Quantification in non-deterministic multi-valued structures

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Abstract

In this paper the concept of a multi-valued non-deterministic (propositional) matrix, in which non-deterministic computations of truth values are allowed, is extended to languages with quantifiers. We describe the difficulties involved in applying the two main classical approaches to interpreting quantifiers, the objectual and the substitutional, and solve the difficulties in the case of the latter. Then we turn to the two-valued case, and explore the effects in this context of each of the four standard Gentzen-type rules for the classical quantifiers. As an example, a sound and complete two-valued non-deterministic semantics is provided for a family of first-order proof systems.

1. Introduction

According to the classical principle of assigning truth values to formulas, the truth value assigned to a complex formula is uniquely determined by the truth values of its subformulas. This principle is also adopted by many non-classical logics and provides the basis for the method of multi-valued matrices, which is a standard method of providing semantics to logical systems. However, an agent acting in the real world often has only incomplete, imprecise or even inconsistent information to guide its decisions. In such cases, this approach is no longer appropriate and other alternatives must be looked for. One such alternative is borrowing the idea of *non-deterministic* computations from automata and computability theory. This idea leads to a quest for structures, where the value assigned by a valuation to a complex formula might be chosen non-deterministically from a certain (non-empty) set of options.

Structures of this sort, named *non-deterministic multi-valued matrices* (henceforth, *Nmatrices*) have been introduced and applied in [1], [2], [3], [4] and [5]. [5] shows that some important propositional logics for reasoning under uncertainty can be characterized by finite Nmatrices although they have only infinite characteristic deterministic matrices. In [4] a strong connection is established between the admissibility of the cut rule in canonical Gentzen-type propositional systems, non-triviality of such systems and the existence of a sound and complete non-deterministic two-valued semantics for them. In [1] and [2] large families of non-classical propositional logics are investigated with the emphasis on the effects of negation, using 3-valued and 4-valued Nmatrices. In [3] general deductive proof-systems based on Nmatrices are developed.

However, all the work on Nmatrices mentioned above has been carried out for the propositional case only. In this paper we investigate Nmatrices in the context of stronger languages, in particular first-order languages. We explore two different semantic approaches for interpreting quantifiers in the context of Nmatrices: *substitutional* quantification and (the standard) *objectual* quantification[1]. We define an n-valued substitutional non-deterministic semantics for languages with quantifiers. As an example, we provide a sound and complete two-valued semantics for a natural family of first-order proof systems. We also explain the difficulties of an objectual quantification in the context of Nmatrices.

The structure of the paper is as follows. Section 2 introduces Nmatrices for languages with quantifiers. Section 3 discusses the problems of objectual quantification in the context of Nmatrices. In section 4 we define a substitutional non-deterministic semantics, discuss the problems arising from a naive extension of the notion of a propositional valuation to languages with quantifiers and propose a solution of these problems. In section 5 we provide a sound and complete two-valued non-deterministic semantics for a family of first-order proof systems, based on the Gentzen-type standard quantifier rules. We show that each quantifier rule imposes a constraint which reduces the non-determinism of the corresponding Nmatrix (this is analogous to the behaviour of the negation rules in [1] and [2]). Section 6 concludes the paper with a summary, directions for future research and remarks on possible applications.

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1 The terminology is taken from [12].

2. Preliminaries

By a first-order language we mean a language containing only the quantifiers $\forall$ and $\exists$. In what follows, L is a language with quantifiers, VAR is its set of variables, F - its set of wffs, F^C - its set of sentences, TR - its set of terms, and TR^C - its set of closed terms. x, y are meta-variables ranging over the variables of L. $\psi, \psi', \varphi, \varphi', A, B$ denote arbitrary formulas, $\mathbf{t}, \mathbf{t}'$ denote arbitrary terms, Γ, Δ denote sets of formulas. $\equiv_\alpha$ is the α-equivalence relation between formulas. We use [ ] for application of functions in the meta-language, leaving the use of ( ) to the object language. $\psi\{\mathbf{t}/x\}$ denotes the formula obtained from ψ by substituting $\mathbf{t}$ for x. Given a set S, we denote the set of all the non-empty subsets of S by $P^+(S)$.

Definition 2.1 *(***Non-deterministic matrix***)*[2] *Given a language L, a non-deterministic matrix (henceforth Nmatrix) for L is a tuple $\mathcal{M} =< \mathcal{V}, \mathcal{D}, \mathcal{O} >$, where:*

- *$\mathcal{V}$ is a non-empty set of truth values.*
- *$\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{T}$.*
- *For every n-ary connective $\diamond$ and for every quantifier Q of L, $\mathcal{O}$ includes the corresponding interpretation functions:*
  - $\tilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to P^+(\mathcal{V})$.
  - $\tilde{Q}_{\mathcal{M}} : P^+(\mathcal{V}) \to P^+(\mathcal{V})$[3]

Definition 2.2 (L-structure) *Let $\mathcal{M}$ be an Nmatrix. An L-structure for $\mathcal{M}$ is a pair $S = \langle D, I\rangle$ where D is a (non-empty) domain and I is a function interpreting constants, predicate symbols and function symbols of L, satisfying the following properties:*

- $I[c] \in D$,
- $I[p^n] : D^n \to \mathcal{V}$ *is an n-ary predicate,*
- $I[f^n] : D^n \to D$ *is an n-ary function.*

I is extended to interpret closed terms of L as follows:

$$I[f(t_1, ..., t_n)] = I[f][I[t_1], ..., I[t_n]]$$

Definition 2.3 *([9]) Let S=$\langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. $L(D)$ is the language obtained from L by adding to it the set of constants $\{\overline{a} \mid a \in D\}$. $S' = \langle D, I'\rangle$ is the $L(D)$-structure, such that I' is an extension of I satisfying: $I'[\overline{a}] = a$.*

Notation and conventions: Given an L-structure $S = \langle D, I\rangle$, we shall refer to the extended $L(D)$-structure $\langle D, I'\rangle$ as S and to I' as I when the meaning is clear from the context. The set of formulas of $L(D)$ is denoted by F_D, the set of sentences of $L(D)$ is denoted by F_D^C and the set of closed terms by TR_D^C.

The standard approach to interpreting first-order formulas is by using *objectual* (or referential) semantics. In objectual quantification, the variable is thought of as ranging over a set of objects from the domain. Usually, open formulas are interpreted using assignments, which map variables to elements of the domain. For different formulations of first-order objectual semantics, see e.g. [7], [8], [9] and [11].

An alternative approach is *substitutional* quantification ([12]), where quantifiers are interpreted substitutionally, i.e. a universal (an existensial) quantification is true if and only if every one (at least one) of its substitution instances is true. Substitutional semantics considers only *Henkin* structures, i.e. structures where every element of the domain has a term referring to it. For formulations of substitutional first-order semantics, see e.g. [13] and [14].

3. The problems of objectual semantics

In this section we discuss the problems encountered at an attempt to introduce an objectual non-deterministic semantics.

Definition 3.1 (S-assignment, I_g) *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. An S-*assignment *is a function $g : VAR \to D$. Given an S-assignment g, $g[x := a]$ is the S-assignment similar to g except that it assigns $a \in D$ to x.*
The interpretation function I is extended to $I_g : TR \to D$ as follows: $I_g[c] = I[c], I_g[x] = g[x]$, $I_g[f(t_1, ..., t_n)] = I[f][I_g[t_1], ..., I_g[t_n]]$

Below is a standard definition of a valuation of formulas (for languages with quantifiers) in case $\mathcal{M}$ is a deterministic matrix.

Definition 3.2 (Deterministic valuation) *Let S be an L-structure for a (deterministic) matrix $\mathcal{M}$ and g an S-assignment. The valuation $v_{S,g} : F \to \mathcal{V}$ is as follows:*

- $v_{S,g}[p(t_1, ..., t_n)] = I[p][I_g[t_1], ..., I_g[t_n]]$
- $v_{S,g}[\diamond[\psi_1, ..., \psi_n]] = \tilde{\diamond}_{\mathcal{M}}[v_{S,g}[\psi_1], ..., v_{S,g}[\psi_n]]$ *for any n-ary connective of L.*
- $v_{S,g}[Qx\psi] = \tilde{Q}_{\mathcal{M}}[\{v_{S,g[x:=a]}[\psi] \mid a \in D\}]$*, for any quantifier Q of L.*

Next we attempt to modify this definition in the context of Nmatrices. Let S be an L-structure for an Nmatrix $\mathcal{M}$ and g an S-assignment. A natural generalization of the definition of a propositional $\mathcal{M}$-legal valuation (see, e.g. [2]) is as follows: an S, g-valuation $v_{S,g}$ is *legal in* $\mathcal{M}$ if it satisfies the following conditions:

1. $v_{S,g}[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$.

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2 This is a generalization of the definition in [5].

3 This is a generalization of the quantifier interpretation by [11].

2. $v_{S,g}[\diamond[\psi_1, ..., \psi_n]] \in \tilde{\diamond}_{\mathcal{M}}[v_{S,g}[\psi_1], ..., v_{s,g}[\psi_n]]$.
3. $v_{S,g}[Qx\psi] \in \tilde{Q}_{\mathcal{M}}[\{v_{S,g[x:=a]}[\psi] \mid a \in D\}]$.

However, condition 3 is not well defined: a valuation $v_{S,g[x:=a]}$ is not necessarily unique! Therefore, we must specify exactly which of the possible $S, g[x := a]$-valuations are to be used.

One alternative is considering *all* possible $\mathcal{M}$-legal $S, g[x := a]$-valuations, i.e. condition 3 can be reformulated as follows: $v[Qx\psi] \in \tilde{Q}_{\mathcal{M}}[\{v'[\psi] \mid a \in D \ and \ v' \ is \ an \ \mathcal{M} - legal \ S, g[x := a] - valuation\}]$. However, this is counter-intuitive, since all the choices of truth values made by v for the subformulas of ψ become irrelevant for the choice of $v[\forall x\psi]$. As a trivial example, take L_a to be the language consisting of $\neg$, $\forall$, a constant c and a unary predicate symbol p. Let $S_a = \langle\{a\}, I_a\rangle$ be the simple L_a-structure, such that $I_a[c_a] = a$ and $I_a[p] = \{a\}$. We use the following Nmatrix $\mathcal{M}_a = \langle\mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ for L_a:

- $\mathcal{V} = \{t, f\}, \mathcal{D} = \{t\}$
- $\tilde{\neg}_{\mathcal{M}_a}[f] = \{t\}, \tilde{\neg}_{\mathcal{M}_a}[t] = \{t, f\}$
- $\tilde{\forall}_{\mathcal{M}_a}[\{t\}] = \{t\}, \tilde{\forall}_{\mathcal{M}_a}[\{f\}] = \tilde{\forall}_{\mathcal{M}_a}[\{t, f\}] = \{f\}$.

Let g be any S_a-assignment (obsiously, $g[x] = a$) and let v be an $\mathcal{M}_a$-legal S, g-valuation satisfying $v[\neg p(c)] = t$. Then $v[\forall x\neg p(c)] = f$, although since the quantification is vacuous, we expect the truth value assigned to $\forall x\neg p(c)$ to be similar to the truth value assigned to $\neg p(c)$.

Whether an adequate purely objectual non-deterministic semantics can be defined, is a question for further investigation. At present, no definition, which is both natural and useful, seems to be available. Hence, we turn to the main alternative - substitutional semantics.

4. Substitutional semantics

Definition 4.1 **(S-substitution)** *Given an L-structure $S = \langle D, I\rangle$ for an Nmatrix $\mathcal{M}$, an S-*substitution *is a function $\sigma : VAR \to TR^C_D$. It is extended to $\sigma : TR_D \cup F_D \to TR^C_D \cup F^C_D$ as follows: for a term t of $L(D)$, $\sigma[t]$ is the closed term obtained from t by replacing every $x \in FV(t)$ by $\sigma[x]$. For a formula φ, $\sigma[\varphi]$ is the sentence obtained from φ by replacing every $x \in FV(\varphi)$ by $\sigma[x]$.*

Notation: Given a set Γ of formulas, we denote the set $\{\sigma[\psi] \mid \psi \in \Gamma\}$ by $\sigma[\Gamma]$.

The most natural approach to interpreting formulas would be to generalize the definition of *an $\mathcal{M}$-legal valuation* (proposed by [1] for the propositional case) as follows:

Definition 4.2 **(Semi-legal valuation)** *Let S be an L-structure for an Nmatrix $\mathcal{M}$. An S-valuation $v : F^C_D \to \mathcal{V}$ is* semi-legal *in $\mathcal{M}$ if it satisfies the following properties:*

1. $v[p(t_1, ..., t_n)] = I[p][I[t_1], ..., I[t_n]]$

2. $v[\diamond(\varphi_1, ..., \varphi_n)] \in \tilde{\diamond}_{\mathcal{M}}[v[\varphi_1], ..., v[\varphi_n]]$ *for any n-ary connective $\diamond$ of L.*

3. $v[Qx\varphi] \in \tilde{Q}_{\mathcal{M}}[\{v[\varphi\{\overline{a}/x\}] \mid a \in D\}]$ *for any quantifier Q of L.*

However, two immediate problems arise from this definition. To demonstrate them, let us return to the language L_a, the simple L_a-structure S_a and the Nmatrix $\mathcal{M}_a$ defined in the previous section. The problems arising from the definition of an $\mathcal{M}_a$-semi-legal S_a-valuation are as follows:

1. The principle of α*-conversion* consists in identifying syntactic objects differing only in the names of bound variables. Such a principle is intuitively justified by the role of bound variables as placeholders, as in $\int_a^b f(x)dx$, which is the same as $\int_a^b f(y)dy$. Thus we expect two α-equivalent sentences to be assigned the same truth value by an $\mathcal{M}_a$-semi-legal S_a-valuation. However, this is not the case here. For example, an S_a-valuation v satisfying $v[\neg\forall xp(x)] = t$ and $v[\neg\forall yp(y)] = f$ is semi-legal in $\mathcal{M}_a$. Since the principle of α-conversion is essential for any logical system, such situation is unacceptable. Therefore, there is a need to impose more limitations on a valuation, so that it will not differentiate between two α-equivalent sentences.
2. Another problem has to do with the nature of identity. Suppose ψ_2 is obtained from ψ_1 by replacing a closed term $\mathbf{t}_1$ by a closed term $\mathbf{t}_2$ denoting the same object as $\mathbf{t}_1$. We expect the same truth value to be assigned to ψ_1 and ψ_2 by any $\mathcal{M}_a$-semi-legal S_a-valuation. However, an S_a-valuation v satisfying $v[\neg p(c_1)] = t$ and $v[\neg p(c_2)] = f$ is semi-legal in $\mathcal{M}_a$.

To solve these problems, we define a notion of *congruence* between terms and formulas of $L(D)$ with respect to an L-structure S for an Nmatrix $\mathcal{M}$. Then we modify the definition of an $\mathcal{M}$-semi-legal S-valuation, so that it will not differentiate between congruent sentences.

Definition 4.3 **(Congruence of terms and formulas)** *Let S be an L-structure for an Nmatrix $\mathcal{M}$. The relation $\sim^S$ between terms of $L(D)$ is defined as follows:*

- $x \sim^S x$
- *For closed terms t, t' of $L(D)$:* *$t \sim^S t'$ when $I[t] = I[t']$.*
- *If $t_1 \sim^S t'_1, ..., t_n \sim^S t'_n$, then* $f(t_1, ..., t_n) \sim^S f(t'_1, ..., t'_n)$.

The relation $Cong^S$ between formulas of $L(D)$ is defined as follows:

- *If $t_1 \sim^S t'_1, t_2 \sim^S t'_2, ..., t_n \sim^S t'_n$, then* $Cong^S(p(t_1, ..., t_n), p(t'_1, ..., t'_n))$.
- *If $Cong^S(\psi_1, \varphi_1), ..., Cong^S(\psi_n, \varphi_n)$, then* $Cong^S(\diamond(\psi_1, ..., \psi_n), \diamond(\varphi_1, ..., \varphi_n))$ *for any n-ary connective $\diamond$ of L.*

- *If $Cong^S(\psi\{z/x\}, \varphi\{z/y\})$, where z is a new variable, then $Cong^S(Qx\psi, Qy\varphi)$ for any quantifier Q of L.*

It is easy to see that for any L-structure S, $Cong^S$ is an congruence relation.

Lemma 4.1 *Let S be an L-structure for an Nmatrix $\mathcal{M}$. Let t, t' be terms and ψ, ψ' formulas of $L(D)$. Let s be a term of $L(D)$ free for x in t and ψ and free for y in t' and ψ'.*

- *If $t\{z/x\} \sim^S t'\{z/y\}$, where z is a new variable, then $t\{s/x\} \sim^S t'\{s/x\}$.*
- *If $Cong^S(\psi\{z/x\}, \psi'\{z/y\})$, where z is a new variable, then $Cong^S(\psi\{s/x\}, \psi'\{s/y\})$.*

Lemma 4.2 *Let S be an L-structure for an Nmatrix $\mathcal{M}$. Let ψ, ψ' be formulas of $L(D)$. Let t, t' be closed terms of $L(D)$, such that $I[t] = I[t']$.*

1. *If $\psi \equiv_\alpha \psi'$, then $Cong^S(\psi, \psi')$.*
2. *$Cong^S(\psi\{t/x\}, \psi\{t'/x\})$.*

Definition 4.4 (Legal valuation) *Let S be an L-structure for an Nmatrix $\mathcal{M}$. An S-valuation $v : F_D^C \to \mathcal{V}$ is* legal in *$\mathcal{M}$ if it is semi-legal in $\mathcal{M}$ and $v[\psi] = v[\psi']$ for every two sentences ψ, ψ' of $L(D)$, such that $Cong^S(\psi, \psi')$.*

From lemma 4.2 it follows, that this definition solves both of the problems discussed above.

Definition 4.5 (Model, $\mathcal{M}$-validity, $\vdash_\mathcal{M}$) *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$, ψ a sentence of $L(D)$ and Γ, Δ sets of formulas of $L(D)$.*

1. *An $\mathcal{M}$-legal S-valuation v is* a model of *ψ (Γ)* in *$\mathcal{M}$, denoted by $S, v \models_\mathcal{M} \psi$ ($S, v \models_\mathcal{M} \Gamma$), if $v[\psi] \in \mathcal{D}$ ($S, v \models_\mathcal{M} \psi$ for every $\psi \in \Gamma$).*
2. *A formula A is $\mathcal{M}$-valid in S if for every S-substitution σ and every $\mathcal{M}$-legal S-valuation v, $S, v \models_\mathcal{M} \sigma[A]$. A formula is $\mathcal{M}$-valid if it is $\mathcal{M}$-valid in every L-structure for $\mathcal{M}$.*
3. *A sequent $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$*-valid in *S if for every S-substitution σ and every S-valuation v legal in $\mathcal{M}$, $S, v \models_\mathcal{M} \sigma[\Gamma]$ implies that there exists some $\varphi \in \Delta$ s.t. $S, v \models_\mathcal{M} \sigma[\varphi]$. A sequent is $\mathcal{M}$*-valid *if it is $\mathcal{M}$-valid in every L-structure for $\mathcal{M}$.*
4. *$\vdash_\mathcal{M}$, the consequence relation induced by $\mathcal{M}$ is defined as follows: $\Gamma \vdash_\mathcal{M} \Delta$ if $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid.*

Definition 4.6 (Refinement) *([2]) Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1\rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2\rangle$ be Nmatrices for a language L. $\mathcal{M}_2$ is a* refinement *of $\mathcal{M}_1$ if $\mathcal{V}_2 \subseteq \mathcal{V}_1, \mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{V}_2$, $\tilde{\diamond}_{\mathcal{M}_2}[a_1, ..., a_n] \subseteq \tilde{\diamond}_{\mathcal{M}_1}[a_1, ...a_n]$ for every n-ary connective $\diamond$ of L and every $a_1, ..., a_n \in \mathcal{V}_2$ and $\tilde{Q}_{\mathcal{M}_2}[H] \subseteq \tilde{Q}_{\mathcal{M}_1}[H]$ for every quantifier Q of L and every $H \subseteq \mathcal{V}_2$.*

Proposition 4.1 *([2]) If $\mathcal{M}_2$ is a refinement of $\mathcal{M}_1$, then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$.*

5. Example: the family QPLK

As an example of the definition of substitutional semantics from the previous section, we use two-valued Nmatrices to provide a sound and complete semantics for QPLK - a family of 16 first-order proof systems, obtained from the propositional system PLK (introduced in [6] under the name PI and used by [2]) by adding to it different combinations of the standard quantifier rules from [10].

Definition 5.1 (PLK) *The system PLK is as follows:*
Axiom:

$$(ax) A \Rightarrow A' \; if \; A \equiv_\alpha A'$$

Structural rules: *Cut, Weakening,Contraction,Exchange*
Logical rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta} (\supset\Rightarrow) \quad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} (\Rightarrow\supset)$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} (\wedge \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} (\Rightarrow \wedge)$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} (\vee \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} (\Rightarrow \vee)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} (\Rightarrow \neg)$$

Let $\mathcal{M}_Q$ be the unique Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ that satisfies the following conditions:

- $\mathcal{V} = \{t, f\}$
- $\mathcal{D} = \{t\}$
- $\tilde{\neg}_{\mathcal{M}_Q}[f] = \{t\}, \tilde{\neg}_{\mathcal{M}_Q}[t] = \{t, f\}$
- $\tilde{\wedge}_{\mathcal{M}_Q}[t, t] = \{t\}, \tilde{\wedge}_{\mathcal{M}_Q}[t, f] = \tilde{\wedge}_{\mathcal{M}_Q}[f, t] = \tilde{\wedge}_{\mathcal{M}_Q}[f, f] = \{f\}$
- $\tilde{\vee}_{\mathcal{M}_Q}[f, f] = \{f\}, \tilde{\vee}_{\mathcal{M}_Q}[t, f] = \tilde{\vee}_{\mathcal{M}_Q}[f, t] = \tilde{\vee}_{\mathcal{M}_Q}[f, f] = \{t\}$
- $\tilde{\supset}_{\mathcal{M}_Q}[t, f] = \{f\}, \tilde{\supset}_{\mathcal{M}_Q}[t, t] = \tilde{\supset}_{\mathcal{M}_Q}[f, t] = \tilde{\supset}_{\mathcal{M}_Q}[f, f] = \{t\}$
- $\tilde{\forall}_{\mathcal{M}_Q}[\{t\}] = \tilde{\forall}_{\mathcal{M}_Q}[\{t, f\}] = \tilde{\forall}_{\mathcal{M}_Q}[\{f\}] = \{t, f\}$
- $\tilde{\exists}_{\mathcal{M}_Q}[\{t\}] = \tilde{\exists}_{\mathcal{M}_Q}[\{t, f\}] = \tilde{\exists}_{\mathcal{M}_Q}[\{f\}] = \{t, f\}$

Proposition 5.1 *PLK is sound for $\mathcal{M}_Q$, i.e. $\vdash_{PLK} \subseteq \vdash_{\mathcal{M}_Q}$.*

The proof can easily be adapted from the propositional case, see e.g. [1]. From proposition 4.1 it follows, that if PLK is sound for $\mathcal{M}_Q$, then it is also sound for any refinement of $\mathcal{M}_Q$.

Definition 5.2 (QR) *Let QR be the following set of rules:*

$$\frac{\Gamma, A\{t/x\} \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta} (\forall \Rightarrow) \quad \frac{\Gamma \Rightarrow A\{y/x\}, \Delta}{\Gamma \Rightarrow \forall x A, \Delta} (\Rightarrow \forall)$$

$$\frac{\Gamma, A\{y/x\} \Rightarrow \Delta}{\Gamma, \exists x A \Rightarrow \Delta}\ (\exists \Rightarrow) \quad \frac{\Gamma \Rightarrow A\{t/x\}, \Delta}{\Gamma \Rightarrow \exists x A, \Delta}\ (\Rightarrow \exists)$$

where y is any variable free for x in A which does not appear free in the conclusion of $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$. The term t is any term free for x in A.

Next we explore the effects of adding the rules from QR to PLK. As we search for an Nmatrix $\mathcal{M}'$ such that all sequents provable in the obtained system are $\mathcal{M}'$-valid, it turns out that each of the rules corresponds to a constraint that leads to a certain refinement of $\mathcal{M}_Q$. (This is analogous to the behavior of the negation rules in [1] and [2].) Let us take $(\forall \Rightarrow)$ for example. This rule corresponds to $\mathcal{M}'$-validity of $\forall x A \supset A\{\mathbf{t}/x\}$, where $\mathbf{t}$ is any term free for x in A. Thus, for any given L-structure S for $\mathcal{M}'$, an S-substitution σ and an $\mathcal{M}'$-legal valuation v, $v[\sigma[\forall x A]] = t$ must imply that $v[\sigma[A\{\mathbf{t}/x\}]] = t$ for any term $\mathbf{t}$ free for x in A. Let $\sigma[\forall x A] = \forall x A'$, where A' is obtained from A by replacing every occurrence of $y \in FV(A) - \{x\}$ by $\sigma[y]$. Suppose there is some term $\mathbf{t}$ free for x in A, such that $v[\sigma[A\{\mathbf{t}/x\}]] = v[A'\{\sigma[\mathbf{t}]/x\}] = f$. Let $I[\sigma[\mathbf{t}]] = b$. Then, by lemma 4.2, $Cong^S(A'\{\sigma[\mathbf{t}]/x\}, A'\{\bar{b}/x\})$. By $\mathcal{M}'$-legality of v, $v[A'\{\bar{b}/x\}] = v[A'\{\sigma[\mathbf{t}]/x\}] = f$. It must be guaranteed that in such case, $\tilde{\forall}_{\mathcal{M}'}[H] = \{f\}$, where $H = \{v[A'\{\bar{a}/x\}] \mid a \in D\}$. Thus the imposed constraint on $\tilde{\forall}_{\mathcal{M}'}$ should be: if $f \in H$, then $\tilde{\forall}_{\mathcal{M}'}[H] = \{f\}$.
Similar analysis can be done for the other three quantifier rules of QR. We summarize the resulting list of refining constraints as follows:

1. $C(\forall \Rightarrow)$: $\tilde{\forall}[\{f\}] = \tilde{\forall}[\{t, f\}] = \{f\}$
2. $C(\Rightarrow \forall)$: $\tilde{\forall}[\{t\}] = \{t\}$
3. $C(\exists \Rightarrow)$: $\tilde{\exists}[\{f\}] = \{f\}$
4. $C(\Rightarrow \exists)$: $\tilde{\exists}[\{t, f\}] = \tilde{\exists}[\{t\}] = \{t\}$

Notation: For $R \subseteq QR$, PLK$[R]$ is the proof system obtained from PLK by adding to it the rules of R. $C(R)$ is the set $\{C(r) \mid r \in R\}$. $\mathcal{M}_Q[R]$ is the weakest refinement of $\mathcal{M}_Q$ in which the conditions of $C(R)$ are satisfied. QPLK is the set $\{\text{PLK}[R] \mid R \subseteq QR\}$.

Soundness and completeness together with cut-elimination are proved for all the proof systems of QPLK simultaneously.

Theorem 5.1 (Soundness) *Let $R \subseteq QR$. If $\Gamma \Rightarrow \Delta$ is PLK$[R]$-provable, then it is $\mathcal{M}_Q[R]$-valid.*

Theorem 5.2 (Completeness and cut-elimination) *Let $R \subseteq QR$ and let $\Gamma \Rightarrow \Delta$ be a sequent such that the set of its bound variables is disjoint from its set of free variables. If $\Gamma \Rightarrow \Delta$ has no cut-free proof in PLK$[R]$, then it is not $\mathcal{M}_Q[R]$-valid.*

Proof outline: Let $\Gamma \Rightarrow \Delta$ be a sequent, such that the sets of its free and bound variables are disjoint, and assume that it has no cut-free proof in PLK$[R]$. We construct an L-structure S, an L-substitution σ and an $\mathcal{M}_Q[R]$-legal S-valuation v, such that $S, v \models_{\mathcal{M}_Q[R]} \sigma_r[\Gamma]$, but there is no $\psi \in \Delta$ such that $S, v \models_{\mathcal{M}_Q[R]} \sigma[\psi]$.

It is easy to see that we can restrict ourselves to the language $L_{\Gamma\cup\Delta}$, the subset of L consisting of all the constants, function and predicate symbols occurring in $\Gamma \Rightarrow \Delta$. Let $\mathbf{T}$ be the set of all the terms of $L_{\Gamma\cup\Delta}$, which do not contain variables occurring bound in $\Gamma \Rightarrow \Delta$. It is a standard matter to show that two (possibly infinite) sets Π and Σ can be constructed, such that (i) $\Gamma \subseteq \Pi$ and $\Delta \subseteq \Sigma$, (ii) for any finite sets $\Gamma' \subseteq \Pi$ and $\Delta' \subseteq \Sigma$, $\Gamma' \Rightarrow \Delta'$ has no cut-free proof in PLK$[R]$ and (iii) the sets Π and Σ satisfy the following conditions:

1. There are no $A \in \Pi$ and $B \in \Sigma$ such that $A \equiv_\alpha B$.
2. If $\psi_1 \wedge \psi_2 \in \Pi$, then $\psi_1 \in \Pi$ and $\psi_2 \in \Pi$. If $\psi_1 \wedge \psi_2 \in \Sigma$, then either $\psi_1 \in \Sigma$ or $\psi_2 \in \Sigma$. Similar properties hold for $\supset$ and $\vee$.
3. If $\neg\psi \in \Sigma$, then $\psi \in \Pi$.
4. If $(\forall \Rightarrow) \in R$ and $\forall x \varphi \in \Pi$, then for every $\mathbf{t} \in \mathbf{T}$, $\varphi\{\mathbf{t}/x\} \in \Pi$.
5. If $(\Rightarrow \forall) \in \Sigma$ and $\forall x \varphi \in \Sigma$, then there exists some $\mathbf{t} \in \mathbf{T}$, such that $\varphi\{\mathbf{t}/x\} \in \Sigma$.
6. Similar properties hold for $\exists$.

The $L_{\Gamma\cup\Delta}$-structure $S_r = \langle D, I \rangle$ is defined as follows:

- $D = \mathbf{T}$,
- I satisfies the following conditions:
  - $I[c] = c$
  - $I[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$
  - $I[p][\mathbf{t}_1, ..., \mathbf{t}_n] = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Pi$.

Let σ_r be any S_r-substitution satisfying: $\sigma_r[x] = \overline{x}$ for every $x \in \mathbf{T}$. Note that every $x \in \mathbf{T}$ is also a member of the domain and thus has a constant $\overline{x}$ referring to it in $L_{\Gamma\cup\Delta}(D)$.

Let v_r be the S_r-valuation defined as follows:

1. $v_r[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$.
2. $v_r[\varphi_1 \diamond \varphi_2] = \iota z.z \in \tilde{\diamond}_{\mathcal{M}_Q[R]}[v_r[\varphi_1], v_r[\varphi_2]]$ for $\diamond \in \{\vee, \wedge, \supset\}$.
3. $v_r[\neg\varphi] = t$ iff $v_r[\varphi] = f$ or there is some formula $A \in \Pi$ such that $Cong^{S_r}(\sigma_r[A], \neg\varphi)$.
4. $v_r[\forall x \varphi] = t$ iff one of the following conditions holds:
   (a) for every $a \in D$, $v_r[\varphi\{\overline{a}/x\}] = t$, and if $(\Rightarrow \forall) \notin R$ then there exists no formula $A \in \Sigma$ such that $Cong^{S_r}(\sigma_r[A], \forall x \varphi)$.
   (b) $(\forall \Rightarrow) \notin R$, and there exists some formula $A \in \Pi$ such that $Cong^{S_r}(\sigma_r[A], \forall x \varphi)$.
5. $(\exists \Rightarrow), (\Rightarrow \exists)$ are treated symmetrically.

Proposition 5.2 *1. For every two sentences ψ, ψ' of $L_{\Gamma\cup\Delta}(D)$: if $Cong^{S_r}(\psi, \psi')$, then $v_r[\psi] = v_r[\psi']$*

2. If $\varphi \in \Pi$, then $v_r[\sigma_r[\varphi]] = t$, and if $\varphi \in \Sigma$, then $v_r[\sigma_r[\varphi]] = f$.

It is easy to verify that v_r is legal in $\mathcal{M}_Q[R]$. Recall that $\Gamma \subseteq \Pi$ and $\Delta \subseteq \Sigma$. Then by the above proposition, for every $\psi \in \Gamma$: $S_r, v_r \models_{\mathcal{M}_Q[R]} \sigma_r[\psi]$ and for every $\psi \in \Delta$: $S_r, v_r \not\models_{\mathcal{M}_Q[R]} \sigma_r[\psi]$. Therefore, $\Gamma \Rightarrow \Delta$ is not $\mathcal{M}_Q[R]$-valid in S_r and thus is not $\mathcal{M}_Q[R]$-valid.

Note that we have shown completeness and cut-elimination under the limitation that the set of all bound variables of the sequent is disjoint from its set of free variables. (Recall that first-order classical logic has the same limitation.) These results can be extended to general sequents by adding an explicit α-conversion inference rule to the proof systems of QPLK.

6. Conclusions and future research

In this paper we have explored Nmatrices in the context of languages with quantifiers, in particular first-order languages. We have presented two different approaches to quantification and their problematic aspects in the context of Nmatrices. We have defined a susbtitutional non-deterministic semantics, providing as an example a sound and complete semantics based on 2Nmatrices for the QPLK family of first-order proof systems, based on the standard quantifier rules.

Some of the most immediate directions for future research:

- Developing a general theory of 2-valued non-deterministic semantics for Gentzen-type and other proof systems, thus extending the results of [5].
- Extending the results to n- and infinitely-valued Nmatrices, along the lines of [1] and [2].
- Proposing non-deterministic semantics for more general quantifiers.
- Searching for a systematic approach to constructing general proof systems for n- and infinitely-valued Nmatrices for languages with quantifiers, possibly along the lines of [3].
- Investigating applications of Nmatrices by exploring the connections between non-determinism in logic and in computer science. One possible application might be in verification of non-deterministic programs. Other, less immediate applications might be: providing logical models for non-deterministic computations, verification of logical circuits, dealing with incosistency in large databases etc.

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