Canonical Calculi: Invertibility, Axiom expansion and (Non)-determinism

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Abstract. We apply the semantic tool of non-deterministic matrices to characterize two important properties of canonical Gentzen-type calculi: invertibility of rules and axiom expansion. We show that in every canonical calculus G satisfying a natural condition, the following are equivalent: (i) the connectives of G admit axiom expansion, (ii) the rules of G are invertible, and (iii) G has a characteristic finite deterministic matrix.

1 Introduction

Canonical systems are sequent calculi which in addition to the standard axioms and structural rules have only logical rules in which exactly one occurrence of a connective is introduced and no other connective is mentioned. Intuitively, the term "canonical systems" refers to systems in which the introduction rules of a logical connective determine the semantic meaning of that connective[3]. It was shown in [1, 2] that such systems are semantically characterized by two-valued *non-deterministic matrices (2Nmatrices)*. These structures form a natural generalization of the standard multi-valued matrices, in which the truth-value assigned to a complex formula is chosen *non-deterministically* out of a given set of options. Moreover, there is a remarkable triple correspondence between the existence of a characteristic 2Nmatrix for a canonical system, the ability to eliminate cuts in it and a constructive syntactic criterion called coherence. Here we show that in the context of canonical systems, 2Nmatrices play a prominent role not only in the phenomena of cut-elimination, but also in two other important properties of sequent calculi: invertibility of logical rules and completeness of atomic axioms (axiom expansion). The former is a key property in many deduction formalisms, such as Rasiowa-Sikorski (R-S) systems [12, 10] (also known as dual tableaux), where it induces an algorithm for finding a proof of a complex formula, if such a proof exists. The latter is also often considered crucial when designing "well-behaved" systems (see e.g. [9]). There are a number of works providing syntactic and semantic criteria for these properties in various calculi. Syntactic sufficient conditions for invertibility and axiom expansion in sequent calculi possibly without structural rules and with quantifier rules were

[3] This is according to a long tradition in the philosophy of logic, established by Gentzen in his classical paper *"Investigations Into Logical Deduction"* ([8]).

introduced in [6] and [11]. A semantic characterization of axiom expansion in single-conclusioned sequent calculi with arbitrary structural rules was provided in [7] in the framework of phase spaces. In the context of labeled sequent calculi (of which canonical calculi are a particular instance), [5] shows that the existence of a finite deterministic matrix is a necessary condition for axiom expansion. In this paper we extend these results by showing that the existence of a finite deterministic matrix for a coherent canonical calculus is also a *sufficient* condition for axiom expansion. Furthermore, we prove that it is also a necessary condition for invertibility. For coherent canonical calculi G in *normal form* (to which every canonical calculus can be transformed), an even stronger correspondence is established: (i) the connectives of G admit axiom expansion, iff (ii) the rules of G are invertible, iff (iii) G has a two-valued deterministic characteristic matrix.

2 Preliminaries

Henceforth $\mathcal{L}$ is a propositional language and $Frm_{\mathcal{L}}$ the set of its wffs. We use the metavariables $\Gamma, \Delta, \Sigma, \Pi$ for sets of $\mathcal{L}$-formulas. By a *sequent* we shall mean an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are *finite* sets of $\mathcal{L}$-formulas. A *clause* is a sequent consisting of atomic formulas. We use the metavariable Θ for sets of sequents, and the metavariable Ω for sequents.

Non-deterministic Matrices and Canonical Calculi

Below we shortly reproduce the basic definitions of the framework of Nmatrices and of canonical Gentzen-type systems from [1, 2, 4].

Definition 1. *A* non-deterministic matrix (Nmatrix) *for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where (i) $\mathcal{V}$ is a non-empty set of truth values, (ii) $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$, and (iii) for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding function $\tilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$. A valuation $v : Frm_{\mathcal{L}} \to \mathcal{V}$ is* legal in an Nmatrix $\mathcal{M}$ *if for every n-ary connective $\diamond$ of $\mathcal{L}$: $v(\diamond(\psi_1, ..., \psi_n)) \in \widetilde{\diamond}(v(\psi_1), ..., v(\psi_n))$*

Ordinary (deterministic) matrices correspond to the case when each $\widetilde{\diamond}$ is a function taking singleton values only. Thus in such matrices the truth-value assigned to $\diamond(\psi_1, ..., \psi_n)$ is uniquely determined by the truth-values of its subformulas: $v(\psi_1), ..., v(\psi_n)$. This, however, is not the case in Nmatrices, as v makes a non-deterministic choice out of the set of options $\tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$.

Definition 2. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be some Nmatrix for $\mathcal{L}$.*

1. *A valuation v satisfies a formula ψ (a set of formulas Γ) in $\mathcal{M}$, denoted by $v \models_{\mathcal{M}} \psi$ ($v \models_{\mathcal{M}} \Gamma$), if $v(\psi) \in \mathcal{D}$ ($v(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma$).*
2. *A valuation v satisfies a sequent $\Omega = \Gamma \Rightarrow \Delta$ in $\mathcal{M}$ if whenever $v \models_{\mathcal{M}} \Gamma$, there is some $\psi \in \Delta$, such that $v \models_{\mathcal{M}} \psi$. v satisfies a set of sequents if it satisfies every sequent in this set.*
3. *For two sets of formulas Γ, Δ, we write $\Gamma \vdash_{\mathcal{M}} \Delta$ if for every $\mathcal{M}$-legal valuation v, $v \models_{\mathcal{M}} \Gamma$ implies that $v \models_{\mathcal{M}} \psi$ for some $\psi \in \Delta$.*

Notation 1. *Let G be any Gentzen-type calculus. We denote $\Gamma \vdash_G \Delta$ when a sequent $\Gamma_0 \Rightarrow \Delta_0$ is provable in G for some $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$. For a set of sequents Θ and a sequent Ω, we denote $\Theta \vdash_G \Omega$ if Ω has a proof in G from Θ.*

Definition 3. *An Nmatrix $\mathcal{M}$ is* characteristic *for a calculus G if for every two sets of formulas Γ, Δ: $\Gamma \vdash_G \Delta$ iff $\Gamma \vdash_{\mathcal{M}} \Delta$. An Nmatrix $\mathcal{M}$ is* strongly characteristic *for G if for every set of sequents Θ and every sequent Ω: $\Theta \vdash_G \Omega$ iff $\Theta \vdash_{\mathcal{M}} \Omega$.*

As shown by the next theorem, Nmatrices can be used for characterizing logics that cannot be characterized by finite ordinary matrices.

Theorem 1. *([2]) Let $\mathcal{M}$ be a two-valued Nmatrix which has at least one proper non-deterministic operation. Then there is no finite deterministic matrix P, such that for every two sets of formulas Γ, Δ: $\Gamma \vdash_{\mathcal{M}} \Delta$ iff $\Gamma \vdash_P \Delta$.*

Definition 4. *A canonical rule of arity n is an expression $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/C$, where $m \geq 0$, C is either $\diamond(p_1, ..., p_n) \Rightarrow$ or $\Rightarrow \diamond(p_1, ..., p_n)$ for some n-ary connective $\diamond$, and for all $1 \leq i \leq m$, $\Pi_i \Rightarrow \Sigma_i$ is a clause such that $\Pi_i, \Sigma_i \subseteq \{p_1, ..., p_n\}$. An application of a canonical left introduction rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \diamond (p_1, ..., p_n) \Rightarrow$ is any inference step of the form:*

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \diamond(\psi_1, ..., \psi_n) \Rightarrow \Delta}$$

where Π_i^* and Σ_i^* are obtained from Π_i and Σ_i respectively by substituting ψ_j for p_j for all $1 \leq j \leq n$ and Γ, Δ are arbitrary sets of formulas. An application of a right introduction rule is defined similarly.
We call an application an identity application *when $\Sigma_i^* = \Sigma_i$ and $\Pi_i^* = \Pi_i$ for all $1 \leq i \leq n$.*

Definition 5. *A Gentzen-type calculus G is canonical if in addition to the standard axioms: $\psi \Rightarrow \psi$ (for any formula ψ), the cut rule*

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta}\ (cut)$$

and the structural rule of weakening, G has only canonical logical rules.

Definition 6. *An* extended axiom *is any sequent of the form $\Gamma \Rightarrow \Delta$, where $\Gamma \cap \Delta \neq \emptyset$. An extended axiom is atomic if $\Gamma \cap \Delta$ contains an atomic formula.*

Definition 7. *A canonical calculus G is* coherent *if for every two rules of the forms $\Theta_1/ \Rightarrow \diamond(p_1, ..., p_n)$ and $\Theta_2/ \diamond (p_1, ..., p_n) \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent (i.e., the empty set can be derived from it using cuts).*

The following well-known fact follows from the completeness of propositional resolution:

Proposition 1. *A set of clauses is satisfiable iff it is consistent.*

Notation 2. *Denote the clause* $\Rightarrow p_i$ *by* S_i^t *and the clause* $p_i \Rightarrow$ *by* S_i^f. *Let* $\overline{a} = \langle a_1, ..., a_n \rangle \in \{t, f\}^n$. *We denote* $\mathsf{C}_{\overline{a}} = \{S_i^{a_i}\}_{1 \leq i \leq n}$.

Lemma 1. *Let* Θ *be a set of clauses over* $\{p_1, ..., p_n\}$. *Let* $\overline{a} = \langle a_1, ..., a_n \rangle \in \{t, f\}^n$ *and let* v *be any valuation, such that* $v(p_i) = a_i$ *for all* $1 \leq i \leq n$. *Then* $\Theta \cup \mathsf{C}_{\overline{a}}$ *is consistent iff* v *is a (classical) model of* Θ.

Definition 8. *([4]) Let* G *be a coherent canonical calculus. The Nmatrix* $\mathcal{M}_G$ *is defined as follows for every* n*-ary connective* $\diamond$ *and* $\overline{a} = \langle a_1, ..., a_n \rangle \in \{t, f\}^n$:

$$\tilde{\diamond}_{\mathcal{M}_G}(a_1, ..., a_n) = \begin{cases} \{t\} & \text{if } \Theta / \Rightarrow \diamond(p_1, ..., p_n) \in G \text{ and } \Theta \cup \mathsf{C}_{\overline{a}} \text{ is consistent.} \\ \{f\} & \text{if } \Theta / \diamond(p_1, ..., p_n) \Rightarrow \in G \text{ and } \Theta \cup \mathsf{C}_{\overline{a}} \text{ is consistent.} \\ \{t, f\} & \text{otherwise} \end{cases}$$

Theorem 2. *([4])* $\mathcal{M}_G$ *is a strongly characteristic Nmatrix for* G.

The following theorem from [2, 4] establishes an exact correspondence between cut-elimination, two-valued Nmatrices, coherence of canonical calculi and their non-triviality, where a consequence relation $\vdash_G$ between sets of formulas is said to be *trivial* if for every two non-empty Γ, Δ: $\Gamma \vdash_G \Delta$.

Theorem 3. *([2, 4]) Let* G *be a canonical calculus. Then the following statements are equivalent: (1)* G *is coherent, (2)* $\vdash_G$ *is non-trivial, (3)* G *has a strongly characteristic two-valued Nmatrix, (4)* G *has a characteristic two-valued Nmatrix, (5)* G *admits cut-elimination.*

Proposition 2. *Let* G *be a coherent canonical calculus. Then the following is equivalent: (i)* $\mathcal{M}_G$ *is deterministic, (ii)* G *has a finite characteristic two-valued deterministic matrix, (iii)* G *has a finite characteristic deterministic matrix.*

Proof. $((i) \Rightarrow (ii))$ and $((ii) \Rightarrow (iii))$ are trivial. For $((iii) \Rightarrow (i))$, assume that $\mathcal{M}_G$ has at least one non-deterministic operation. Then by Theorem 1, there is no finite ordinary matrix P, such that $\vdash_P = \vdash_{\mathcal{M}}$. Hence, there is no characteristic finite deterministic matrix for G.

Equivalence of Calculi

Definition 9. *Two sets of canonical rules* S_1 *and* S_2 *are* equivalent *if for every application of* $R \in S_1$*, its conclusion is derivable from its premises using rules from* S_2 *together with structural rules, and vice versa. Two canonical calculi* G_1 *and* G_2 *are* cut-free equivalent *if their rules are equivalent.*

Proposition 3. *For every two coherent canonical calculi* G_1 *and* G_2 *which are cut-free equivalent,* $\mathcal{M}_{G_1} = \mathcal{M}_{G_2}$.

Proof. First we shall need the following technical propositions and notations:

Notation 3. *For a set of formulas Γ, denote by $\mathsf{At}(\Gamma)$ the set of atomic formulas occurring in Γ. For a sequent $\Omega = \Gamma \Rightarrow \Delta$, denote by $\mathsf{At}(\Omega)$ the sequent $\mathsf{At}(\Gamma) \Rightarrow \mathsf{At}(\Delta)$. For a clause Ω (a set of clauses Θ), denote by $mod(\Omega)$ the set of all the atomic valuations*[4] *which satisfy Ω (Θ).*

Lemma 2. *Let $R = \Theta/C$ be a canonical rule, where $\Theta = \{\Sigma_i \Rightarrow \Pi_i\}_{1\leq i\leq m}$. Consider an identity application (Defn. 4) of R with premises $\Omega_1, \ldots, \Omega_m$ and conclusion Ω. Then $(\bigcap_{1\leq i\leq m} mod(\mathsf{At}(\Omega_i))) \setminus mod(\mathsf{At}(\Omega)) \subseteq mod(\Theta)$.*

Proof. Let Ω be either $\Gamma \Rightarrow \Delta, \diamond(p_1, \ldots, p_n)$ or $\diamond(p_1, \ldots, p_n), \Gamma \Rightarrow \Delta$. Let $\Omega_i = \Gamma, \Sigma_i \Rightarrow \Pi_i, \Delta$. Let $v \in (\bigcap_{1\leq i\leq m} mod(\mathsf{At}(\Omega_i))) \setminus mod(\mathsf{At}(\Omega))$. $v \notin mod(\mathsf{At}(\Gamma \Rightarrow \Delta))$ (otherwise it would be the case that $v \in mod(\mathsf{At}(\Omega))$). Thus v satisfies $\mathsf{At}(\Gamma)$ but does not satisfy any of the formulas in $\mathsf{At}(\Delta)$. Let $1 \leq i \leq m$. If v satisfies Σ_i, then since v satisfies $\mathsf{At}(\Omega_i) = \mathsf{At}(\Gamma), \Sigma_i \Rightarrow \mathsf{At}(\Delta), \Pi_i$, there is some $\psi \in \Pi_i$, of which v is a model. Thus v satisfies $\Sigma_i \Rightarrow \Pi_i$ for all $1 \leq i \leq m$ and so $v \in mod(\Theta)$.

Corollary 1. *Let G be a canonical calculus. Suppose that Ω has a derivation in G from extended atomic axioms, which consists only of identity applications of canonical rules. If an atomic valuation v does not satisfy $\mathsf{At}(\Omega)$, then there is some canonical rule Θ/C applied in this derivation, such that $v \in mod(\Theta)$.*

Proof. By induction on the length l of the derivation of Ω. For $l = 1$ the claim trivially holds (v satisfies $\mathsf{At}(\Omega)$). Otherwise, consider the last rule applied in the derivation, which must be an identity application of some canonical rule Θ/C, where $\Theta = \{\Sigma_i \Rightarrow \Pi_i\}_{1\leq i\leq m}$. Denote its premises by $\Omega_1, \ldots, \Omega_m$ and its conclusion by Ω. Let $v \notin mod(\mathsf{At}(\Omega))$. If v satisfies $\mathsf{At}(\Omega_i)$ for all $1 \leq i \leq m$, then by Lemma 2, $v \in mod(\Theta)$. Otherwise there is some $1 \leq i \leq m$, such that v does not satisfy $\mathsf{At}(\Omega_i)$. By the induction hypothesis, v satisfies Θ' for some canonical rule Θ'/C' applied in the derivation of Ω_i.

Back to the proof of Proposition 3, let G_1 and G_2 be two coherent canonical calculi that are cut-free equivalent. Let $\diamond$ be some n-ary connective and $\overline{a} = \langle a_1, \ldots, a_n\rangle \in \{t, f\}^n$. Suppose that $\tilde{\diamond}_{\mathcal{M}_{G_1}}(\overline{a}) = \{t\}$. Then there is a rule in G_1 of the form $R = \Theta/ \Rightarrow \diamond(p_1, \ldots, p_n)$, such that $\Theta \cup \mathsf{C}_{\overline{a}}$ is consistent. Consider the application of R with premises Θ and conclusion $\Rightarrow \diamond(p_1, \ldots, p_n)$. Let v be any atomic valuation, such that $v(p_i) = a_i$ for all $1 \leq i \leq n$. Since $\Theta \cup \mathsf{C}_{\overline{a}}$ is consistent, by Lemma 1, $v \in mod(\Theta)$. Now since G_1 and G_2 are cut-free equivalent, there is a derivation D of $\Rightarrow \diamond(p_1, \ldots, p_n)$ from Θ using the rules of G_2 and weakening. Since $\mathsf{At}(\Rightarrow \diamond(p_1, \ldots, p_n)) = \emptyset$, $v \notin \mathsf{At}(\Rightarrow \diamond(p_1, \ldots, p_n))$, and by Corollary 1, there is some rule Θ'/S of G_2 applied in D, such that $v \in mod(\Theta')$. Since the derivation of $\Rightarrow \diamond(p_1, \ldots, p_n)$ from Θ is cut-free, it must be the case that this application is an identity application and S is the sequent $\Rightarrow (p_1, \ldots, p_n)$. By Lemma 1, $\Theta' \cup \mathsf{C}_{\overline{a}}$ is consistent. Hence, $\tilde{\diamond}_{\mathcal{M}_{G_2}}(\overline{a}) = \{t\}$. The case when $\tilde{\diamond}_{\mathcal{M}_{G_1}}(\overline{a}) = \{f\}$ is handled similarly. If $\tilde{\diamond}_{\mathcal{M}_{G_2}}(\overline{a}) = \{t\}$ (or $\tilde{\diamond}_{\mathcal{M}_{G_2}}(\overline{a}) = \{f\}$), the

[4] By an atomic valuation we mean any mapping from the atomic formulas of $\mathcal{L}$ to $\{t, f\}$.

proof that $\tilde{\diamond}_{\mathcal{M}_{G_1}}(\overline{a}) = \{t\}$ (or $\tilde{\diamond}_{\mathcal{M}_{G_1}}(\overline{a}) = \{f\}$) is symmetric to the previous case.

We leave the following easy proposition to the reader:

Proposition 4. *If a canonical calculus G is coherent, so is any canonical calculus G' which is cut-free equivalent to G.*

Canonical Calculi in Normal Form

A canonical calculus may have a number of right (and left) introduction rules for the same connective. However, below we show (an adaptation of proofs from [3] and [5]) that any canonical calculus can be transformed (normalized) into a cut-free equivalent calculus with at most one right and one left introduction rule for each connective.

Definition 10. *We say that sequent $\Gamma \Rightarrow \Delta$ is* subsumed *by a sequent $\Gamma' \Rightarrow \Delta'$ if $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. A canonical calculus G is in* normal form *if (i) G has at most one left and at most one right introduction rule for each connective, (ii) its introduction rules have no extended axioms as their premises, and (iii) its introduction rules have no clauses in their premises which are subsumed by some other clause in their premises.*

Lemma 3. *Let R be a canonical rule having an extended axiom as one of its premises. The rule obtained by discarding this premise is equivalent to R.*

Proposition 5. *Every canonical calculus G has a cut-free equivalent calculus G^n in normal form.*

Proof. Let us describe the transformation of G into a calculus G^n in normal form. Take a pair of rules in G of the forms $R_1 = \{\Sigma_i^1 \Rightarrow \Pi_i^1\}_{1\leq i\leq m}/ \Rightarrow \diamond(p_1 \dots p_n)$ and $R_2 = \{\Sigma_j^2 \Rightarrow \Pi_j^2\}_{1\leq j\leq l}/ \Rightarrow \diamond(p_1 \dots p_n)$. Replace R_1 and R_2 in G by $R = \{\Sigma_i^1, \Sigma_j^2 \Rightarrow \Pi_i^1, \Pi_j^2\}_{1\leq i\leq m, 1\leq j\leq l}/ \Rightarrow \diamond(p_1 \dots p_n)$. Clearly, any application of R can be simulated by applying R_1 and R_2. Moreover, any application of R_1 and of R_2 can be simulated by weakening and R. Hence, $\{R\}$ and $\{R_1, R_2\}$ are cut-free equivalent. By repeatedly applying this step, we get at most one left and one right introduction rule for each connective. Next, in the obtained rules discard the premises which are extended axioms. By Lemma 3, G and the resulting calculus G' are cut-free equivalent. Finally, in each rule of G' discard any premise $\Gamma \Rightarrow \Delta$ subsumed by any other premise $\Gamma' \Rightarrow \Delta'$. The resulting calculus is cut-free equivalent to G' as $\Gamma \Rightarrow \Delta$ can be derived from $\Gamma' \Rightarrow \Delta'$ using weakening.

Example 1. Consider the canonical calculus G_X with four introduction rules for the binary connective X (representing XOR):

$$\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\}/ \Rightarrow p_1 X p_2 \quad \{\Rightarrow p_2 \ ; \ p_1 \Rightarrow\}/ \Rightarrow p_1 X p_2$$

$$\{\Rightarrow p_1 \ ; \ \Rightarrow p_2\}/p_1 X p_2 \Rightarrow \quad \{p_1 \Rightarrow \ ; \ p_2 \Rightarrow\}/p_1 X p_2 \Rightarrow$$

This calculus can be transformed into a cut-free equivalent calculus G_X^n in normal form as follows. We start by replacing the first two rules by the following rule:

$$\{\Rightarrow p_1, p_2 \ ; \ p_1, p_2 \Rightarrow; \ p_1 \Rightarrow p_1 \ ; \ p_2 \Rightarrow p_2\}/ \Rightarrow p_1 X p_2$$

The second pair of rules can be replaced by:

$$\{p_1 \Rightarrow p_2 \ ; \ p_2 \Rightarrow p_1; \ p_1 \Rightarrow p_1 \ ; \ p_2 \Rightarrow p_2\}/p_1 X p_2 \Rightarrow$$

Finally, by Lemma 3, the axioms in the premises can be discarded and we get the following cut-free equivalent calculus G_X^n in normal form:

$$\{\Rightarrow p_1, p_2 \ ; \ p_1, p_2 \Rightarrow\}/ \Rightarrow p_1 X p_2 \quad \{p_1 \Rightarrow p_2 \ ; \ p_2 \Rightarrow p_1\}/p_1 X p_2 \Rightarrow$$

3 Investigating Invertibility

In this section we investigate the connection between invertibility and determinism in coherent canonical calculi. We show that the latter is a necessary condition for invertibility, which turns out to be also sufficient for calculi in normal form. The usual definition of invertibility of rules is the following:

Definition 11. *A rule R is* invertible in a calculus G *if for every application of R it holds that whenever its conclusion is provable in G, also each of its premises is provable in G.*

Notation 4. *Henceforth we use the metavariable R to refer to a canonical rule of the form $\{\Sigma_i \Rightarrow \Pi_i\}_{1 \leq i \leq m}/ \Rightarrow \diamond(p_1, ..., p_n)$.*

We now introduce a useful notion which is equivalent to invertibility in the context of canonical calculi.

Definition 12. *Let G be a canonical calculus. A rule R is* canonically invertible in G *if for every $1 \leq i \leq m$: $\Sigma_i \Rightarrow \Pi_i$ has a proof in G from $\Rightarrow \diamond(p_1, ..., p_n)$. Canonical invertibility for left introduction rules is defined similarly.*

Remark 1. It is important to note that unlike standard invertibility, canonical invertibility is defined for *rules*, and not their instances.

Proposition 6. *A canonical rule is invertible in a canonical calculus G iff it is canonically invertible in G.*

Proof. ($\Leftarrow$) Assume w.l.o.g. that a rule R is canonically invertible in G. Consider an application of R with premises $\Gamma, \Sigma_1^* \Rightarrow \Delta, \Pi_1^*; \ \dots \ ; \ \Gamma, \Sigma_m^* \Rightarrow \Delta, \Pi_m^*$ and conclusion $\Gamma \Rightarrow \Delta, \diamond(\psi_1, ..., \psi_n)$ where for all $1 \leq j \leq m$, Σ_j^*, Π_j^* are obtained from Σ_j, Π_j by replacing each p_k by ψ_k for all $1 \leq k \leq n$. Suppose that $\vdash_G \Gamma \Rightarrow \Delta, \diamond(\psi_1, ..., \psi_n)$. We need to show that $\vdash_G \Gamma, \Sigma_j^* \Rightarrow \Delta, \Pi_j^*$ for all $1 \leq j \leq m$. Being R canonically invertible, there is a proof of $\Sigma_j \Rightarrow \Pi_j$ from $\Rightarrow \diamond(p_1, ..., p_n)$. By replacing in this proof each p_k by ψ_k and adding the contexts Γ and Δ everywhere, we obtain a proof of $\Gamma, \Sigma_j^* \Rightarrow \Delta, \Pi_j^*$ from $\Gamma \Rightarrow \Delta, \diamond(\psi_1, ..., \psi_n)$. Thus if $\Gamma \Rightarrow \Delta, \diamond(\psi_1, ..., \psi_n)$ is provable, so is $\Gamma, \Sigma_j^* \Rightarrow \Delta, \Pi_j^*$. Hence R is invertible. ($\Rightarrow$) Assume that R is invertible in G. Consider the application of R with conclusion $\diamond(p_1, ..., p_n) \Rightarrow \diamond(p_1, ..., p_n)$. Being G canonical, $\diamond(p_1, ..., p_n) \Rightarrow \diamond(p_1, ..., p_n)$ is provable in G. Since R is invertible, its premises $\Sigma_i, \diamond(p_1, ..., p_n) \Rightarrow \Pi_i$ are provable as well. By applying cut and weakening, we obtain a proof of $\Sigma_i \Rightarrow \Pi_i$ from $\Rightarrow \diamond(p_1, ..., p_n)$ for every $1 \leq i \leq m$ and the claim follows.

Next we introduce the notion of *expandability* of rules, and show that it is equivalent to invertibility in coherent canonical calculi.

Definition 13. *A canonical right introduction rule R is* expandable *in a canonical calculus G if for every $1 \leq i \leq m$: $\diamond(p_1, ..., p_n), \Sigma_i \Rightarrow \Pi_i$ has a* cut-free *proof in G. The notion of expandability in G for a left introduction rule is defined symmetrically.*

Proposition 7. *For any canonical calculus G, every expandable rule is invertible. If G is coherent, then every invertible rule is expandable.*

Proof. Let G be any canonical calculus. Assume w.l.o.g. that the rule R is expandable in G. Hence $\Sigma_i, \diamond(p_1, ..., p_n) \Rightarrow \Pi_i$ is provable for each $1 \leq i \leq m$. By cut, $\Sigma_i \Rightarrow \Pi_i$ is provable from $\Rightarrow \diamond(p_1, ..., p_n)$. Thus R is canonically invertible, and hence invertible by Proposition 6. Now assume that G is coherent and R is invertible in G. By Proposition 6, R is canonically invertible, and so for all $1 \leq i \leq m$: $\Sigma_i \Rightarrow \Pi_i$ is derivable from $\Rightarrow \diamond(p_1, \ldots, p_n)$. By adding $\diamond(p_1, \ldots, p_n)$ on the left side of all the sequents in the derivation, we obtain a derivation of $\diamond(p_1, \ldots, p_n), \Sigma_i \Rightarrow \Pi_i$ in G. Since G is coherent, by Theorem 3 it admits cut-elimination, thus we have a cut-free derivation of $\diamond(p_1, \ldots, p_n), \Sigma_i \Rightarrow \Pi_i$ in G, and hence R is expandable.

Although expandability and invertibility are equivalent for coherent canonical calculi, checking the former is an easier task, as it amounts to checking whether a sequent is cut-free provable.
Not surprisingly, in canonical calculi which are not coherent (and hence do not admit cut-elimination by Theorem 3), expandability is strictly stronger than invertibility. This is demonstrated by the following example.

Example 2. Consider the following non-coherent calculus G_B:

$$R_1 = \{p_1 \Rightarrow p_2\}/ \Rightarrow p_1 \star p_2 \qquad R_2 = \{p_1 \Rightarrow p_2\}/p_1 \star p_2 \Rightarrow$$

Neither $p_1 \star p_2, p_1 \Rightarrow p_2$ nor $p_1 \Rightarrow p_2, p_1 \star p_2$ have a cut-free derivation in G_B. Indeed, while trying to find a proof bottom-up, the only rules which could be applied are either introduction rules for $\star$ or structural rules but these do not lead to (extended) axioms. Thus the above rules are not expandable. However, $p_1 \Rightarrow p_2$ has a derivation[5] (using cuts) in G_B:

$$\dfrac{\dfrac{\dfrac{p_1 \Rightarrow p_1}{p_1, p_2 \Rightarrow p_1}\,(w,l)}{p_1 \Rightarrow p_2 \star p_1}\,(R_1) \qquad \dfrac{\dfrac{p_2 \Rightarrow p_2}{p_2 \Rightarrow p_1, p_2}\,(w,r)}{p_2 \star p_1 \Rightarrow p_2}\,(R_2)}{p_1 \Rightarrow p_2}\,(cut)$$

[5] Note that by Theorem 3, G_B is trivial as it is not coherent. Hence, for any two atoms p, q: $\vdash_{G_B} p \Rightarrow q$.

Thus R_1 and R_2 are invertible, although not expandable.

Proposition 8. *Let G be a coherent canonical calculus. If G has an invertible rule for $\diamond$, then $\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic.*

Proof. Assume w.l.o.g. that R is invertible in G. Suppose by contradiction that $\tilde{\diamond}_{\mathcal{M}_G}$ is not deterministic. Then there is some $\overline{a} = \langle a_1, ..., a_n\rangle \in \{t, f\}^n$, such that $\tilde{\diamond}_{\mathcal{M}_G}(\overline{a}) = \{t, f\}$. Let v be any $\mathcal{M}_G$-legal valuation, such that $v(p_i) = a_i$ and $v(\diamond(p_1, ..., p_n)) = t$ (such v exists since $\tilde{\diamond}_{\mathcal{M}_G}(\overline{a}) = \{t, f\}$). $\Theta \cup \mathsf{C}_{\overline{a}}$ is inconsistent (since otherwise by the definition of $\mathcal{M}_G$, it would be the case that $\tilde{\diamond}_{\mathcal{M}_G}(\overline{a}) = \{t\}$ due to the rule R). Thus $(*)$ there is some $1 \leq j_v \leq m$, for which v does not satisfy the sequent $\Sigma_{j_v} \Rightarrow \Pi_{j_v}$ (otherwise, since v also satisfies $\mathsf{C}_{\langle a_1,...,a_n\rangle}$ the set of clauses $\Theta \cup \mathsf{C}_{\overline{a}}$ would be consistent). Since R is invertible, by Proposition 6 it is also canonically invertible. Then for every $1 \leq i \leq m$, $\Sigma_i \Rightarrow \Pi_i$ is provable in G from $\Rightarrow \diamond(p_1, ..., p_n)$. Since $\mathcal{M}_G$ is strongly characteristic for G, $\Rightarrow \diamond(p_1, ..., p_n) \vdash_{\mathcal{M}_G} \Sigma_i \Rightarrow \Pi_i$ for every $1 \leq i \leq m$. Since v satisfies $\Rightarrow \diamond(p_1, ..., p_n)$, it should also satisfy $\Sigma_{j_v} \Rightarrow \Pi_{j_v}$, which contradicts $(*)$.

The following theorem establishes the correspondence between determinism, invertibility and expandability:

Theorem 4. *Let $\mathcal{L}$ be a propositional language and G a coherent canonical calculus in normal form. The following statements are equivalent:*

1. *G has an invertible rule for $\diamond$.*
2. *G has an expandable rule for $\diamond$.*
3. *$\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic.*
4. *G has a rule for $\diamond$ and all its rules are invertible.*
5. *G has a rule for $\diamond$ and all its rules are expandable.*

Proof. $1 \Rightarrow 3$ follows by Proposition 8. $1 \Leftrightarrow 2$ and $4 \Leftrightarrow 5$ follow by Proposition 7. $4 \Rightarrow 1$ follows trivially. It remains to show that $3 \Rightarrow 5$. Suppose that $\mathcal{M}_G$ is deterministic. By the definition of $\mathcal{M}_G$, there must be at least one rule for $\diamond$, as otherwise $\tilde{\diamond}_{\mathcal{M}_G}(\overline{a}) = \{t, f\}$ for every $\overline{a} \in \{t, f\}^n$. Let R be any such rule w.l.o.g. Suppose by contradiction that R is not expandable in G. Then there is some $1 \leq i \leq m$, such that $\diamond(p_1, ..., p_n), \Sigma_i \Rightarrow \Pi_i$ has no cut-free proof in G. Since G is coherent, by Theorem 3 it admits cut-elimination, and so $\diamond(p_1, ..., p_n), \Sigma_i \Rightarrow \Pi_i$ is not provable in G. Since $\mathcal{M}_G$ is a characteristic Nmatrix for G, $\Sigma_i, \diamond(p_1, ..., p_n) \nvdash_{\mathcal{M}_G} \Pi_i$. Then there is an $\mathcal{M}_G$-legal valuation v, such that $v \models_{\mathcal{M}_G} \{\diamond(p_1, ..., p_n)\} \cup \Sigma_i$ and for every $\psi \in \Pi_i$: $v \not\models_{\mathcal{M}_G} \psi$. Let $\overline{a} = \langle v(p_1), ..., v(p_n)\rangle$. By Lemma 1, $(*)$ $\{\Sigma_i \Rightarrow \Pi_i\}_{1 \leq i \leq m} \cup \mathsf{C}_{\overline{a}}$ is inconsistent. Since $\mathcal{M}_G$ is deterministic, either $\tilde{\diamond}_{\mathcal{M}_G}(\overline{a}) = \{t\}$ or $\tilde{\diamond}_{\mathcal{M}_G}(\overline{a}) = \{f\}$. But the first case is impossible by definition of $\mathcal{M}_G$ and the fact that R is the only right introduction rule for $\diamond$. Thus $\tilde{\diamond}_{\mathcal{M}_G}(\overline{v}) = \{f\}$, in contradiction to our assumption that $v \models_{\mathcal{M}_G} \diamond(p_1, ..., p_n)$. Therefore R is expandable in G.

The next example demonstrates that Theorem 4 does not hold for calculi which are not in normal form.

Example 3. Consider the calculus G_X in Example 1 and its associated (deterministic) Nmatrix $\mathcal{M}_{G_X}$:

X	t	f
t	$\{f\}$	$\{t\}$
f	$\{t\}$	$\{f\}$

It is easy to see that $\Rightarrow p_1 X p_2 \not\vdash_{\mathcal{M}_{G_X}} \Rightarrow p_1$. Hence $\Rightarrow p_1$ is not derivable in G_X from $\Rightarrow p_1 X p_2$ and so the first rule is not canonically invertible. By Proposition 6 it is not invertible, and by Proposition 7, it is also not expandable.

Corollary 2. *If a coherent canonical calculus G in normal form has a right (left) invertible rule for $\diamond$ with a non-empty set of premises, then it also has a left (right) invertible rule for $\diamond$.*

Proof. Let G be a canonical coherent calculus in normal form with an invertible rule $[\Theta / \Rightarrow \diamond(p_1, \ldots, p_n)]$. By Theorem 4, $\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic. Since Θ is non-empty and it cannot be a set of extended axioms (recall that G is in normal form), there is some $v \notin mod(\Theta)$ (cf. Notation 3). But since $\tilde{\diamond}_{\mathcal{M}_G}(v(p_1), \ldots, v(p_n))$ is deterministic, there must be a rule $[\Theta'/C']$, such that $\Theta' \cup \mathsf{C}_{\langle v(p_1),\ldots,v(p_n)\rangle}$ is consistent. Since G is in normal form and $\Theta' \neq \Theta$, this cannot be a right introduction rule for $\diamond$, hence C' is $\diamond(p_1, \ldots, p_n) \Rightarrow$. By Theorem 4, this rule is invertible.

4 Investigating Axiom expansion

Axiom expansion is an important property of deduction systems, which allows for the reduction of logical axioms to the atomic case. We show that for coherent canonical calculi this property fully characterizes the existence for a calculus of a two-valued deterministic characteristic matrix. Furthermore we show that in coherent canonical calculi axiom expansion is a necessary condition for invertibility, which turns out to be also sufficient for calculi in normal form.

Definition 14 ([7]). *An n-ary connective $\diamond$ admits axiom expansion in a calculus G if whenever $\diamond(p_1, ..., p_n) \Rightarrow \diamond(p_1, ..., p_n)$ is provable in G, it has a cut-free derivation in G from atomic axioms of the form $\{p_i \Rightarrow p_i\}_{1\leq i\leq n}$.*

Proposition 9. *Let G be a canonical calculus. If G has an expandable rule for $\diamond$, then $\diamond$ admits axiom expansion in G.*

Proof. Suppose without loss of generality that G has a right introduction rule $R = \{\Sigma_i \Rightarrow \Pi_i\}_{1\leq i\leq m}/ \Rightarrow \diamond(p_1, ..., p_n)$, which is expandable in G. Then $(*)$ $\Sigma_i, \diamond(p_1, ..., p_n) \Rightarrow \Pi_i$ has a cut-free derivation in G for every $1 \leq i \leq m$. Note that $\Sigma_i, \Pi_i \subseteq \{p_1, ..., p_n\}$ and hence the sequents denoted by $(*)$ are derivable from atomic axioms $\{p_i \Rightarrow p_i\}_{1\leq i\leq n}$. By applying R with premises $\{\Sigma_i, \diamond(p_1, ..., p_n) \Rightarrow \Pi_i\}_{1\leq i\leq m}$, we obtain the required cut-free derivation of $\diamond(p_1, ..., p_n) \Rightarrow \diamond(p_1, ..., p_n)$ in G from atomic axioms. Thus $\diamond$ admits axiom expansion in G.

Lemma 4. *Let G be a canonical calculus. If a sequent Ω has a cut-free proof in G from atomic axioms, then Ω also has a cut-free proof in G from atomic (extended) axioms with no application of weakening.*

Theorem 5. *Let G be a coherent canonical calculus. $\diamond$ admits axiom expansion in G iff $\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic.*

Proof. ($\Rightarrow$) If $\diamond$ admits axiom expansion in G then $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ is cut-free derivable from atomic axioms. By Lemma 4, we can assume that the derivation contains only extended atomic axioms and applications of canonical rules. Since there are no cuts, it is easy to see that the applications of canonical rules in this derivation must be identity applications of introduction rules for $\diamond$. Now since $\mathsf{At}(\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n))$ is the empty sequent, Corollary 1 ensures that for every valuation v there is some logical rule Θ/C (where C is either $\Rightarrow \diamond(p_1, \ldots, p_n)$ or $\diamond(p_1, \ldots, p_n) \Rightarrow$) used in this derivation, such that $v \in mod(\Theta)$. By Lemma 1, for every $\overline{a} = \langle a_1, \ldots, a_n \rangle \in \{t, f\}^n$ there is some canonical rule Θ/C for $\diamond$, such that $\Theta \cup \mathsf{C}_{\overline{a}}$ is consistent. Thus $\tilde{\diamond}_{\mathcal{M}_G}(a_1, \ldots, a_n)$ is a singleton, and so $\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic.
($\Leftarrow$) First transform G into a cut-free equivalent calculus G^n in normal form (cf. Proposition 5). By Proposition 4, G^n is coherent, and by Proposition 3, $\mathcal{M}_{G^n}$ is deterministic. By Theorem 4 and Proposition 9, $\diamond$ admits axiom expansion in G^n and therefore also in G, since G is cut-free equivalent to G^n.

Remark 2. An alternative proof of ($\Rightarrow$) is contained in [5] for a generalization of canonical calculi.

Corollary 3. *For a coherent canonical calculus G, every connective admits axiom expansion in G iff G has a two-valued characteristic deterministic matrix.*

Proof. Follows from the theorem above and Proposition 2.

Corollary 4. *If a coherent canonical calculus G has an invertible rule for $\diamond$, then $\diamond$ admits axiom expansion in G.*

Proof. If G has an invertible rule for $\diamond$, then by Proposition 7 it is also expandable. By Proposition 9, $\diamond$ admits axiom expansion in G.

We finish the paper by summarizing the correspondence between determinism, invertibility and axiom expansion:

Corollary 5. *Let $\mathcal{L}$ be a propositional language and G a coherent canonical calculus in normal form with introduction rules for each connective in $\mathcal{L}$. The following are equivalent: (i) The rules of G are invertible, (ii) G has a characteristic two-valued deterministic matrix, and (iii) Every connective of $\mathcal{L}$ admits axiom expansion in G.*

Proof. By Proposition 2, the existence of a two-valued characteristic deterministic matrix for G is equivalent to $\mathcal{M}_G$ being deterministic. The rest follows by Theorem 4, Corollary 2 and Theorem 5.

As shown by the following example the above correspondence does not hold for calculi which are not in normal form.

Example 4. Consider the calculus G_X of Example 1. Although the rules for the connective X are not invertible, X admits axiom expansion:

$$\dfrac{\dfrac{\dfrac{p_1 \Rightarrow p_1}{p_1 \Rightarrow p_1, p_1 X p_2} \qquad \dfrac{\dfrac{p_2 \Rightarrow p_2}{p_2 \Rightarrow p_2, p_1} \quad \dfrac{p_1 \Rightarrow p_1}{p_2, p_1 \Rightarrow p_1}}{p_2 \Rightarrow p_1, p_1 X p_2}}{p_1 X p_2 \Rightarrow p_1, p_1 X p_2} \qquad \dfrac{\dfrac{p_2 \Rightarrow p_2}{p_1 X p_2, p_2 \Rightarrow p_2} \qquad \dfrac{\dfrac{p_1 \Rightarrow p_1}{p_2, p_1 \Rightarrow p_1} \quad \dfrac{p_2 \Rightarrow p_2}{p_2, p_1 \Rightarrow p_2}}{p_1 X p_2, p_2, p_1 \Rightarrow}}{p_1 X p_2, p_2 \Rightarrow p_1 X p_2}}{p_1 X p_2 \Rightarrow p_1 X p_2}$$

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