that by property 6 v is well-defined. We still need to show that v is legal in $\mathcal{M}_B^f$.
Let ψ, ψ' be two L'-sentences, such that $\psi \sim_\alpha^v \psi'$. Then by property 8 and since $\sim_\alpha^v$ is symmetric, $\psi \in \Gamma^*$ iff $\psi' \in \Gamma^*$. Also, $\neg\psi \sim_\alpha^v \neg\psi'$ and again by property 8 $\neg\psi \in \Gamma^*$ iff $\neg\psi' \in \Gamma^*$. Thus $v[\psi] = v[\psi']$ and so v respects the $\sim_\alpha^v$ relation. It remains to show that v respects the interpretations of the connectives and quantifiers in $\mathcal{M}_B^f$. The proof for propositional connectives is similar to the proof in (Avron 2006). We show the proof for $\forall$ and $\exists$:

- Let $\forall x\psi$ be an L'-sentence, such that $f \in \{v[\psi\{\mathbf{t}/x\}] \mid \mathbf{t} \in Trm_{L'}^{\mathsf{cl}}\}$. Denote the set $\{v[\psi\{\mathbf{t}/x\}] \mid \mathbf{t} \in Trm_{L'}^{\mathsf{cl}}\}$ by H_ψ. Suppose by contradiction that $v[\forall x\psi] = t$ or $v[\forall x\psi] = I$. If the latter holds, by definition of v, $\forall x\psi \in \Gamma^*$. If the former holds, $\neg\forall x\psi \notin \Gamma^*$ and by property 6, again $\forall x\psi \in \Gamma^*$. By property 9, for every closed L'-term $\mathbf{t}$: $\psi\{\mathbf{t}/x\} \in \Gamma^*$ and so $v[\psi\{\mathbf{t}/x\}] \neq f$, in contradiction to our assumption.
- Let $\forall x\psi$ be an L'-sentence, such that $f \notin H_\psi$. Suppose by contradiction that $v[\forall x\psi] = f$. Then by definition of v, $\forall x\psi \notin \Gamma^*$. By property 10, there is some closed L'-term $\mathbf{t}$, such that $\psi\{\mathbf{t}/x\} \notin \Gamma^*$. Then $v[\psi\{\mathbf{t}/x\}] = f$, in contradiction to our assumption.
- Let $\exists x\psi$ be an L'-sentence, such that $\mathcal{D} \cap H_\psi \neq \emptyset$. Suppose by contradiction that $v[\exists x\psi] = f$. Then $\exists x\psi \notin \Gamma^*$. By property 10, for every closed L'-term $\mathbf{t}$: $\psi\{\mathbf{t}/x\} \notin \Gamma^*$, and so $v[\psi\{\mathbf{t}/x\}] = f$, in contradiction to our assumption.
- Let $\exists x\psi$ be an L'-sentence, such that $\mathcal{D} \cap H_\psi = \emptyset$. Suppose by contradiction that $v[\exists x\psi] \in \mathcal{D}$. Then $\exists x\psi \in \Gamma^*$ and by property 10, there is some closed term $\mathbf{t}$, such that $\psi\{\mathbf{t}/x\} \in \Gamma^*$. Then $v[\{\mathbf{t}/x\}] \in \mathcal{D}$, in contradiction to our assumption.

It is easy to see that for every L'-sentence ψ: $v[\psi] \in \mathcal{D}$ iff $\psi \in \Gamma^*$. So $S, v \models_{\mathcal{M}_B^f} \Gamma$ (recall that $\Gamma \subseteq \Gamma^*$), but $S, v \not\models_{\mathcal{M}_B^f} \mu$. $\square$

Now we handle the various extensions of the basic system $\mathbf{B}^f$ by the combinations of propositional schemata from S (see defn. 12).

Definition 18 **($\mathbf{B}^f\mathbf{s}$)** *For $s \in S$, the system $\mathbf{B^f}s$ is obtained from $\mathbf{B^f}$ by adding the schemata in s.*

Definition 19 **($\mathcal{M}_{\mathbf{Bs}}^{\mathbf{f}}$)** *For $s \in S$, $\mathcal{M}_{Bs}^f$ is the $\mathbf{B}^f$-Nmatrix which satisfies $Cond(x)$ for every x occurring in s.*

Theorem 4 *Let Γ be a set of L-sentences and μ - an L-sentence. For any $s \in S$, $\Gamma \vdash_{\mathbf{Bs}} \mu$ iff $\Gamma \vdash_{\mathcal{M}_{Bs}^f} \mu$.*

Proof: The proof of soundness is not hard and is left to the reader. The proof of completeness is similar to the proof of theorem 3. It remains to show that the new conditions imposed by the schemata in s are respected by the valuation v. The proof is similar to the proof in (Avron 2006) (see theorem 3).

Consistency propagation for quantifiers

Next we focus on consistency propagation in quantified formulas and analyze the semantic effects of different styles of propagation. First recall the two propositional schemata for consistency propagation discussed above:

for $\diamond \in \{\vee, \wedge, \rightarrow\}$:
$(\mathbf{a}_\diamond)$ $\circ\varphi \wedge \circ\psi \rightarrow \circ(\varphi \diamond \psi)$
$(\mathbf{o}_\diamond)$ $\circ\varphi \vee \circ\psi \rightarrow \circ(\varphi \diamond \psi)$

In $(\mathbf{a}_\diamond)$ the consistency of both ψ and φ implies the consistency of $\psi \diamond \varphi$. In $(\mathbf{o}_\diamond)$ the consistency of *one* of the sentences ψ, φ implies the consistency of $\psi \diamond \varphi$. In order to generalize the above schemata to quantified formulas, we can think of a universally (existentially) quantified formula as a "conjunction" ("disjunction") of all of its substitution instances. Then we get the following quantificational schemata (some of them have already been considered in the literature of LFIs, see e.g. (Carnielli 2006)):

$(\mathbf{a}_\forall)$: $\forall x \circ \psi \rightarrow \circ\forall x\psi$
$(\mathbf{a}_\exists)$: $\forall x \circ \psi \rightarrow \circ\exists x\psi$
$(\mathbf{o}_\forall)$: $\exists x \circ \psi \rightarrow \circ\forall x\psi$
$(\mathbf{o}_\exists)$: $\exists x \circ \psi \rightarrow \circ\exists x\psi$

Like in the propositional case, we do not consider here systems which include the schemata $(\mathbf{a_Q})$ or $(\mathbf{o_Q})$ for $\mathbf{Q} \in \{\forall, \exists\}$, but not the schema **(i)**.[2] The schemata $(\mathbf{a_Q})$[3] and $(\mathbf{o_Q})$ for $\mathbf{Q} \in \{\forall, \exists\}$ are a generalization of the $(\mathbf{a}_\diamond)$ and $(\mathbf{o}_\diamond)$ propagation schemata respectively. $(\mathbf{a_Q})$ implies that the consistency of a quantified formula follows from the consistency of *all* of its substitution instances. $(\mathbf{o_Q})$ implies that the consistency of a quantified formula follows from the consistency of *at least one* of its substitution instances.
We can also consider other variations of quantificational propagation:

[2]Similarly to the propositional case, five-valued modular semantics can be defined for such first-order systems.

[3]$(\mathbf{a}_\forall)$ and $(\mathbf{a}_\forall)$ are included in da-Costa's predicate calculus C_1^*, as will be explained in the next subsection.

$(\mathbf{w}_\forall): \circ\forall x\psi \rightarrow \forall x \circ \psi$
$(\mathbf{w}_\exists): \circ\exists x\psi \rightarrow \forall x \circ \psi$
$(\mathbf{r}_\forall): \circ\forall x\psi \rightarrow \exists x \circ \psi$
$(\mathbf{r}_\exists): \circ\exists x\psi \rightarrow \exists x \circ \psi$

$(\mathbf{w_Q})$ implies that from the consistency of the quantified formula follows the consistency of *all* of the substitution instances. $(\mathbf{r_Q})$ implies that from the consistency of the quantified formula, follows the consistency of *at least one* of its substitution instances.

Definition 20 *Let* $PR = \{\mathbf{a}_\forall, \mathbf{a}_\exists, \mathbf{o}_\forall, \mathbf{o}_\exists, \mathbf{r}_\forall, \mathbf{r}_\exists, \mathbf{w}_\exists\}$.

Definition 21 *The refining conditions induced by the above schemata are:*

$Cond(\mathbf{a}_\forall): H \subseteq \mathcal{T} \cup \mathcal{F} \Rightarrow \tilde{\forall}[H] \subseteq \mathcal{T} \cup \mathcal{F}$
$Cond(\mathbf{a}_\exists): H \subseteq \mathcal{T} \cup \mathcal{F} \Rightarrow \tilde{\exists}[H] \subseteq \mathcal{T} \cup \mathcal{F}$
$Cond(\mathbf{o}_\forall): H \cap (\mathcal{T} \cup \mathcal{F}) \neq \emptyset \Rightarrow \tilde{\forall}[H] \subseteq \mathcal{T} \cup \mathcal{F}$
$Cond(\mathbf{o}_\exists): H \cap (\mathcal{T} \cup \mathcal{F}) \neq \emptyset \Rightarrow \tilde{\exists}[H] \subseteq \mathcal{T} \cup \mathcal{F}$
$Cond(\mathbf{w}_\forall): H \cap \mathcal{I} \neq \emptyset \Rightarrow \tilde{\forall}[H] \subseteq \mathcal{I}$
$Cond(\mathbf{w}_\exists): H \cap \mathcal{I} \neq \emptyset \Rightarrow \tilde{\exists}[H] \subseteq \mathcal{I}$
$Cond(\mathbf{r}_\forall): H \subseteq \mathcal{I} \Rightarrow \tilde{\forall}[H] \subseteq \mathcal{I}$
$Cond(\mathbf{r}_\exists): H \subseteq \mathcal{I} \Rightarrow \tilde{\exists}[H] \subseteq \mathcal{I}$

Remark: Note that we do not include $(\mathbf{w}_\forall)$ in PR. The reason for this is that no refinement $\mathcal{M}'$ of M_B^f can satisfy $Cond(\mathbf{w}_\forall)$, since $\tilde{\forall}_{\mathcal{M}'}[\{f, I\}]$ would then be an empty set! For simplicity of presentation, we do not consider here systems including this schema (in order to provide semantics for such systems, at least one non-designated "inconsistent" truth-value needs to be added).

Now let us explain how the above conditions are derived. A refutation of $(\mathbf{a}_\forall)$ $\forall x \circ \psi \rightarrow \circ\forall x\psi$ would be a valuation v, such that $v[\circ\forall x\psi] \in \mathcal{F}$, but $v[\forall x \circ \psi] \in \mathcal{D}$. $v[\forall x \circ \psi] \in \mathcal{D}$ implies that for every closed term **t**: $v[\circ\psi\{\mathbf{t}/x\}] \in \mathcal{D}$. Then $H_\psi = \{v[\psi\{\mathbf{t}/x\}] \quad \mathbf{t} \in Trm_L^{\mathsf{cl}}\} \subseteq \mathcal{T} \cup \mathcal{F}$. Also, since $v[\circ\forall x\psi] \in \mathcal{F}$, $v[\forall x\psi]) \in \mathcal{I}$. This will be impossible if $\tilde{\forall}[H_\psi] \subseteq \mathcal{T} \cup \mathcal{F}$, and so $v[\forall x\psi] \in \mathcal{F} \cup \mathcal{T}$ and $v[\circ\forall x\psi] \in \mathcal{D}$. Similarly for the case of $(\mathbf{a}_\exists)$.

A refutation of $(\mathbf{o}_\forall)$ $\exists x \circ \psi \rightarrow \circ\forall x\psi$ would be a valuation v, such that $v[\circ\forall x\psi] \in \mathcal{F}$, and $v[\exists x \circ \psi] \in \mathcal{D}$. This means that there is some closed term **t**, such that $v[\psi\{\mathbf{t}/x\}] \in \mathcal{V} \cup \mathcal{F}$. If for every $H \subseteq \mathcal{V}$, such that $H \cap (\mathcal{V} \cup \mathcal{F}) \neq \emptyset$: $\tilde{\forall}[H] \subseteq \mathcal{V} \cup \mathcal{F}$, $v[\forall x\psi] \notin \mathcal{I}$ and so $v[\circ\forall x\psi] \notin \mathcal{F}$. Similarly for the case of $(\mathbf{o}_\exists)$.

A refutation of $(\mathbf{w}_\forall)$ $\circ\forall x\psi \rightarrow \forall x \circ \psi$ would be a valuation v, such that $v[\circ\forall x\psi] \in \mathcal{D}$, but $v[\forall x \circ \psi] \in \mathcal{F}$. It means that for some closed L-term **t**, $v[\circ\psi\{\mathbf{t}/x\}] \in \mathcal{F}$ and thus $v[\circ\psi\{\mathbf{t}/x\}] \in \mathcal{I}$. If for every H, such that $I \in H$: $\tilde{\forall}[H] \subseteq \mathcal{I}$, then it would not be possible to choose $v[\circ\forall x\psi] \in \mathcal{D}$. Recall, however, that in the three-valued Nmatrix $\mathcal{M}_B^f$, for $H = \{f, I\}$: $\tilde{\forall}[H] \subseteq \mathcal{F}$, and so the above condition cannot hold in $\mathcal{M}_B^f$ or any of its refinements. This problem does not arise in the case of $(\mathbf{w}_\exists)$, since the requirement that $H \subseteq \mathcal{I}$ implies $\tilde{\exists}[H] \subseteq \mathcal{I}$ is compatible with the interpretation of $\exists$ in $\mathcal{M}_B^f$.

A refutation of $(\mathbf{r}_\forall)$ $\circ\forall x\psi \rightarrow \exists x \circ \psi$ would be a valuation v, such that $v[\circ\forall x\psi] \in \mathcal{D}$, but $v[\exists x \circ \psi] \in \mathcal{F}$. It means that for every closed L-term **t**, $v[\circ\psi\{\mathbf{t}/x\}] \in \mathcal{F}$ and thus $v[\circ\psi\{\mathbf{t}/x\}] \in \mathcal{I}$. If for every H, such that $H \subseteq \mathcal{I}$: $\tilde{\forall}[H] \subseteq \mathcal{I}$, then it would not be possible to choose $v[\circ\forall x\psi] \in \mathcal{D}$. Similarly for the case of $(\mathbf{r}_\exists)$.

Definition 22 $(\mathbf{B^f s[P]})$ *For* $s \in S$ *and* $P \subseteq PR$, *let* $\mathbf{B}^f s[P]$ *be the system obtained from* $\mathbf{B}^f s$ *by adding the schemata in* P.

Definition 23 $(\mathcal{M}^{\mathbf{f}}_{\mathbf{Bs}}[\mathbf{P}])$ *For* $s \in S$ *and* $P \subseteq PR$, *let* $\mathcal{M}^f_{Bs}[P]$ *be the refinement of* $\mathcal{M}^f_{Bs}$, *which satisfies the refining conditions of the schemata from* P.

Theorem 5 *Let* $P \subseteq PR$ *and* $s \in S$, *such that* $\mathbf{i} \in s$. *Let* Γ *be a set of* L-*sentences and* μ *- an* L-*sentence. Then* $\Gamma \vdash_{\mathbf{B}s[P]} \mu$ *iff* $\Gamma \vdash_{\mathcal{M}^f_{Bs[P]}} \mu$.

Proof: a straightforward modification of the proof of theorem 3. We only have to check that the conditions imposed by the schemata in P are respected by the valuation v defined in the proof.

- Suppose that $\mathbf{a}_\forall \in s$ and let $\forall x\psi$ be an L-sentence, such that $H_\psi = \{v[\psi\{\mathbf{t}/x\}] \mid \mathbf{t} \in Trm^{\mathsf{cl}}_{L'}\} \subseteq \mathcal{V} \cup \mathcal{F}$. Suppose by contradiction that $v[\forall x\psi] = I$. Then, by definition of v, $\forall x\psi \in \Gamma^*$ and $\neg\forall x\psi \in \Gamma^*$. By property 7, $\circ\forall x\psi \notin \Gamma^*$. By the schema $(\mathbf{a}_\forall)$, $\forall x \circ \psi \notin \Gamma^*$. By property 9, there is some closed L'-term **t**, such that $\circ\psi\{\mathbf{t}/x\} \notin \Gamma^*$. By property 6, $\neg \circ \psi\{\mathbf{t}/x\} \in \Gamma^*$. By schema **(i)** and property 4, $\psi\{\mathbf{t}/x\} \in \Gamma^*$ and $\neg\psi\{\mathbf{t}/x\} \in \Gamma^*$ and so $v[\psi\{\mathbf{t}/x\}] = I$, in contradiction to our assumption.
- The proof for $(\mathbf{a}_\exists)$ is similar to the previous case.
- Suppose that $\mathbf{o}_\forall \in s$ and let $\forall x\psi$ be an L-sentence, such that $H_\psi \cap (\mathcal{V} \cup \mathcal{F}) \neq \emptyset$. Suppose by contradiction that $v[\forall x\psi] = I$. Then, by definition of v, $\forall x\psi \in \Gamma^*$ and $\neg\forall x\psi \in \Gamma^*$. By property 7, $\circ\forall x\psi \notin \Gamma^*$. By the schema $(\mathbf{o}_\forall)$, $\exists x \circ \psi \notin \Gamma^*$. By property 9, for every closed L'-term **t**: $\circ\psi\{\mathbf{t}/x\} \notin \Gamma^*$. By property 6, $\neg \circ \psi\{\mathbf{t}/x\} \in \Gamma^*$. By schema **(i)** and property 4, $\psi\{\mathbf{t}/x\} \in \Gamma^*$ and $\neg\psi\{\mathbf{t}/x\} \in \Gamma^*$. Hence, for every closed L'-term **t**: $v[\psi\{\mathbf{t}/x\}] = I$, in contradiction to our assumption. The proof for $(\mathbf{o}_\exists)$ is similar to the previous case.
- Suppose that $\mathbf{w}_\exists \in s$ and let $\exists x\psi$ be an L-sentence, such that $H_\psi \cap \mathcal{I} \neq \emptyset$. Suppose by contradiction that $v[\exists x\psi] \notin \mathcal{I}$. Then either $\exists x\psi \notin \Gamma^*$, or $\neg\exists x\psi \notin \Gamma^*$. Either way, by property 4, $\exists x\psi \wedge \neg\exists x\psi \notin \Gamma^*$. By the schema **(i)**, $\neg \circ \exists x\psi \notin \Gamma^*$. By property 6, $\circ\exists x\psi \in \Gamma^*$. By the schema $(\mathbf{w}_\exists)$, $\forall x \circ \psi \in \Gamma^*$. By property 9, for every closed L'-term **t**: $\circ\psi\{\mathbf{t}/x\} \in \Gamma^*$. By property 7, every closed L'-term **t**: either $\psi\{\mathbf{t}/x\} \notin \Gamma^*$, or $\neg\psi\{\mathbf{t}/x\} \notin \Gamma^*$. Hence for every closed L'-term **t**: either $v[\psi\{\mathbf{t}/x\}] \in \mathcal{T}$ or $v[\psi\{\mathbf{t}/x\}] \in \mathcal{F}$, in contradiction to our assumption.
- Suppose that $\mathbf{r}_\forall \in s$ and let $\forall x\psi$ be an L-sentence, such that $H_\psi \subseteq \mathcal{I}$. Suppose by contradiction that $v[\forall x\psi] \notin \mathcal{I}$. Then either $\forall x\psi \notin \Gamma^*$, or $\neg\forall x\psi \notin \Gamma^*$. Either way, by property 4, $\forall x\psi \wedge \neg\forall x\psi \notin \Gamma^*$. By the schema **(i)**, $\neg\circ\forall x\psi \notin \Gamma^*$. By property 6, $\circ\forall x\psi \in \Gamma^*$. By the schema $(\mathbf{r}_\forall)$, $\exists x \circ \psi \in \Gamma^*$. By property 9, there is some closed L'-term **t**: $\circ\psi\{\mathbf{t}/x\} \in \Gamma^*$. By property 7, either $\psi\{\mathbf{t}/x\} \notin$

Γ^*, or $\neg\psi\{\mathbf{t}/x\} \notin \Gamma^*$. Hence either $v[\psi\{\mathbf{t}/x\}] \in \mathcal{T}$ or $v[\psi\{\mathbf{t}/x\}] \in \mathcal{F}$, in contradiction to our assumption. The proof for $(\mathbf{r}_\exists)$ is similar to the previous case. □

Note how the modularity of the semantic framework of Nmatrices allows for a flexible control of the consistency propagation: one can choose different propagation styles for different quantifiers.

Systems with infinitely-valued semantics

Now we handle first-order extensions of systems including the schemata **(l)**, **(d)** and **(b)**. Again, for $z \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$, $s \in S$ and $P \subseteq PR$, the systems $\mathbf{B^f}z$, $\mathbf{B^f}sz$ and $\mathbf{B^f}sz[P]$ are defined as before.

Definition 24 ($\mathcal{M}^{\mathbf{f}}_{\mathbf{Bz}}$) *The Nmatrix $\mathcal{M}^f_{Bz}$ where $z \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$ for L is defined like the Nmatrix $\mathcal{M}_{Bz}$ (defn. 14) with the addition of the following interpretations of quantifiers for every $H \subseteq P^+(\mathcal{V})$:*

$$\widetilde{\forall}[H] = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$\widetilde{\exists}[H] = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

where:

$$\mathcal{F} = \{f\},\ \mathcal{D} = \{t_i^j \mid i \geq 0, j \geq 0\} \cup \{I_i^j \mid i \geq 0, j \geq 0\}$$

Theorem 6 *Let Γ be a set of L'-sentences and μ - an L'-sentence. For $z \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$: $\Gamma \vdash_{\mathbf{B}^f_z} \mu$ iff $\Gamma \vdash_{\mathcal{M}^f_{Bz}} \mu$.*

Proof: An simple generalization of the proof of (Avron 2006) for the propositional case.

Definition 25 ($\mathcal{M}^{\mathbf{f}}_{\mathbf{Bsz}}[\mathbf{P}]$) *For $z \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$, $s \in S$ and $P \subseteq PR$, the Nmatrix $\mathcal{M}^f_{Bsz}[P]$ is the weakest refinement of $\mathcal{M}^f_{Bsz}$, satisfying the conditions of P. In other words, $\mathcal{M}^f_{Bsz}[P]$ is obtained from $\mathcal{M}^f_{Bz}$ by the following modifications:*

1. *If $(\mathbf{a}_\forall) \in P$, then for every $H \subseteq \mathcal{T}$: $\tilde{\forall}[H] = \mathcal{T}$.*
2. *If $(\mathbf{a}_\exists) \in P$, then for every $H \subseteq \mathcal{T} \cup \mathcal{F}$, such that $H \cap \mathcal{T} \neq \emptyset$: $\tilde{\exists}[H] = \mathcal{T}$.*
3. *If $(\mathbf{o}_\forall) \in P$, then for every $H \subseteq \mathcal{D}$, such that $H \cap \mathcal{T} \neq \emptyset$: $\tilde{\forall}[H] = \mathcal{T}$.*
4. *If $(\mathbf{o}_\exists) \in P$, then for every $H \subseteq \mathcal{V}$, such that $H \cap \mathcal{T} \neq \emptyset$: $\tilde{\exists}[H] = \mathcal{T}$.*
5. *If $(\mathbf{w}_\exists) \in P$, then for every $H \subseteq \mathcal{V}$, such that $H \cap \mathcal{I} \neq \emptyset$: $\tilde{\exists}[H] = \mathcal{I}$.*
6. *If $(\mathbf{r}_\forall) \in P$, then for every $H \subseteq \mathcal{I}$: $\tilde{\forall}[H] = \mathcal{I}$.*
7. *If $(\mathbf{r}_\exists) \in P$, then for every $H \subseteq \mathcal{I}$: $\tilde{\exists}[H] = \mathcal{I}$.*

Theorem 7 *Let Γ be a set of L-sentences and μ - an L-sentence. For $z \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$, $s \in S$ and $P \subseteq PR$: $\Gamma \vdash_{\mathbf{B}^f_{sz}[P]} \mu$ iff $\Gamma \vdash_{\mathcal{M}^f_{Bsz}[P]} \mu$.*

Proof: a simple adaptation of the proof of theorem 6.

For instance, consider the system $\mathbf{B^f cial}[\{(\mathbf{a}_\forall), (\mathbf{a}_\exists)]$ equivalent[4] to da-Costa's predicate calculus C_1^* ((da Costa. 1974)). It includes the schemata of C_1, $(\forall_\mathbf{t})$, $(\forall_\mathbf{f})$, $(\exists_\mathbf{t})$, $(\exists_\mathbf{f})$, $(\mathbf{a}_\forall)$ and $(\mathbf{a}_\exists)$. Also, if A and B are α-equivalent, or one is obtained from the other by the suppression of void quantifiers, then $A \to B$ and $B \to A$ are axioms of C_1^* (this is captured by the $(\mathbf{v}_\alpha)$ axiom).

The Nmatrix $\mathcal{M}^f_{Bcial[\{(\mathbf{a}_\forall),(\mathbf{a}_\exists)\}]}$ is defined similarly to $\mathcal{M}_{Bcial}$, with the addition of the following interpretation functions:

$$\widetilde{\forall}[H] = \begin{cases} \mathcal{T} & \text{if } H \subseteq \mathcal{T} \\ \mathcal{F} & \text{if } H \cap \mathcal{F} \neq \emptyset \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\widetilde{\exists}[H] = \begin{cases} \mathcal{T} & \text{if } H \cap \mathcal{T} \neq \emptyset \text{ and } H \subseteq (\mathcal{T} \cup \mathcal{F}) \\ \mathcal{F} & \text{if } H \subseteq \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

Using the above Nmatrix, it is easy to semantically verify the following theorem of da Costa.:

Theorem 8 *((da Costa. 1974))*

$\nvdash_{C_1^*} \neg\exists x \neg A(x) \leftrightarrow \forall x A(x)$
$\nvdash_{C_1^*} \neg\forall x \neg A(x) \leftrightarrow \exists x A(x)$
$\nvdash_{C_1^*} \neg\exists x A(x) \leftrightarrow \forall x \neg A(x)$
$\nvdash_{C_1^*} \exists x \neg A(x) \leftrightarrow \neg\forall x A(x)$

Let us show for the first case. It is easy to see that an L-structure S and an $\mathcal{M}^f_{Bcial[\{(\mathbf{a}_\forall),(\mathbf{a}_\exists)\}]}$-legal S-valuation v can be defined, which satisfies:

- For every closed term **t** of L: $v[p(\mathbf{t})] \in \mathcal{I}$ and $v[\neg p(\mathbf{t})] \in \mathcal{I}$ (for the legality of v in $\mathcal{M}^f_{Bcial[\{(\mathbf{a}_\forall),(\mathbf{a}_\exists)\}]}$, if $v[p(\mathbf{t})] = I_i^j$, we take $v[\neg p(\mathbf{t})] = I_i^{j+1}$).
- $v[\forall x p(x)] \in \mathcal{I}$.
- $v[\exists x \neg p(x)] \in \mathcal{T}$.
- $v[\neg\exists x \neg p(x)] \in \mathcal{F}$.

Thus, $\nvdash_{\mathbf{B^f cial}[\{(\mathbf{a}_\forall),(\mathbf{a}_\exists)]} \forall x p(x) \to \neg\exists x \neg p(x)$.

Nmatrices as a proof-theoretical tool

In this section we demonstrate how the tool of Nmatrices can be applied to prove a proof-theoretical property of the systems discussed above.

Definition 26 (Logical indistinguishability) *Two sentences A, B are* logically indistinguishable *in a system* C, *denoted by $A \equiv_\mathsf{C} B$, if $\vdash_\mathsf{C} A \leftrightarrow B$ implies $\vdash_\mathsf{C} \psi(A) \leftrightarrow \psi(B)$ for any formula ψ.*

Theorem 9 *Let A, B be L-sentences. For any system* C *weaker than $\mathbf{B^f ciaeb}[PR]$: $A \equiv_\mathsf{C} B$ iff $A \sim^v_\alpha B$.*

[4] In (Carnielli & Marcos 2002; Carnielli 2006) it is shown that C_1 and **Bcial** (which is called there **Cila**) are equivalent in the sense that **Bcial** is a conservative extension of C_1 and it is also interpretable in C_1.

Proof: Let C be a system weaker than $\mathbf{B^f ciaeb}[PR]$. Suppose $A \sim^v_\alpha B$. Then since $\psi(A) \rightarrow \psi(B)$ and $\psi(B) \rightarrow \psi(A)$ are axioms of C for any ψ, $A \equiv_{\mathsf{C}} B$.
Conversely, suppose $A \not\sim^v_\alpha B$. Let q be an atomic closed formula which does not occur in A or B. It is easy to see that there exists an L-structure S and an S-valuation v legal in $\mathcal{M}^f_{Bciaeb}[PR]$, satisfying:

$$v[q] = I$$
$$v[q \rightarrow (B \rightarrow B)] = I$$
$$v[\neg(q \rightarrow (B \rightarrow B))] = I$$
$$v[q \rightarrow (A \rightarrow A)] = t$$
$$v[\neg(q \rightarrow (A \rightarrow A))] = f$$

Thus $\neg(q \rightarrow (B \rightarrow B)) \not\vdash_{\mathcal{M}^f_{Bciaeb}[PR]} \neg(q \rightarrow (A \rightarrow A))$. By theorem 5:

$$\neg(q \rightarrow (B \rightarrow B)) \not\vdash_{\mathbf{Bciaeb}[PR]} \neg(q \rightarrow (A \rightarrow A))$$

Since C is weaker than $\mathbf{B^f ciaeb}[PR]$:

$$\neg(q \rightarrow (B \rightarrow B)) \not\vdash_{\mathsf{C}} \neg(q \rightarrow (A \rightarrow A))$$

Hence, $A \not\equiv_{\mathsf{C}} B$. $\square$

Corollary 1 *Two L-sentences A, B are logically indistinguishable in $\mathbf{B^f cial}[(\mathbf{a}_\forall), (\mathbf{a}_\exists)]$ iff $A \sim^v_\alpha B$.*

Proof: Follows directly from the above theorem, since **(b)** implies **(l)**.

Remark: Note that the above result is not straightforwardly extended to other systems discussed above. For instance, it can be shown that $\circ(A \rightarrow A)$ and $\circ(B \rightarrow B)$ are logically indistinguishable in $\mathbf{B}^f$**cio** (and already in **Bcio**) for any formulas A and B (see (Carnielli & Marcos 2002) for further details).

These results are deeply connected to the results of (Mortensen 1980; Carnielli & Marcos 2002) on *algebraization* of LFIs. The idea of algebraization is in partitioning the algebra of formulas in a given logic into a quotient algebra of equivalence classes by some relation $\sim$ holding between formulas. The relation should satisfy the requirement of *intersubstitutability*: if $A \sim B$, then $\psi(A) \sim \psi(B)$ for any ψ. The most standard way of algebraizing a given logic is by using an equivalence relation $\sim$ induced by the consequence relation of the logic. In Lindenbaum-Tarski algebraization for a system $\mathcal{L}$, one sets $A \sim B$ iff $\vdash_{\mathcal{L}} A \leftrightarrow B$. Hence, to confirm that two formulas are intersubstitutable, one has to show that they are logically indistinguishable. However, in most LFIs, and in particular in all of the systems discussed above, $\vdash A \leftrightarrow B$ *does not* imply $\vdash \psi(A) \leftrightarrow \psi(B)$, and so no Lindenbaum-Tarski-style algebraizations for them are available.
One of the most investigated systems in this context is **Bcial** (which is called **Cila** in (Carnielli & Marcos 2002; Carnielli 2006) and is equivalent to da-Costa's C_1 in the sense explained above). Several attempts have been made to find some (more general) congruence relation $\sim$ on its set of formulas. The final blow to this search was delivered by Mortensen ((Mortensen 1980)), who showed, using valuation semantics, that no non-trivial[5] quotient algebra is definable for **Bcial**[6] or any logic weaker than **Bcial**. Theorem 9 extends this result in two aspects: (i) extension to first-order languages, and (ii) extension to the schemata **(e)** and **(b)**. The use of Nmatrices also makes the proof simpler.

Summary and further research

Nmatrices are an attractive semantic framework for characterizing paraconsistent logics due to their simplicity and modularity. In this paper the modular approach based on Nmatrices was extended to first-order and applied to characterize a large family of first-order LFIs. We provided three-valued modular non-deterministic semantics for most of the LFIs and infinite-valued semantics for the rest of them. We showed that the provided semantics can also be easily tailored to different styles of consistency propagation. Finally, we demonstrated the usefulness of the tool of Nmatrices for proving an important property of the discussed LFIs.

It is clear from our case-study that the presented method has a large range of applications far beyond the LFIs discussed in this paper. The semantic framework of Nmatrices needs to be extended to more complex proof systems, more general quantifiers and higher-order languages. Another direction is to increase the degree of non-determinism of Nmatrices by allowing also non-deterministic interpretations of predicate and function symbols of the language. Such a generalization seems particularly useful for representing fuzzy notions. In addition we intend to investigate concrete applications of Nmatrices. The fields for which Nmatrices seem particularly useful are inconsistency-tolerant reasoning, commonsense reasoning, reasoning under uncertainty and representation of incomplete and/or inconsistent information. Another promising direction is combining Nmatrices and probabilistic reasoning. The first step is to generalize Nmatrices by allowing weights to be assigned to each of the optional truth-values for $v[\psi]$ (so that the sum of these weights is 1).

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[5]By a trivial quotient algebra we mean an algebra defined by a congruence relation $\sim$, such that $A \sim B$ iff A and B are identical.

[6]More precisely, the result of (Mortensen 1980) was formulated for C_1.

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