Non-deterministic Multi-valued Logics - A Tutorial

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Abstract

Non-deterministic multi-valued matrices (Nmatrices) are a new, fruitful and quickly expanding field of research first introduced a few years ago. Since then it has been rapidly developing towards a foundational logical theory and has found numerous applications. The novelty of Nmatrices is in extending the usual many-valued algebraic semantics of logical systems by importing the idea of non-deterministic computations, and allowing the truth-value of a formula to be chosen non-deterministically out of a given set of options. Nmatrices have proved to be a powerful tool, the use of which preserves all the advantages of ordinary many-valued matrices, but is applicable to a much wider range of logics. Indeed, there are many useful (propositional) non-classical logics, which have no finite multi-valued characteristic matrices, but do have finite Nmatrices, and thus are decidable. In this tutorial we introduce the reader to the concept of Nmatrices, and demonstrate their usefulness by providing modular non-deterministic semantics for a well-known family of logics for reasoning under uncertainty.

1. Introduction

The principle of truth-functionality is a basic principle in many-valued logic. According to this principle, the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas. However, real-world information is inescapably incomplete, uncertain, vague, imprecise or inconsistent. The sources of non-determinism may vary: partially unknown information, faulty behavior of devices and ambiguity of natural languages are just a few cases in point. Truth-functional semantics, induced by ordinary many-valued matrices, cannot capture non-deterministic behavior. A possible solution is borrowing the idea of non-deterministic computations from computability theory and applying it to evaluations of formulas. This leads to the concept of *non-deterministic matrices (Nmatrices)*, introduced in [3]. Nmatrices are a natural generalization of ordinary matrices, where the truth-value of a formula is chosen non-deterministically from some set of options. There are many natural motivations for introducing non-determinism into the truth-tables of logical connectives. We discuss some of them below. They give rise to two different ways in which non-determinism can be incorporated: the *dynamic* and the *static*. In both the value $v(\diamond(\psi_1, \ldots, \psi_n))$ assigned to the formula $\diamond(\psi_1, ..., \psi_n)$ is selected from a set $\widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ (where $\widetilde{\diamond}$ is the interpretation of $\diamond$). In the dynamic approach this selection is made separately and independently for each tuple $\langle\psi_1, \ldots, \psi_n\rangle$. Thus the choice of one of the possible values is made at the lowest possible (local) level of computation, or on-line, and $v(\psi_1), \ldots, v(\psi_n)$ do not uniquely determine $v(\diamond(\psi_1, \ldots, \psi_n))$. In contrast, in the static semantics this choice is made globally, system-wide, and the interpretation of $\diamond$ is a function, which is selected before any computation begins. This function is a "determinisation" of the non-deterministic interpretation $\widetilde{\diamond}$, to be applied in computing the value of any formula under the given valuation. This limits non-determinism, but still leaves the freedom of choosing the above function among all those that are compatible with the non-deterministic interpretation $\widetilde{\diamond}$ of $\diamond$.

Below we present some cases in which the need for non-deterministic interpretation of connectives naturally arises.
Inherent non-deterministic behavior of circuits: Nmatrices can be applied to model non-deterministic behavior of various elements of electrical circuits in cases when the measured behavior of a circuit may deviate from the expected behavior (for reasons such as faulty gates, presence of disturbing noise sources, temperature changes and other conditions). The exact mathematical form of the relation between input and output in a given logical gate is not always known, and so it can be approximated by a non-deterministic truth-table. For instance, suppose that the circuit C given in Figure 1 consists of a standard OR gate and a faulty AND gate, which responds correctly if the inputs are similar, and unpredictably otherwise. The behavior of the gate can be described by the following truth-table, equipped with the *dynamic* semantics:

		AND
t	**t**	$\{\mathbf{t}\}$
t	**f**	$\{\mathbf{f}, \mathbf{t}\}$
f	**t**	$\{\mathbf{f}, \mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$

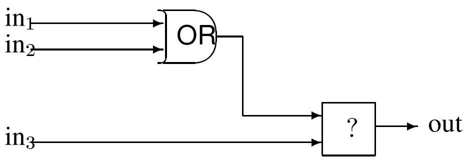

Figure 1. The circuit C

Computation with unknown functions: Let us return to Figure 1, and suppose that this time it represents a circuit about which only some partial information is known. Namely, it is known that the gate labelled with "?" is either a XOR gate or an OR gate, but it is not known which one. Thus the function describing the second gate is deterministic, but unknown to us. This situation can be represented by using the following non-deterministic truth-table for $\vee$, equipped with the *static* semantics:

		$\vee$
t	**t**	$\{\mathbf{t},\mathbf{f}\}$
t	**f**	$\{\mathbf{t}\}$
f	**t**	$\{\mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$

Verification with unknown evaluation models: There are two well-known three-valued logics for describing different types of computational models. The first, which captures parallel evaluation, was described in the context of computational mathematics by Kleene ([11]); the second, programming oriented method, in which evaluation proceeds sequentially, was proposed by McCarthy ([13]). Below are the corresponding truth-tables for $\vee$:

(Kleene)

$\tilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	**t**
t	**t**	**t**	**t**

(McCarthy)

$\tilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	**e**
t	**t**	**t**	**t**

Now suppose we are sending an expression $\psi \vee \varphi$ for evaluation to some distant computer, for which it is not known whether it performs parallel or sequential computations. Hence we know that $\psi \vee \varphi$ will be evaluated using a deterministic function $\tilde{\vee}$, defined by either Kleene's or McCarthy's truth-table for $\vee$, but we have no information which of the two. Again this can be captured by using a *static* interpretation of the following "truth-table":

$\tilde{\vee}$	**f**	**e**	**t**
f	$\{\mathbf{f}\}$	$\{\mathbf{e}\}$	$\{\mathbf{t}\}$
e	$\{\mathbf{e}\}$	$\{\mathbf{e}\}$	$\{\mathbf{e},\mathbf{t}\}$
t	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

According to this static interpretation, the function $f_\vee : \{\mathbf{t},\mathbf{f},\mathbf{e}\}^2 \to \{\mathbf{t},\mathbf{f},\mathbf{e}\}$ used by the computer satisfies either $f_\vee(\mathbf{t},\mathbf{e}) = \mathbf{t}$ (in case the computation is parallel) or $f_\vee(\mathbf{t},\mathbf{e}) = \mathbf{e}$ (in case it is sequential). However, it is not known which of these two conditions is satisfied.

Nmatrices have already proved to be a powerful tool, the use of which preserves all the advantages of ordinary many-valued matrices, but is applicable to a much wider range of logics. Indeed, there are many useful (propositional) non-classical logics, which have no finite many-valued characteristic matrices, but *do* have finite Nmatrices, and thus are decidable. Another important advantage of the framework of Nmatrices is its *modularity*. Each syntactic rule in a proof system corresponds to a certain semantic condition, leading to a refinement of some basic Nmatrix. In many cases the semantics of a complex system can be obtained by straightforwardly combining the semantic effects of each of the added rules. As a result, the semantic effect of a syntactic rule can be analyzed separately. This is impossible in standard multi-valued matrices, where the semantics of a system can only be presented as a whole. In this tutorial we present the basic theory of Nmatrices and demonstrate the general method by providing modular non-deterministic semantics for a family of paraconsistent logics of da Costa. For a more extensive survey of Nmatrices and their applications, see [5].

2. Logics and Matrices

As noted above, in many cases the use of Nmatrices leads to simple and finite characterizations of *logics* which have no finite characteristic standard *matrices*. Before moving further, we define in precise terms what we mean by a 'logic', and remind the reader of the basic definitions of standard matrices. In what follows, $\mathcal{L}$ is a propositional language and $Frm_{\mathcal{L}}$ is its set of wffs.

A logic is essentially a *consequence relation* between sets of formulas and formulas in some language $\mathcal{L}$, satisfying some natural properties. This relation determines which propositions should follow (or be inferred) from which sets of assumptions.

Definition 2.1 A (Tarskian) consequence relation (tcr) $\vdash$ for a language $\mathcal{L}$ is a binary relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, that satisfies:

strong reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
transitivity (*cut*): if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$.

A tcr $\vdash$ for $\mathcal{L}$ is *structural* if for every uniform $\mathcal{L}$-substitution σ and every Γ and ψ, if $\Gamma \vdash \psi$ then $\sigma(\Gamma) \vdash \sigma(\psi)$. $\vdash$ is *consistent* (or *non-trivial*) if there exist some non-empty Γ and some ψ s.t. $\Gamma \not\vdash \psi$. *A propositional logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash_{\mathbf{L}}$ is a structural and consistent tcr for $\mathcal{L}$.

There are several ways of defining consequence relations for a language $\mathcal{L}$. Two of the most common ones are the proof-theoretical and the model-theoretical approaches. In the former, the definition of a consequence relation is based on some notion of a *proof* in some formal calculus. In the latter approach, the definition is based on a notion of a *semantics* for $\mathcal{L}$. The general notion of an abstract semantics is rather opaque. One usually starts by defining a notion of a *valuation* as a certain type of partial functions from $Frm_{\mathcal{L}}$ to some set (of truth-values). Then ones defines what it means for a valuation to *satisfy* a formula (or to be a *model* of a formula). A semantics is then some set S of valuations, and the consequence relation induced by S is defined as follows: $\Gamma \vdash_{\mathsf{S}} \psi$ if every *total* valuation in S which satisfies all the formulas in Γ, satisfies ψ as well (note that this always defines a tcr). We say that a semantics S is *analytic* if every partial valuation in S, whose domain is some set of formulas closed under subformulas, can be extended to a full (i.e. total) valuation in S. This implies that the exact identity of the language $\mathcal{L}$ is not important, since analycity allows us to focus on some subset of its connectives (see Note 2.3 below for another important consequence of analycity). Both ordinary many-valued semantics and the semantics based on Nmatrices can easily be shown to be analytic. However, as we shall see below, this is not always the case in general.

One of the most well-known semantic methods for defining propositional logics is by using many-valued (deterministic) matrices (see, e.g. [14] for a comprehensive survey):

Definition 2.2 A *matrix* for $\mathcal{L}$ is a tuple $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding function $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$.
A *valuation* in $\mathcal{P}$ is a function $v : Frm_{\mathcal{L}} \to \mathcal{V}$, such that for each n-ary connective $\diamond$ of $\mathcal{L}$: $v(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$. A valuation v in $\mathcal{P}$ *satisfies* an $\mathcal{L}$-formula ψ if $v(\psi) \in \mathcal{D}$. A valuation v in $\mathcal{P}$ *satisfies* a set of $\mathcal{L}$-formulas Γ if it satisfies every formula in Γ.
The *consequence relation induced by a matrix* $\mathcal{P}$ is defined as follows: $\Gamma \vdash_{\mathcal{P}} \psi$ if whenever a valuation in $\mathcal{P}$ satisfies all the formulas of Γ, it satisfies also ψ. $\mathcal{P}$ is a *characteristic matrix* for a logic $\mathbf{L}$ if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{P}}$. For a family of matrices F, we say that $\Gamma \vdash_F \psi$ if $\Gamma \vdash_{\mathcal{P}} \psi$ for every $\mathcal{P}$ in F.

It is easy to verify that for every matrix $\mathcal{P}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{P}} = \langle \mathcal{L}, \vdash_{\mathcal{P}} \rangle$ is a propositional logic. In the converse direction, every propositional logic is induced by some family of matrices ([15]). Whenever a propositional logic is also *uniform*[1], it can be characterized by a *single* matrix ([12]). However, these matrices are often infinite and are sometimes hard to find and use. As we show below, in many cases *finite* characteristic Nmatrices exist for logics which have only infinite characteristic matrices.

Note 2.3 (Analycity of matrices) It is easy to show that matrices-based semantics is *analytic*, i.e., any partial valuation in a matrix $\mathcal{P}$ for $\mathcal{L}$, which is defined on a set of $\mathcal{L}$-formulas closed under subformulas, can be extended to a full valuation in $\mathcal{P}$. Due to this property, $\vdash_{\mathcal{P}}$ is decidable whenever $\mathcal{P}$ is a finite matrix. Moreover, analycity guarantees semi-decidability of non-theoremhood even if a matrix $\mathcal{P}$ is infinite, provided that $\mathcal{P}$ is effective (i.e, the set of truth-values is countable, the interpretation functions of the connectives are computable, and the set of designated truth-values is decidable). Note that this implies decidability in case $\vdash_{\mathcal{P}}$ also has a corresponding sound and complete proof system.

One of the main shortcomings of matrix-based semantics is its lack of modularity with respect to proof systems. To use this type of semantics, the rules and axioms of a system which are related to a given connective should be considered as a whole, and there is no method for separately determining the semantic effects of each rule alone. Consider, for instance, the standard Gentzen-type rules for negation:

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}\ (\neg \Rightarrow) \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}\ (\Rightarrow \neg)$$

The first rule corresponds to the classical explosive non-contradiction principle ($\varphi, \neg\varphi \vdash \psi$), implying that the negation of a "true" proposition is a "false" proposition. The second rule corresponds to the law of excluded middle ($\vdash \varphi, \neg\varphi$), implying that the negation of a "false" proposition must be "true". Thus the corresponding truth-table is the classical one, and the rules $(\neg \Rightarrow)$ and $(\Rightarrow \neg)$ dictate the first and the second lines of the truth-table respectively:

	$\neg$
t	**f**
f	**t**

Suppose now that the rule $(\Rightarrow \neg)$ is discarded. The resulting system has no finite characteristic matrix (this is a special case of Theorem 3.6 below). Intuitively, this is due to underspecification: there is now no rule concerning $\tilde{\neg}\mathbf{f}$. This problem can be solved by moving to Nmatrices: if $\tilde{\neg}\mathbf{f}$ is not dictated by any of the rules, we should put there all possible options. Indeed, it can be shown that the obtained system has a characteristic two-valued Nmatrix, where:

	$\neg$
t	{**f**}
f	{**t**,**f**}

Hence in contrast to standard matrices, the semantics of Nmatrices does allow a high degree of modularity, and in many cases the effect of each syntactic rule or axiom alone can easily be determined.

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1 We say that a logic $\langle \mathcal{L}, \vdash \rangle$ is *uniform* if $\Gamma \vdash \psi$ whenever $\Gamma, \Delta \vdash \psi$, $\mathsf{Atoms}(\Gamma \cup \{\psi\}) \cap \mathsf{Atoms}(\Delta) = \emptyset$, and there exists some φ such that $\Delta \not\vdash \varphi$.

3. Introducing Nmatrices

The results in this section are mainly taken from [3, 5].

Definition 3.1 1. A *non-deterministic matrix (Nmatrix)* for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$.

2. A *dynamic valuation* in an Nmatrix $\mathcal{M}$ for $\mathcal{L}$ is any function $v : Frm_{\mathcal{L}} \to \mathcal{V}$, which satisfies: $v(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

3. A *static valuation* in $\mathcal{M}$ is a dynamic valuation which satisfies also the following compositionality principle: $v(\diamond(\psi_1, \ldots, \psi_n)) = v(\diamond(\varphi_1, \ldots, \varphi_n))$ if $v(\psi_i) = v(\varphi_i)$ $(i = 1 \ldots n)$.

Ordinary (deterministic) multi-valued matrices correspond to the case when each $\tilde{\diamond}$ is a function taking singleton values only (then it can be treated as a function $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$). Also note that unlike the static semantics, the dynamic semantics is non-truth-functional: the truth-value of a complex formula is not completely determined by the truth-values of its subformulas. However, in the deterministic case there is no difference between the static and dynamic valuations.
The property of analycity is as easy to prove for Nmatrices as it is for standard matrices, and so Note 2.3 applies to Nmatrices as well.

Example 3.2 (2-valued Nmatrices) Let $\mathcal{L}$ be a language over $\{\vee, \wedge, \supset, \neg\}$, $\mathcal{V} = \{\mathbf{f}, \mathbf{t}\}$, $\mathcal{D} = \{\mathbf{t}\}$. Suppose that $\vee, \wedge$ and $\supset$ are interpreted classically, while $\neg$ satisfies the law of excluded middle ($\neg\varphi \vee \varphi$), but not necessarily the law of contradiction ($\neg(\varphi \wedge \neg\varphi)$). This leads to the Nmatrix $\mathcal{M}^2 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where $\mathcal{O}$ is:

		$\tilde{\vee}$	$\tilde{\wedge}$	$\tilde{\supset}$
t	**t**	{**t**}	{**t**}	{**t**}
t	**f**	{**t**}	{**f**}	{**f**}
f	**t**	{**t**}	{**f**}	{**t**}
f	**f**	{**f**}	{**f**}	{**t**}

	$\tilde{\neg}$
t	{**t**, **f**}
f	{**t**}

Note that any dynamic valuation satisfies $\neg\psi \vee \psi$ but not necessarily $\psi \supset \neg\neg\psi$. Any static valuation satisfies both $\neg\psi \vee \psi$ and $\psi \supset \neg\neg\psi$ (it admits only two interpretations of $\neg$: the classical one and $\lambda x.\mathbf{t}$). However, it satisfies neither $\neg\neg\psi \supset \psi$ nor $(\varphi \wedge \neg\varphi) \supset \psi$ (take $\tilde{\neg} = \lambda x.\mathbf{t}$, $v(\varphi) = \mathbf{t}$, $v(\psi) = \mathbf{f}$).

Example 3.3 (3-valued Nmatrices) Consider the following 3-valued Nmatrices $\mathcal{M}_3^L$ and $\mathcal{M}_S^3$ for the language $\mathcal{L} = \{\wedge, \vee, \supset, \neg\}$. In both we have $\mathcal{V} = \{\mathbf{f}, \top, \mathbf{t}\}$ and $\mathcal{D} = \{\top, \mathbf{t}\}$. Also the interpretations of disjunction, conjunction and implication are the same in both of them, corresponding to those in classical logic. For $\vee$, e.g., we have:

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{V} - \mathcal{D} & \text{if } a, b \in \mathcal{V} - \mathcal{D} \end{cases}$$

However, negation is interpreted differently: more liberally in $\mathcal{M}_L^3$, and more strictly in $\mathcal{M}_S^3$:

$\mathcal{M}_L^3$:

	$\tilde{\neg}$
t	**f**
$\top$	$\mathcal{V}$
f	**t**

$\mathcal{M}_S^3$:

	$\tilde{\neg}$
t	**f**
$\top$	$\mathcal{D}$
f	**t**

Nmatrices can be used to characterize logics which are defined in syntactic terms. In such cases finding a finite characteristic Nmatrix is useful for showing that some formulas are or are not theorems of the logic (by showing that they are or are not valid in the corresponding Nmatrix). The decidability of the logic also immediately follows.

Definition 3.4 Let $\mathcal{M}$ be an Nmatrix for $\mathcal{L}$. The notion of a dynamic (static) valuation in $\mathcal{M}$ satisfying a formula and a set of formulas is defined as in Defn. 2.2. A formula is *dynamically (statically) valid* in $\mathcal{M}$ if every dynamic (static) valuation in $\mathcal{M}$ satisfies it. $\vdash^d_{\mathcal{M}}$ ($\vdash^s_{\mathcal{M}}$), the *dynamic (static) consequence relation induced by* $\mathcal{M}$, is defined as follows: $\Gamma \vdash^d_{\mathcal{M}} \psi$ ($\Gamma \vdash^s_{\mathcal{M}} \psi$) if every dynamic (static) valuation in $\mathcal{M}$ which satisfies Γ, also satisfies ψ. $\Gamma \vdash^d_{\mathcal{M}} \psi$ ($\Gamma \vdash^s_{\mathcal{M}} \psi$), if every dynamic (static) valuation v in $\mathcal{M}$ which satisfies Γ, also satisfies ψ. $\mathcal{M}$ is *dynamically (statically) characteristic* for a logic $\mathbf{L}$ if $\vdash^d_{\mathcal{M}} = \vdash_{\mathbf{L}}$ ($\vdash^s_{\mathcal{M}} = \vdash_{\mathbf{L}}$).

Obviously, the static consequence relation includes the dynamic one, i.e. $\vdash^d_{\mathcal{M}} \subseteq \vdash^s_{\mathcal{M}}$. Also, for a deterministic matrix $\mathcal{M}$: $\vdash^s_{\mathcal{M}} = \vdash^d_{\mathcal{M}}$.

Example 3.5 The following examples are taken from [2]:

1. The dynamic semantics and the static semantics of the Nmatrix $\mathcal{M}^2$ from Example 3.2 generate two different paraconsistent logics: the logic $CLuN$ of Batens ([6]) and Carnot's logic CAR ([10]) respectively.
2. It is interesting to note that the dynamic semantics of two different Nmatrices $\mathcal{M}_3^L$ and $\mathcal{M}_S^3$ from Example 3.3 both characterize the basic paraconsistent logic C_{min} ([8]), which was originally defined only proof-theoretically.

The dynamic semantics of finite Nmatrices can characterize logics which have no corresponding finite matrices:

Theorem 3.6 *Let $\mathcal{M}$ be a two-valued Nmatrix which has at least one proper non-deterministic operation. Then there is no finite family of finite ordinary matrices F, such that*

$\vdash^d_{\mathcal{M}} = \vdash_F$. *If in addition $\mathcal{M}$ includes the classical implication, then there is no finite family of ordinary matrices F, such that $\vdash^d_{\mathcal{M}} \psi$ iff $\vdash_F \psi$.*

The expressive power of static semantics, however, is the same as of families of ordinary matrices:

Theorem 3.7 *For every (finite) Nmatrix $\mathcal{M}$, there is a (finite) family of ordinary matrices, such that $\vdash^s_{\mathcal{M}} = \vdash_F$.*

For this reason our focus henceforth will be on the dynamic semantics, and we will write simply $\vdash_{\mathcal{M}}$ instead of $\vdash^d_{\mathcal{M}}$.

The method for providing modular non-deterministic semantics to logical systems described in the next section is based on two main operations on Nmatrices: *expansion* and *refinement*. Intuitively, expanding a given Nmatrix means "duplicating" its elements, i.e. constructing a new Nmatrix which is completely equivalent to the original one by replacing each element by some nonempty set of its "copies", and defining the operations in the new Nmatrix to be "the same" as in the original one, without distiguishing between two copies of the same element. Refining a given Nmatrix usually leads to deleting some of the options in its truth-tables, thus reducing the level of its non-determinism. This can be formalized as follows:

Definition 3.8 Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}$.

1. $\mathcal{M}_1$ is a *refinement*[2] of $\mathcal{M}_2$ if $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, and $\tilde{\diamond}_{\mathcal{M}_1}(\overline{x}) \subseteq \tilde{\diamond}_{\mathcal{M}_2}(\overline{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every tuple $\overline{x} \in \mathcal{V}_1^n$.
2. Let F be a function that assigns to each $x \in \mathcal{V}$ a nonempty set $F(x)$, such that $F(x_1) \cap F(x_2) = \emptyset$ if $x_1 \neq x_2$. The *F-expansion* of $\mathcal{M}_1$ is the following Nmatrix $\mathcal{M}_1^F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F \rangle$: $\mathcal{V}_F = \bigcup_{x \in \mathcal{V}} F(x)$, $\mathcal{D}_F = \bigcup_{x \in \mathcal{D}} F(x)$, and $\tilde{\diamond}_{\mathcal{M}_1^F}(y_1, \ldots, y_n) = \bigcup_{z \in \tilde{\diamond}_{\mathcal{M}_1}(x_1,\ldots,x_n)} F(z)$ whenever $\diamond$ is an n-ary connective of $\mathcal{L}$, and $x_i \in \mathcal{V}$, $y_i \in F(x_i)$ for every $1 \leq i \leq n$. We say that $\mathcal{M}_2$ is an expansion of $\mathcal{M}_1$ if $\mathcal{M}_2$ is the F-expansion of $\mathcal{M}_1$ for some function F.

Expanding an Nmatrix $\mathcal{M}$ does not change the original logic (induced by $\mathcal{M}$), while refining $\mathcal{M}$ may do so:

Proposition 3.9 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be Nmatrices for $\mathcal{L}$.*

1. *If $\mathcal{M}_1$ is an expansion of $\mathcal{M}_2$, then $\vdash_{\mathcal{M}_1} = \vdash_{\mathcal{M}_2}$.*
2. *If $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.*

Example 3.10 Consider again the Nmatrices $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$ from Example 3.3. $\mathcal{M}^3_S$ is a refinement of $\mathcal{M}^3_L$. The classical two-valued matrix $\mathcal{M}_{cl}$ is a refinement of $\mathcal{M}^3_S$ (and also of $\mathcal{M}^3_L$). The $\{\vee, \wedge, \supset\}$-fragment of $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$ is an expansion of (the positive fragment) of $\mathcal{M}_{cl}$.

Finally, we note that in this tutorial Nmatrices are discussed on the purely propositional level. However, no semantic framework can be considered really useful unless it can be naturally extended to the first-order level and beyond. See e.g. [4] for an extension of the theory of Nmatrices to languages with quantifiers.

4. Nmatrices for Paraconsistent Logics

Let us now demonstrate the modular method of Nmatrices by providing non-deterministic semantics for a family of well-known *paraconsistent logics*. The results of this section are mainly taken from [1].

In classical logic the explosive non-contradiction principle ($\varphi, \neg\varphi \vdash \psi$) allows us to derive any formula from a contradiction. This makes any inconsistent theory trivial, and so no sensible reasoning is possible in the presence of contradictions. A paraconsistent logic is a logic which allows non-trivial inconsistent theories. One of the oldest and best known approaches to the problem of designing useful paraconsistent logics is da Costa's approach. It is based on two main ideas. The first is to limit the applicability of the principle $\varphi, \neg\varphi \vdash \psi$ to the case where φ is "consistent". The second is to express this assumption of consistency of φ within the language. The easiest way to implement these ideas is to include in the language a special connective $\circ$, with the intended meaning of $\circ\varphi$ being "φ is consistent". Then one can explicitly add the assumption of the consistency of φ to the problematic rule, getting the rule $\varphi, \neg\varphi, \circ\varphi \vdash \psi$ instead. Other rules concerning $\neg$ and $\circ$ can then be added, leading to a large family of logics known as "Logics of Formal Inconsistency" (LFIs, see, e.g. [7]). Although the syntactic formulations of the propositional LFIs are relatively simple, the known semantic interpretations are more complicated: the vast majority of LFIs cannot be characterized by means of finite multi-valued matrices. The first systems of da Costa have been introduced only proof-theoretically, and only some years later bivaluations semantics and possible translations semantics have been proposed for their interpretation (see, e.g. [8]). However, no general theorem of analycity is available for this type of semantics (in contrast to Nmatrices), and it has to be proven from scratch for any instance of them. Moreover, unlike Nmatrices-based semantics, it is not clear how to extend them to the first-order case.

We will now provide modular non-deterministic semantics for a selection of LFIs considered in the literature. We start with the basic paraconsistent logic B over $\mathcal{L}_C = \{\wedge, \vee, \supset, \neg, \circ\}$. This is the minimal logic in $\mathcal{L}_C$ which extends positive classical logic and satisfies the conditions $\vdash \neg\varphi \vee \varphi$ and $\circ\varphi, \neg\varphi, \varphi \vdash \psi$. Let HCL^+ be some ax-

2 This notion is called a 'simple refinement' in [1, 5], while 'refinement' is used for a more general notion.

iomatization of positive classical logic with MP as the only rule of inference. The corresponding Hilbert-style system $\mathbf{B}$ for the logic B is obtained by adding the following two axioms to HCL^+:

$$(\mathbf{t})\ \neg\varphi \vee \varphi \qquad (\mathbf{b})\ (\circ\varphi \wedge \varphi \wedge \neg\varphi) \supset \psi$$

The basic system $\mathbf{B}$ can be further extended by various schemata. Below we use some examples considered in the literature of LFIs:

(c) $\neg\neg\varphi \supset \varphi$	**(e)** $\varphi \supset \neg\neg\varphi$
(i$_1$) $\neg\circ\varphi \supset \varphi$	**(i$_2$)** $\neg\circ\varphi \supset \neg\varphi$
(k$_1$) $\circ\varphi \vee \varphi$	**(k$_2$)** $\circ\varphi \vee \neg\varphi$
(a$_\sharp$) $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi\sharp\psi)$	for $\sharp \in \{\wedge, \vee, \supset\}$
(o$_\sharp$) $(\circ\varphi \vee \circ\psi) \supset \circ(\varphi\sharp\psi)$	for $\sharp \in \{\wedge, \vee, \supset\}$

Denote the set of the above axioms by $\mathbf{Ax}$. For any $X \subseteq \mathbf{Ax}$, denote by $\mathbf{B}(X)$ the system obtained from $\mathbf{B}$ by adding the axioms in X. For simplicity, we refer to $\mathbf{B}(X)$ by concatenating the names of the schemata in X to "$\mathbf{B}$". For instance, $\mathbf{Bce}$ is the system obtained from $\mathbf{B}$ by adding the schemata $(\mathbf{c})$ and $(\mathbf{e})$. Moreover, we write $\mathbf{Bi}$ instead of $\mathbf{Bi_1i_2}$ and similarly for $\mathbf{k}, \mathbf{a}$ and $\mathbf{o}$. We also identify below the Hilbert-style systems with the logics induced by them.

Let us now explain the basic idea in providing semantics for $\mathbf{B}(X)$ (where $X \subseteq \mathbf{Ax}$). In classical logic we have only two truth-values: **t** and **f**, which when assigned to a proposition, provide information about its being "true" or "false" respectively. In paraconsistent logic, however, (since the classical rule $\varphi, \neg\varphi \vdash \psi$ is rejected) it can be the case that a proposition φ is both "true" and "false" at the same time (it simply means that φ is inconsistent). The key idea is to let the value assigned to a sentence φ provide information not only on the "truth"/"falsity" of φ, but also on the "truth"/"falsity" of $\neg\varphi$, as well as of $\circ\varphi$. This leads to the use of elements from $\{0,1\}^3$ as our truth-values[3], where the intended intuitive meaning of $v(\varphi) = \langle x, y, z\rangle$ is now:

- $x = 1$ iff φ is "true" (i.e. $v(\varphi) \in \mathcal{D}$).
- $y = 1$ iff $\neg\varphi$ is "true" (i.e. $v(\neg\varphi) \in \mathcal{D}$).
- $z = 1$ iff $\circ\varphi$ is "true" (i.e. $v(\circ\varphi) \in \mathcal{D}$).

Denote $P_1(\langle x_1, x_2\rangle) = x_1$, and $P_2(\langle x_1, x_2\rangle) = x_2$. Then such interpretation of truth-values dictates the following constraints on any valuation v: $P_1(v(\neg\varphi)) = P_2(v(\varphi))$ and $P_1(v(\circ\varphi)) = P_3(v(\varphi))$. For Nmatrices this constraint translates into the conditions: (NEG) $\tilde{\neg}a \subseteq \{y \mid P_1(y) = P_2(a)\}$ and (CON) $\tilde{\circ}a \subseteq \{y \mid P_1(y) = P_3(a)\}$. Hence our starting point for a semantics for $\mathbf{B}(X)$ is $\mathcal{M}_8^B$, the most general Nmatrix, which satisfies both (NEG) and (CON), while its $\{\vee, \wedge, \supset\}$-fragment is simply an expansion (see Defn. 3.8) of the classical two-valued matrix:

3 For some extensions of $\mathbf{B}$ it is enough to consider only elements from $\{0,1\}^2$, see [1] for details.

Definition 4.1 *The Nmatrix $\mathcal{M}_8^B = \langle\mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is defined as follows: $\mathcal{V}_8 = \{0,1\}^3$, $\mathcal{D}_8 = \{a \in \mathcal{V}_8 \mid P_1(a) = 1\}$. Let $\mathcal{F} = \mathcal{V} - \mathcal{D}$. The operations in $\mathcal{O}$ are:*

$$\tilde{\neg}a = \begin{cases} \mathcal{D} & \text{if } P_2(a) = 1 \\ \mathcal{F} & \text{if } P_2(a) = 0 \end{cases} \quad \tilde{\circ}a = \begin{cases} \mathcal{D} & \text{if } P_3(a) = 1 \\ \mathcal{F} & \text{if } P_3(a) = 0 \end{cases}$$

$$a\tilde{\vee}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a\tilde{\supset}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a\tilde{\wedge}b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

By Proposition 3.9 it easily follows that the $\{\vee, \wedge, \supset\}$-fragment of $\mathcal{M}_8^B$ characterizes positive classical logic. Now we turn to the axioms $(\mathbf{t})$ and $(\mathbf{b})$ of $\mathbf{B}$. $(\mathbf{t})$ rules out the truth-values $\langle 0,0,0\rangle$ and $\langle 0,0,1\rangle$, as it cannot be the case that both the proposition and its negation are "false". According to the axiom **(b)**, it cannot be the case that φ, $\neg\varphi$ and $\circ\varphi$ are all "true". This rules out the truth-value $\langle 1,1,1\rangle$. We are left with five truth-values: $\mathbf{t} = \langle 1,0,1\rangle$, $\mathbf{t_I} = \langle 1,0,0\rangle$, $\mathbf{I} = \langle 1,1,0\rangle$, $\mathbf{f} = \langle 0,1,1\rangle$ and $\mathbf{f_I} = \langle 0,1,0\rangle$. Denote the Nmatrix obtained from $\mathcal{M}_8^B$ by deleting the three rejected truth-values by $\mathcal{M}_5^B$. E.g., the interpretation of $\neg$ in $\mathcal{M}_5^B$ is:

$\tilde{\neg}$	$\mathbf{f}$	$\mathbf{f_I}$	$\mathbf{I}$	$\mathbf{t}$	$\mathbf{t_I}$
	$\{\mathbf{I}, \mathbf{t}, \mathbf{t_I}\}$	$\{\mathbf{I}, \mathbf{t}, \mathbf{t_I}\}$	$\{\mathbf{I}, \mathbf{t}, \mathbf{t_I}\}$	$\{\mathbf{f}, \mathbf{f_I}\}$	$\{\mathbf{f}, \mathbf{f_I}\}$

Theorem 4.2 *$\mathcal{M}_5^B$ is characteristic for* $\mathbf{B}$.

Proof outline: Soundness is easy to verify. We give here just a hint of how to prove completeness, which applies also to the rest of the completeness proofs below. Suppose that $\mathbf{T}$ is a theory and φ_0 a sentence such that $\mathbf{T} \not\vdash_{\mathbf{B}} \varphi_0$. We construct a model of $\mathbf{T}$ in $\mathcal{M}_5^B$ which is not a model of φ_0. For this extend $\mathbf{T}$ to a maximal theory $\mathbf{T}^*$ such that $\mathbf{T}^* \not\vdash_{\mathbf{B}} \varphi_0$. $\mathbf{T}^*$ has the following properties: (i) $\psi \notin \mathbf{T}^*$ iff $\psi \supset \varphi_0 \in \mathbf{T}^*$, (ii) If $\psi \notin \mathbf{T}^*$ then $\psi \supset \varphi \in \mathbf{T}^*$, (iii) $\varphi \vee \psi \in \mathbf{T}^*$ iff either $\varphi \in \mathbf{T}^*$, or $\psi \in \mathbf{T}^*$, (iv) $\varphi \wedge \psi \in \mathbf{T}^*$ iff both $\varphi \in \mathbf{T}^*$ and $\psi \in \mathbf{T}^*$, (v) $\varphi \supset \psi \in \mathbf{T}^*$ iff either $\varphi \notin \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$, (vi) For every sentence φ of $\mathcal{L}_C$, either $\varphi \in \mathbf{T}^*$ or $\neg\varphi \in \mathbf{T}^*$, and (vii) If both $\varphi \in \mathbf{T}^*$ and $\neg\varphi \in \mathbf{T}^*$ then $\circ\varphi \notin \mathbf{T}^*$. Next we define a valuation v by $v(\varphi) = \langle x, y, z\rangle$ for $x, y, z \in \{0,1\}$, where $x = 1$ iff $\varphi \in \mathbf{T}^*$, $y = 1$ iff $\neg\varphi \in \mathbf{T}^*$, and $z = 1$ iff $\circ\varphi \in \mathbf{T}^*$. It is easy to see that v is a refuting valuation, i.e. v satisfies $\mathbf{T}^*$ (and so also $\mathbf{T}$), but not φ_0. It remains to show that v is legal in $\mathcal{M}_5^B$, which is guaranteed by the properties of $\mathbf{T}^*$.

Suppose now that we are interested in a semantics for the system $\mathbf{Bc}$. Luckily, we do not need to start the process from scratch, as it is easily obtained by refining the Nmatrix $\mathcal{M}_5^B$. To see this, first note that $\neg\neg\varphi \supset \varphi$ is not $\mathcal{M}_5^B$-valid. Indeed, by taking any $\mathcal{M}_5^B$-valuation v such that $v(\varphi) \in \mathcal{F}$,

$v(\neg\varphi) \in \mathcal{D}$ and $v(\neg\neg\varphi) \in \mathcal{D}$, this schema is refuted. Thus we must forbid valuations, in which φ can be assigned a non-designated value, while $\neg\neg\varphi$ is assigned a designated one. This can be achieved by imposing the following condition on the interpretation of $\neg$:

$\mathbf{Cond}(\mathbf{c})$ If $P_1(a) = 0$ then $\tilde{\neg}(a) \subseteq \{x \mid P_2(x) = 0\}$

(in other words, if $a \in \{\mathbf{f}, \mathbf{f_I}\}$, then $\tilde{\neg}(a) \subseteq \{\mathbf{t}, \mathbf{t_I}\}$).
A characteristic Nmatrix for $\mathbf{Bc}$ is obtained by taking the weakest refinement of $\mathcal{M}_5^B$, in which $\mathbf{Cond}(\mathbf{c})$ holds. It is similar to $\mathcal{M}_5^B$, except that it interprets $\neg$ as follows:

$\tilde{\neg}$	$\mathbf{f}$	$\mathbf{f_I}$	$\mathbf{I}$	$\mathbf{t}$	$\mathbf{t_I}$
	$\{\mathbf{t}, \mathbf{t_I}\}$	$\{\mathbf{t}, \mathbf{t_I}\}$	$\{\mathbf{I}, \mathbf{t}, \mathbf{t_I}\}$	$\{\mathbf{f}, \mathbf{f_I}\}$	$\{\mathbf{f}, \mathbf{f_I}\}$

Similarly we derive the rest of the semantic conditions for the other schemata from $\mathbf{Ax}$:

$\mathbf{Cond}(\mathbf{c})$ If $P_1(a) = 1$ then $\tilde{\neg}a \subseteq \{x \mid P_2(x) = 1\}$

$\mathbf{Cond}(\mathbf{k1})$**:** If $P_1(a) = 0$ then $P_3(a) = 1$

$\mathbf{Cond}(\mathbf{k2})$**:** If $P_2(a) = 0$ then $P_3(a) = 1$

$\mathbf{Cond}(\mathbf{i1})$**:** If $P_1(a) = 0$ then $\tilde{\circ}a \subseteq \{x \mid P_2(x) = 0\}$

$\mathbf{Cond}(\mathbf{i2})$**:** If $P_2(a) = 0$ then $\tilde{\circ}a \subseteq \{x \mid P_2(x) = 0\}$

$\mathbf{Cond}(\mathbf{a}_\diamond)$**:** If both $P_3(a) = 1$ and $P_3(b) = 1$ then $a\tilde{\diamond}b \subseteq \{x \mid P_3(x) = 1\}$

$\mathbf{Cond}(\mathbf{o}_\diamond)$**:** If either $P_3(a) = 1$ or $P_3(b) = 1$ then $a\tilde{\diamond}b \subseteq \{x \mid P_3(x) = 1\}$

For $X \subseteq \mathbf{Ax}$, denote by $\mathcal{M}_5^B(X)$ the the weakest refinement of $\mathcal{M}_5^B$ in which the conditions for X are satisfied.

Theorem 4.3 *For $X \subseteq \mathbf{Ax}$, $\mathcal{M}_5^B(X)$ is characteristic for* $\mathbf{B}(X)$.

Example 4.4 The Nmatrix $\mathcal{M}_5^B(\{(\mathbf{k_1}), (\mathbf{k_2})\})$ is three-valued, as the conditions $\mathbf{Cond}(\mathbf{k_1})$ and $\mathbf{Cond}(\mathbf{k_2})$ lead to the deletion of the truth-values $\mathbf{f_I} = \langle 0, 1, 0\rangle$ and $\mathbf{t_I} = \langle 1, 0, 0\rangle$ respectively. The interpretations of $\neg, \circ$ and $\vee$ in it are as follows:

$\tilde{\neg}$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{f}\}$

$\tilde{\circ}$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
	$\{\mathbf{t}, \mathbf{I}, \mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \mathbf{I}, \mathbf{f}\}$

$\tilde{\vee}$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$
$\mathbf{I}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$

The addition of *either* $(\mathbf{a}_\vee)$ *or* $(\mathbf{o}_\vee)$ to $\mathbf{Bk}$ leads to the following truth-tables for $\vee$ respectively:

$\tilde{\vee}$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{t}\}$
$\mathbf{I}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{t}\}$

$\tilde{\vee}$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\mathbf{I}$	$\{\mathbf{t}\}$	$\{\mathbf{I}, \mathbf{t}\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

An important axiom was intentionally left out so far: $(\mathbf{l})$ $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$. This is one of the axioms of the well-known paraconsistent logic C_1 of da Costa ([9]), which is the $\circ$-free fragment of $\mathbf{Bcila}$. This axiom is problematic due to the following:

Theorem 4.5 *No system between* $\mathbf{Bl}$ *and* $\mathbf{Bcilo}$ *has a finite characteristic Nmatrix (and so none of them has a finite characteristic ordinary matrix).*

It easily follows from this theorem (as $\mathbf{Bcilo}$ is an extension of $\mathbf{Bcila}$) that C_1 has no finite characteristic Nmatrix. Hence the method used in the previous subsection cannot work for logics like $\mathbf{Bcila}$. However, as a reasonable useful substitute, *infinite* Nmatrices can be provided for a family of such systems, which can still be used to provide decision procedures for the logics they characterize. As before, we start with the basic system $\mathbf{Bl}$, and find first a characteristic Nmatrix for it. To see the motivation for an infinite supply of truth-values here, note that the validity of $(\mathbf{l})$ in an Nmatrix means that whenever $\circ\varphi$ is "false", so is $\neg(\varphi \wedge \neg\varphi)$. Accordingly, Nmatrices appropriate for $\mathbf{Bl}$ should be able to "distinguish" between conjunctions of an "inconsistent" formula with its negation from other types of conjunctions. Therefore such Nmatrices should enforce an intimate connection between the truth-value of an "inconsistent" formula and the truth-value of its negation. This in turn requires a supply of infinitely many truth-values, corresponding to the potentially infinite number of "inconsistent" formulas. But from where will we take these truth-values, and how should we define the operations on them? A key observation in our path to solve these problems is that **(k1)** and **(k2)** are theorems of $\mathbf{Bl}$, and so $\mathbf{Bl}$ extends the system $\mathbf{Bk}$. Accordingly, to characterize $\mathbf{Bl}$ we first create an expansion (see Definition 3.8) of $\mathcal{M}_5^B(\{(\mathbf{k1}), (\mathbf{k2})\})$ from Example 4.4 with infinitely many truth-values (recall that by Proposition 3.9 we obtain the same logic as that induced by $\mathcal{M}_5^B(\{(\mathbf{k1}), (\mathbf{k2}))$). Then we shall look for the weakest refinement of the obtained Nmatrix in which axiom $(\mathbf{l})$ becomes valid.

Definition 4.6 Let $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{F} = \{f\}$. The Nmatrix $\mathcal{M}_{\mathbf{Bl}} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is defined as follows: $\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$, $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$, and:

$$a\tilde{\vee}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a\tilde{\supset}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a\tilde{\wedge}b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\tilde{\neg}a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{if } a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases} \qquad \tilde{\circ}a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

Theorem 4.7 *$\mathcal{M}_{\mathbf{Bl}}$ is characteristic for* $\mathbf{Bl}$.

The system $\mathbf{Bl}$ can be extended with further axioms[4] from $\mathbf{Ax}$. As before, each axiom corresponds to some easily computed semantic condition, this time on refinements of the basic Nmatrix $\mathcal{M}_{\mathbf{Bl}}$:

Cond(c) : $\tilde{\neg} f \subseteq \mathcal{T}$ **Cond(e):** $\tilde{\neg}(I_i^j) = \{I_i^{j+1}\}$

Cond(i$_1$) : $\tilde{\circ} f \subseteq \mathcal{T}$ **Cond(i$_2$):** If $a \in \mathcal{T}$ then $\tilde{\circ} a \subseteq \mathcal{T}$

Cond(a$_\sharp$) : if $a, b \in \mathcal{T} \cup \mathcal{F}$, then $a \tilde{\sharp} b \subseteq \mathcal{T} \cup \mathcal{F}$

Cond(o$_\sharp$) : if $a \in \mathcal{T} \cup \mathcal{F}$ or $b \in \mathcal{T} \cup \mathcal{F}$, then $a \tilde{\sharp} b \subseteq \mathcal{T} \cup \mathcal{F}$

For $X \subseteq \mathbf{Ax}$, denote by $\mathbf{Bl}(X)$ the system obtained from $\mathbf{Bl}$ by adding the schemata in $\mathbf{Ax}$. Denote by $\mathcal{M}_{\mathbf{Bl}}(X)$ the weakest refinement of $\mathcal{M}_{\mathbf{Bl}}$, in which the conditions for axioms in X are satisfied.

Theorem 4.8 *For $X \subseteq \mathbf{Ax}$, $\mathcal{M}_{\mathbf{Bl}}(X)$ is characteristic for* $\mathbf{Bl}(X)$.

At this point it is important to stress that although the semantics defined above for systems with the axiom $(\mathbf{l})$ are infinite, decidability is still preserved. This is due to the fact that to check whether a given formula φ is provable in $\mathbf{Bl}(X)$, by the proof of the above theorem in [1], it suffices to check all legal *partial* valuations v in $\mathcal{M}_{\mathbf{Bl}(X)}$ which assign to subformulas of φ values in the set $\{f\} \cup \{t_i^j \mid 0 \leq i \leq n(\varphi), 0 \leq j \leq k(\varphi)\} \cup \{I_i^j \mid 0 \leq i \leq n(\varphi),\ 0 \leq j \leq k(\varphi)\}$, where $n(\varphi)$ is the number of subformulas of φ which do not begin with $\neg$, and $k(\varphi)$ is the maximal number of consecutive negation symbols occurring in φ. This is a finite process.

Corollary 4.9 *da Costa's system C_1 is decidable, and it has a characteristic Nmatrix $\mathcal{M}_{C_1}$, in which the sets of truth-values and designated truth-values are like in $\mathcal{M}_{\mathbf{Bl}}$, and the interpretations of the connectives are*[5]*:*

$$a \tilde{\supset} b = \begin{cases} \mathcal{F} & a \in \mathcal{D}, b \in \mathcal{F} \\ \mathcal{T} & a \in \mathcal{F}, b \notin \mathcal{I} \\ \mathcal{T} & b \in \mathcal{T}, a \notin \mathcal{I} \\ \mathcal{D} & \textit{otherwise} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & a \in \mathcal{T} \\ \mathcal{T} & a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & a = I_i^j \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & a \in \mathcal{F} \textit{ or } b \in \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T}, b \in \mathcal{T} \\ \mathcal{T} & a = I_i^j, b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \textit{otherwise} \end{cases}$$

$$a \tilde{\vee} b = \begin{cases} \mathcal{F} & a \in \mathcal{F}, b \in \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T}, b \notin \mathcal{I} \\ \mathcal{T} & b \in \mathcal{T}, a \notin \mathcal{I} \\ \mathcal{D} & \textit{otherwise} \end{cases}$$

4 Except for $(\mathbf{k_1})$ and $(\mathbf{k_2})$, which are already theorems of $\mathbf{Bl}$.

5 The interpretation of $\circ$ is discarded, as C_1 is the $\circ$-free fragment of $\mathbf{Bcila}$.

Acknowledgement

This research was supported by The Israel Science Foundation (grant No 809-06).

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