Systematic Construction of Analytic Calculi for Logics of Formal Inconsistency

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ABSTRACT.
This paper makes a substantial step towards automatization of paraconsistent reasoning by providing a method for a systematic generation of analytic calculi for thousands of Logics of Formal (In)consistency. The method relies on non-deterministic three-valued semantics for these logics, and produces in a modular way uniform Gentzen-type rules, corresponding to a variety of schemata considered in the literature of LFIs.

1   Introduction

A *paraconsistent logic* is a logic which allows non-trivial inconsistent theories. One of the oldest and best known approaches to paraconsistency is da Costa's approach ([da Costa, 1974; da Costa *et al.*, 1995; D'Ottaviano, 1990]), which seeks to allow to use of classical logic whenever it is safe to do so, but behaves completely differently when contradictions are involved. This approach has led to the introduction of the family of *Logics of Formal (In)consistency (LFIs)* by W.A. Carnielli ([Carnielli *et al.*, 2007; Carnielli and Marcos, 2002]). This family is based on two key ideas. The first is that propositions should be divided into two sorts: the "normal" (or consistent), and the "abnormal" (or inconsistent) ones. While classical logic can be applied freely to normal propositions, its use is restricted for the abnormal ones. The second idea is to reflect this classification within the language used. In the important class of LFIs called *C-systems* ([Carnielli and Marcos, 2002]), this is done by employing a specific formula constructor $C(\varphi)$ which is available in the language, where the intuitive meaning of $C(\varphi)$ is "φ is consistent". Usually this is done by simply introducing a special unary connective $\circ$, and taking $C(\varphi)$ to be $\circ\varphi$.

For a long time the class of C-systems (which is the most important and useful class of LFIs) had two major shortcomings, which in our opinion prevented it from becoming a widely-used logical formalism for reasoning with inconsistent data and theories. The first is that originally these systems lacked a corresponding intuitive and useful semantics, which would provide real insight into them. Later bivaluations semantics and possible translations semantics were introduced for them ([Carnielli, 1998; Marcos, 2004; Carnielli *et al.*, 2007; Carnielli and Marcos, 2002]). However, both of these types of semantics are problematic from the crucial point of view of *analyticity*. A semantics is *analytic* if for determining whether φ follows from T it always suffices to check only *partial* models, in which only subformulas of $T \cup \{\varphi\}$ are involved. Unfortunately, neither bivaluations semantics, nor possible translations semantics

are satisfactory in this respect, as their analyticity is not apriorily guaranteed. Accordingly, a corresponding proposition should be proved from scratch (if it is true at all) for any potentially useful instance of these types of semantics. This unfortunate state of affairs was finally remedied in [Avron, 2007b; Avron, 2005a; Avron, 2007a], where simple, modular and analytic semantics for practically all the propositional C-systems considered in the literature was introduced. This semantics is based on the use of *non-deterministic matrices (Nmatrices)*. This type of structures provides a natural generalization (which is still analytic) of the class of a many-valued matrices.[1] In this generalization the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options. The analyticity of this kind of semantics guarantees that a logic which has a finite characteristic Nmatrix is necessarily decidable.

The second shortcoming of C-systems was that their formulation was originally given in terms of Hilbert-type calculi, and for many years no analytic[2] calculi were available for most of them. At first most of the efforts for finding such calculi concentrated on the historical system C_1 of da Costa. After an aborted attempt by Raggio in the sixties ([Raggio, 1968]), Beziau proposed in [Béziau, 1993] somewhat peculiar sequent rules for C_1, using an intuitive translation of certain semantical conditions. Later he proved a general completeness theorem which explains why this intuitive translation works. Proving cut-elimination using his "monstrous" rules (as he himself describes them in [Béziau, 2001]) was another non-trivial task. At about the same time, Carnielli et al. introduced a tableau system for C_1 ([Carnielli and Lima-Marques, 1992; Carnielli *et al.*, 2007]). Since then further analytic systems were provided for other LFIs. This includes systems for mbC and Ci and LFI1 ([Carnielli and Marcos, 2001]), mCi [Neto and Finger, 2007], and recently for bC, Ci and Cil [Gentilini, 2011]. However, since each of these calculi has been tailored for some specific LFI, their rules have been introduced in a sort of an ad-hoc manner, and they have no uniform structure. Therefore even a slight modification of any of these LFIs practically means starting the search for a corresponding analytic calculus all over again.

In this paper we provide a *uniform* and *modular* method for a *systematic* generation of cut-free sequent calculi for a large family of thousands of LFIs. The method exploits an algorithm given in [Avron *et al.*, 2006] for constructing an analytic Gentzen-type system for a logic which has a characteristic finite-valued Nmatrix, and its language is sufficiently expressive for that Nmatrix. The resulting sequent calculus automatically enjoys cut-admissibility, and its rules have a uniform form, closely related to that used in classical logic and other well known calculi. We believe that these results can open the door to efficient implementations of theorem provers based on this type of paraconsistent logics, and that this in its turn will lead to useful applications of them for reasoning under uncertainty.

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[1] In turn, bivaluations semantics and especially possible translations semantics can be viewed as a generalization of the semantics of Nmatrices (see [Carnielli and Coniglio, 2005]) —a generalization in which the property of analyticity is *lost*.

[2] Note that in this context we mean the usual syntactic analyticity of a calculus, as opposed to the semantic analyticity described above.

2 Preliminaries

In what follows, $\mathcal{L}$ is a propositional language, and $Frm_\mathcal{L}$ is its set of wffs. The metavariables T, S range over theories of $\mathcal{L}$-formulas, and Γ, Δ range over finite theories of $\mathcal{L}$-formulas.

2.1 A Taxonomy of LFIs

Let $\mathcal{L}_{cl}^+ = \{\wedge, \vee, \supset\}$, and $\mathcal{L}_C = \{\wedge, \vee, \supset, \neg, \circ\}$.

DEFINITION 1. Let HCL^+ be some standard Hilbert-style system which has MP as the only inference rule, and is sound and strongly complete for the $\mathcal{L}_{cl}^+$-fragment of classical propositional logic. The system **B** over $\mathcal{L}_C$ is obtained by adding to HCL^+ the following axioms:

$$(\mathbf{t})\ \neg\varphi \vee \varphi \qquad\qquad (\mathbf{b})\ \circ\varphi \supset (\varphi \wedge \neg\varphi \supset \psi)$$

The system **Bk** is obtained from **B** by adding the following axiom:

$$(\mathbf{k})\ \circ\varphi \vee (\varphi \wedge \neg\varphi)$$

REMARK 2. It is sometime useful to split $(\mathbf{k})$ into the following two axioms:

$$(\mathbf{k_1})\ \circ\varphi \vee \varphi \qquad\qquad (\mathbf{k_2})\ \circ\varphi \vee \neg\varphi$$

REMARK 3. The system **B** is frequently taken as the most basic LFI (see e.g. [Carnielli *et al.*, 2007; Carnielli and Marcos, 2002], where it is called **mbC**). Nevertheless, we find it much more appropriate to take **Bk** as the most basic C-system. Our reasons for this choice are the following:

1. Given the intended meaning of $\circ\varphi$ as "φ is consistent", the meaning of axiom $(\mathbf{b})$ is that no formula is both consistent and contradictory. Axiom $(\mathbf{k})$ complements this by saying that every formula is either consistent or contradictory. This seems to be as essential for the intended meaning of $\circ\varphi$ as the intended content of axiom $(\mathbf{b})$.

2. As we shall see later, the intended meaning of both $(\mathbf{b})$ and $(\mathbf{k})$ we have just explained is well reflected in the three-valued non-deterministic semantics used in this paper for deriving cut-free sequent systems. Moreover, the semantic effect of including $(\mathbf{k})$ is that of $\circ\psi$ can be read as "ψ behaves classically" (i.e., it is assigned a classical truth-value).

3. A particularly strong indication that **Bk** is the most natural basic C-system is given by the Gentzen-type system for it (and for all its extensions) provided below. Exactly like in the case of $\wedge, \vee$ and $\supset$, that system includes exactly two rules for $\circ$: one for introducing it on the left hand side of a sequent, the other for introducing it on the right hand side. The first is a direct translation of axiom $(\mathbf{b})$, the other is a direct translation of axiom $(\mathbf{k})$. What is important here is that like in the cases of $\wedge, \vee$ and $\supset$, these two rules are *dual* to each other in the sense that each of them is equivalent (using cuts) to the joint inverses of the other. It follows that the rules for $(\mathbf{b})$ and $(\mathbf{k})$ are strongly related to each other, in exactly the

same way that the rules for $\vee, \wedge$, and $\supset$ do. Moreover this implies that both rules are invertible in the Gentzen-type system of **Bk** (see Remark 35 below) — something which is not true in the system for B.

4. Finally, we note that **(k)** is anyway a theorem of almost every important LFI ever studied. This is due to the fact that it is derivable in **B** from either of the following axioms (each of which is of a central importance in LFIs: **(i)**, **(l)**, and **(d)** (see Definition 4 below and the remark that follows it, and the discussion in Section 4). These dependencies are easily established; we demonstrate the case of **(l)** in Example 34.

Next we provide a list of axioms which are frequently used for defining LFIs:

DEFINITION 4. Let **A** be the following set of axioms for $\sharp \in \{\wedge, \supset, \vee\}$:

$$
\begin{array}{llll}
\textbf{(c)} & \neg\neg\varphi \supset \varphi & \textbf{(e)} & \varphi \supset \neg\neg\varphi \\
\textbf{(i}_1\textbf{)} & \neg\circ\varphi \supset \varphi & \textbf{(i}_2\textbf{)} & \neg\circ\varphi \supset \neg\varphi \\
\textbf{(o}_\sharp^1\textbf{)} & \circ\varphi \supset \circ(\varphi\sharp\psi) & \textbf{(o}_\sharp^2\textbf{)} & \circ\psi \supset \circ(\varphi\sharp\psi) \\
\textbf{(a}_\sharp\textbf{)} & (\circ\varphi \wedge \circ\psi) \supset \circ(\varphi\sharp\psi) & &
\end{array}
$$

REMARK 5. In the literature on LFIs one usually finds what is called axiom **(i)**. This axiom is the conjunction of our **(i$_1$)** and **(i$_2$)**. Similarly, the axiom **(o$_\sharp$)** which is frequently mentioned in the literature is the conjunction of our **(o$_\sharp^1$)** and **(o$_\sharp^2$)**. Note also that the extensions of **B** with **(c)**, and with **(c)** and **(i)**, are called **bC** and **Ci** respectively in [Gentilini, 2011] (and elsewhere).

DEFINITION 6. For $A \subseteq \mathbf{A}$, **Bk**$[A]$ is the system obtained from **Bk** by adding to it 'the axioms in A. [3]

2.2   Non-deterministic Matrices

Our main semantic tool in what follows will be the following generalization of the concept of a many-valued matrix introduced in [Avron and Lev, 2001; Avron and Lev, 2005] (for a comprehensive survey on non-deterministic matrices, see also [Avron and Zamansky, 2011]):

DEFINITION 7.

1. A non-deterministic matrix (Nmatrix) for a language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where: $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$ and $\mathcal{O}$ includes an interpretation function $\tilde{\diamond}_\mathcal{M} : \mathcal{V}^n \to P^+(\mathcal{V})$ for every n-ary connective $\diamond$. We say that $\mathcal{M}$ is *finite* if so is $\mathcal{V}$.

2. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix. Let F be some set of $\mathcal{L}$-formulas closed under subformulas. An $\mathcal{M}$-valuation on F is a function $v : F \to \mathcal{V}$ which satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in F$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}_\mathcal{M}(v(\psi_1), \ldots, v(\psi_n))$$

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[3]In the sequel we shall usually omit the various brackets, and write e.g. **Bkci$_1$** instead of **Bk**$[\{\mathbf{c}, \mathbf{i_1}\}]$. Moreover: we shall write e.g. **Bkci** instead of **Bk**$[\{\mathbf{c}, \mathbf{i_1}, \mathbf{i_2}\}]$, and use similar abbreviations in the cases of **(a)** and **(o)**.

A *full $\mathcal{M}$-valuation* is an $\mathcal{M}$-valuation on $Frm_{\mathcal{L}}$.

3. Let F be as above, and let $\psi \in F$. An $\mathcal{M}$-valuation v on F *satisfies* ψ, denoted by $v \models_{\mathcal{M}} \psi$, if $v(\psi) \in \mathcal{D}$. v satisfies a set $\Gamma \subseteq F$ of formulas, denoted by $v \models_{\mathcal{M}} \Gamma$, if it satisfies every formula of Γ.

4. Let F be as above, and let v be an $\mathcal{M}$-valuation on F. A sequent $\Gamma \Rightarrow \Delta$ such that $\Gamma \cup \Delta \subseteq F$ is *true in v* if either there is some $\psi \in \Delta$, such that $v \models_{\mathcal{M}} \psi$, or for every $\psi \in \Gamma$, $v \not\models_{\mathcal{M}} \psi$. A sequent is *valid in $\mathcal{M}$* if it is true in every full $\mathcal{M}$-valuation.

5. $\vdash_{\mathcal{M}}$, the consequence relation induced by $\mathcal{M}$, is defined by: $T \vdash_{\mathcal{M}} \psi$ if $v \models_{\mathcal{M}} \psi$ for every full $\mathcal{M}$-valuation v such that $v \models_{\mathcal{M}} T$.

Notation. Below we shall frequently write just $\diamond$ instead of writing $\tilde{\diamond}_{\mathcal{M}}$, relying on the context to determine whether we mean the connective or its interpretation in some Nmatrix $\mathcal{M}$.

Nmatrices enjoy many of the attractive properties of usual (deterministic) finite-valued matrices. This includes the following:

PROPOSITION 8. **(Compactness)** *If $\mathcal{M}$ is finite, then $T \vdash \psi$ implies that there is some finite $\Gamma \subseteq T$, such that $\Gamma \vdash_{\mathcal{M}} \psi$.*

PROPOSITION 9. **(Semantic Analyticity)** *Let F be some set of $\mathcal{L}$-formulas closed under subformulas. Let $\mathcal{M}$ be an Nmatrix for $\mathcal{L}$. Any $\mathcal{M}$-valuation on F can be extended to a full $\mathcal{M}$-valuation.*

COROLLARY 10. *If $T \cup \{\psi\} \subseteq F$, then $T \vdash_{\mathcal{M}} \psi$ iff ψ is satisfied by every $\mathcal{M}$-valuation on F which satisfies T.*

COROLLARY 11. **(Decidability)** *For every finite Nmatrix $\mathcal{M}$, the question whether $\Gamma \vdash_{\mathcal{M}} \psi$ is decidable for every finite theory Γ and every sentence ψ.*

The following notion of *simple refinements* is going to be useful in the sequel:

DEFINITION 12. *Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$. $\mathcal{M}_2$ is a simple refinement of $\mathcal{M}_1$ if $\mathcal{V}_1 = \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2$ and for every n-ary connective $\diamond$ and every $a_1, \ldots, a_n \in \mathcal{V}_1$, $\tilde{\diamond}_{\mathcal{M}_2}(a_1, \ldots, a_n) \subseteq \tilde{\diamond}_{\mathcal{M}_1}(a_1, \ldots, a_n)$.*

PROPOSITION 13. ([Avron, 2007b]) *If $\mathcal{M}_2$ is a simple refinement of $\mathcal{M}_1$, then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$.*

3 A Systematic Generation of Analytic Calculi

Our method for a systematic construction of Gentzen-type calculi for all the LFIs presented in this paper is based on the following two facts:

1. All LFIs presented above, as well as many other paraconsistent logics, have a semantic characterization in terms of finite-valued (in fact, three-valued) Nmatrices. These characterizations can be obtained in a modular way within the finite-valued non-deterministic semantic framework that has been developed in [Avron, 2005a; Avron, 2007a; Avron, 2007b].

2. [Avron *et al.*, 2006] provides an algorithm for constructing cut-free Gentzen-type systems for logics which have a characteristic finite-valued Nmatrix $\mathcal{M}$, and the language of which is *sufficiently expressive* with respect to $\mathcal{M}$. This means that from any formula φ we can construct in a uniform way a finite set of formulas $S(\varphi)$, such that it is possible to determine the truth-value assigned by a valuation v in $\mathcal{M}$ to φ once we know which are the formulas in $S(\varphi)$ which v satisfies (note there is no need to know the exact truth-values that v assigns to the formulas of $S(\varphi)$).

We will shortly see that the language of the LFIs studied in this paper is sufficiently expressive with respect to all the three-valued Nmatrices used in the papers cited in the first fact. Hence we can indeed exploit the algorithm mentioned in the second fact in order to construct cut-free Gentzen-type systems for all the paraconsistent logics presented above.

We start by recalling some basic relevant definitions:

DEFINITION 14. The consequence relation $\vdash_{\mathbf{G}}$, induced by a Gentzen-type system $\mathbf{G}$, is defined as follows: $T \vdash_{\mathbf{G}} \psi$ if there is some finite $\Gamma \subseteq T$, such that $\vdash_{\mathbf{G}} \Gamma \Rightarrow \psi$.

DEFINITION 15. We say that an Nmatrix $\mathcal{M}$ is *characteristic* for a Gentzen-type system $\mathbf{G}$, if for every Γ and Δ it holds that $\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$ iff $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}$.

REMARK 16. If $\mathcal{M}$ is characteristic for $\mathbf{G}$, then $\vdash_{\mathbf{G}} \Gamma \Rightarrow \psi$ iff $\Gamma \vdash_{\mathcal{M}} \psi$. By the compactness theorem (Proposition 8), if $\mathcal{M}$ is finite, then this implies that $\vdash_{\mathcal{M}} = \vdash_{\mathbf{G}}$.

REMARK 17. A note is in order here on the relationship between the Gentzen-type systems provided below and the corresponding Hilbert-style systems using which the LFIs above were formulated. So each such Gentzen-type system $\mathbf{G}$ is equivalent to the corresponding Hilbert-type system H in the sense that $T \vdash_H \psi$ iff $T \vdash_G \psi$ (it is a standard matter to show this using cuts). In particular, ψ is a theorem of H iff $\vdash_{\mathbf{G}} \Rightarrow \psi$.

Below we define non-deterministic three-valued semantics for $\mathbf{Bk}[A]$ for all $A \subseteq \mathbf{A}$, and then introduce their corresponding Gentzen-type systems.

3.1 The Non-deterministic Semantics

Our non-deterministic semantics is based on the following three truth-values, the intuition being that when a formula φ is assigned truth-value of the form $\langle x, y \rangle$, $x = 1$ iff φ is "true", while $y = 1$ iff $\neg\varphi$ is "true".

$$t = \langle 1, 0 \rangle, f = \langle 0, 1 \rangle, \top = \langle 1, 1 \rangle$$

We start by defining the Nmatrix $\mathcal{M}_0^3$ for the most basic system $\mathbf{B}$:

DEFINITION 18. The Nmatrix $\mathcal{M}_0^3 = (\{t, f, \top\}, \{t, \top\}, \mathcal{O})$ for $\mathcal{L}_C$ is defined as follows:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t, \top, f\}$
$\top$	$\{t, \top\}$	$\{f\}$
f	$\{t, \top\}$	$\{t, \top, f\}$

$\wedge$	t	$\top$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

$\vee$	t	$\top$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$

$\supset$	t	$\top$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

PROPOSITION 19. ([Avron, 2007b]) $T \vdash_{\mathcal{M}_0^3} \psi$ *iff* $T \vdash_{\mathbf{B}} \psi$.

For obtaining semantics for the system $\mathbf{Bk}$, the basic Nmatrix $\mathcal{M}_0^3$ is refined according to the semantic conditions induced by the (**k**)-schemata (see e.g. [Avron, 2005a], where ($\mathbf{k_1}$) and ($\mathbf{k_2}$) are called ($\mathbf{d_1}$) and ($\mathbf{d_2}$), respectively):

DEFINITION 20. Let:

$$\mathsf{C}(\mathbf{k_1}): \quad \circ t = \{t, \top\}$$

$$\mathsf{C}(\mathbf{k_2}): \quad \circ f = \{t, \top\}$$

The Nmatrix $\mathcal{M}^3$ is the weakest refinement of $\mathcal{M}_0^3$, in which both of these conditions hold. In other words, $\mathcal{M}^3$ is similar to $\mathcal{M}_0^3$, except for its interpretation of $\circ$, which is as follows: $\circ t = \circ f = \{t, \top\}$, $\circ \top = \{f\}$.

PROPOSITION 21. ([Avron, 2005a]) $T \vdash_{\mathcal{M}^3} \psi$ *iff* $T \vdash_{\mathbf{Bk}} \psi$.

We turn to provide non-deterministic semantics for the extensions of $\mathbf{Bk}$ with axioms from $\mathbf{A}$. Our semantics is *modular* in the following sense: like in the case of (**k**)-axioms, each axiom $ax \in \mathbf{A}$ corresponds to certain semantic conditions $\mathsf{C}(ax)$. These conditions lead to simple refinements of the basic Nmatrix $\mathcal{M}^3$ (which amounts to reducing the level of non-determinism in it). The semantics of $\mathbf{Bk}[A]$ is then obtained by straightforwardly combining the semantic effects of all the schemata from A.

Tables 1 and 2 include the various semantic conditions that correspond to the axioms in $\mathbf{A}$. Most of them are taken from the papers cited above, or are easily derivable using the method used in those papers.

EXAMPLE 22. We explain how $\mathsf{C}(\mathbf{o}_\vee)$ was derived as an example. For this, assume that v is a valuation in $\mathcal{M}^3$. If $v(\varphi) = \top$ then v certainly satisfies $\circ \varphi \supset \circ(\varphi \vee \psi)$. Otherwise it satisfies this sentence iff it satisfies $\circ(\varphi \vee \psi)$, which is the case iff $v(\varphi \vee \psi) \neq \top$. This again necessarily holds if $v(\varphi) = v(\psi) = f$. In the remaining five cases all we know is that $v(\varphi \vee \psi) \in \{t, \top\}$. Hence to ensure it is indeed not $\top$, we have to force it in these cases to be t. This requires five basic semantic conditions, which can conveniently be grouped as follows: (i) $t \vee t = t \vee \top = t \vee f = \{t\}$ (i.e. $t \vee x = \{t\}$ for $x \in \{t, \top, f\}$), and (ii) $f \vee t = f \vee \top = \{t\}$. These are the elements of $\mathsf{C}(\mathbf{o}_\vee)$ given in Table 2.

DEFINITION 23. For $A \subseteq \mathbf{A}$, the Nmatrix $\mathcal{M}^3[A]$ is the weakest simple refinement of $\mathcal{M}^3$ in which $\mathsf{C}(ax)$ (from Tables 1 and 2) holds for every $ax \in A$.

	ax	$\mathsf{C}(ax)$	$\mathsf{R}(ax)$
(c)	$\neg\neg\varphi \supset \varphi$	$\neg f = \{t\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$
(e)	$\varphi \supset \neg\neg\varphi$	$\neg\top = \{\top\}$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$
($\mathbf{i_1}$)	$\neg\circ\varphi \supset \varphi$	$\circ f = \{t\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\circ\varphi \Rightarrow \Delta}$
($\mathbf{i_2}$)	$\neg\circ\varphi \supset \neg\varphi$	$\circ t = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg\circ\varphi \Rightarrow \Delta}$
($\mathbf{a_\wedge}$)	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$	$t \wedge t = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow, \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
($\mathbf{a_\vee}$)	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi)$	$t \vee t = t \vee f = \{t\}$ $t \vee t = f \vee t = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
($\mathbf{a_\supset}$)	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \supset \psi)$	$f \supset t = f \supset f = \{t\}$ $f \supset t = t \supset t = \{t\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \neg\psi, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \neg\varphi, \varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$

Table 1. Axioms, semantic conditions and Gentzen-type rules (for $x \in \{t, \top, f\}$)

	ax	$\mathsf{C}(ax)$	$\mathsf{R}(ax)$
$(\mathbf{o}_\wedge^1)$	$\circ\varphi \supset \circ(\varphi \wedge \psi)$	$t \wedge t = t \wedge \top = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\mathbf{o}_\wedge^2)$	$\circ\psi \supset \circ(\varphi \wedge \psi)$	$t \wedge t = \top \wedge t = \{t\}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\mathbf{o}_\vee^1)$	$\circ\varphi \supset \circ(\varphi \vee \psi)$	$t \vee x = \{t\}$ $f \vee t = f \vee \top = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{o}_\vee^2)$	$\circ\psi \supset \circ(\varphi \vee \psi)$	$x \vee t = \{t\}$ $t \vee f = \top \vee f = \{t\}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{o}_\supset^1)$	$\circ\varphi \supset \circ(\varphi \supset \psi)$	$t \supset t = t \supset \top = \{t\}$ $f \supset x = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$
$(\mathbf{o}_\supset^2)$	$\circ\psi \supset \circ(\varphi \supset \psi)$	$x \supset t = \{t\}$ $f \supset f = \{t\}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$

Table 2. Axioms, semantic conditions and rules (for $x \in \{t, \top, f\}$) - Continued

REMARK 24. It is easy to check that none of the combinations of the conditions in Tables 1 and 2 is contradictory. Hence $\mathcal{M}^3[A]$ is well-defined for every $A \subseteq \mathbf{A}$.

REMARK 25. The semantic conditions of $(\mathbf{c})$, $(\mathbf{e})$, $\mathbf{i_1}$, $\mathbf{i_2}$ (as well as those for $\mathbf{k_1}$ and $\mathbf{k_2}$) correspond to their respective axioms already in the framework of the system $\mathbf{B}$. For the other conditions this holds only for $\mathbf{Bk}$.

PROPOSITION 26. *For $A \subseteq \mathbf{A}$, $T \vdash_{\mathcal{M}^3[A]} \psi$ iff $T \vdash_{\mathbf{Bk}[A]} \psi$.*

Proof. The proof is practically identical to the proof of Theorem 3 in [Avron, 2007b], with some slight modifications (due to the fact that each of the axioms $(\mathbf{i})$, $(\mathbf{a})$, and $(\mathbf{o})$ has been split here into simpler axioms). ∎

COROLLARY 27. *For $\sharp \in \{\wedge, \vee, \supset\}$, the conjunction of $(\mathbf{o_\sharp^1})$ and $(\mathbf{o_\sharp^2})$ implies $(\mathbf{a_\sharp})$ in $\mathbf{Bk}$.*

REMARK 28. It is a mechanical matter to check, using the semantic conditions of Tables 1,2, that there are no other dependencies among the axioms in $\mathbf{A}$.

EXAMPLE 29.

- The truth-tables for $\circ$ and $\neg$ in the Nmatrix $\mathcal{M}^3[\{(\mathbf{c}), (\mathbf{e})\}]$ (which is characteristic for the system $\mathbf{Bkce}$) are defined as follows:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t, \top\}$
$\top$	$\{\top\}$	$\{f\}$
f	$\{t\}$	$\{t, \top\}$

- The system called $\mathbf{Cie}$ in [Carnielli and Marcos, 2002; Carnielli *et al.*, 2007] is equivalent to our $\mathbf{Bkcei_1i_2}$ (or in short $\mathbf{Bkcei}$, or just $\mathbf{Bcei}$, by item 4 of Remark 3). In the corresponding characteristic Nmatrix $\mathcal{M}^3[\{(\mathbf{c}), (\mathbf{e}), (\mathbf{i_1}), (\mathbf{i_2})\}]$, the truth-tables for $\circ$ and $\neg$ are as follows:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t\}$
$\top$	$\{\top\}$	$\{f\}$
f	$\{t\}$	$\{t\}$

- A characteristic Nmatrix $\mathcal{M}_{Bka}$ for the LFI $\mathbf{Bka}$ (i.e., the extension $\mathbf{Bk}$ with $(\mathbf{a_\wedge})$, $(\mathbf{a_\vee})$ and $(\mathbf{a_\supset})$), can be defined as follows:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$
f	$\{t, \top\}$	$\{t, \top\}$

$\wedge$	t	$\top$	f
t	$\{t\}$	$\{t, \top\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

$\vee$	t	$\top$	f
t	$\{t\}$	$\{t, \top\}$	$\{t\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
f	$\{t\}$	$\{t, \top\}$	$\{f\}$

$\supset$	t	$\top$	f
t	$\{t\}$	$\{t, \top\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
f	$\{t\}$	$\{t, \top\}$	$\{t\}$

3.2   The Corresponding Gentzen-type Systems

Before applying the algorithm of [Avron *et al.*, 2006], we first need the following proposition and its proof:

PROPOSITION 30. *$\mathcal{L}_C$ is sufficiently expressive for every simple refinement of $\mathcal{M}_0^3$ (and so for every simple refinement of $\mathcal{M}^3$).*

Proof. Let $S(\psi) = \{\psi, \neg\psi\}$ for every formula ψ. Then the following holds in $\mathcal{M}_0^3$ and any simple refinement of it:

- $v(\psi) = t$ iff v does not satisfy $\neg\psi$ (i.e., $v(\neg\psi)\notin\mathcal{D}$).

- $v(\psi) = f$ iff v does not satisfy ψ (i.e., $v(\psi)\notin\mathcal{D}$).

- $v(\psi) = \top$ iff v satisfies both ψ and $\neg\psi$ (i.e., $v(\psi) \in \mathcal{D}$ and $v(\neg\psi) \in \mathcal{D}$).

It follows that $S(\psi)$ can be used to characterize in $\mathcal{M}_0^3$ all its three truth-values, and so $\mathcal{L}_C$ is sufficiently expressive for every simple refinement of $\mathcal{M}_0^3$.   ■

Now the method of [Avron *et al.*, 2006] for constructing a cut-free, sound and complete Gentzen-type system for a given finite Nmatrix $\mathcal{M}$ involves two stages. In the first (and more important) stage every entry of every truth-table of $\mathcal{M}$ is translated into a rule. In the second stage certain streamlining principles are used to combine and simplify rules in order to get an optimal set of rules. The process can significantly be simplified in the present case, because just three truth values are used (and also because of the simplicity of the set $S(\psi)$ used here). All we need are the following six facts about valuations in $\mathcal{M}_0^3$ (corresponding to the nontrivial subsets of the set of truth values we employ) :

- $v(\psi) = t$ iff $\neg\psi \Rightarrow$ is true in v.

- $v(\psi) = f$ iff $\psi \Rightarrow$ is true in v.

- $v(\psi) = \top$ iff $\Rightarrow \psi$ and $\Rightarrow \neg\psi$ are both true in v.

- $v(\psi) \in \{f, \top\}$ iff $\Rightarrow \neg\psi$ is true in v.

- $v(\psi) \in \{t, \top\}$ iff $\Rightarrow \psi$ is true in v.

- $v(\psi) \in \{t, f\}$ iff $\psi, \neg\psi \Rightarrow$ is true in v.

To see how these facts are used to derive rules, take for example the truth table for $\vee$ in $\mathcal{M}_0^3$. The entry $f \vee f = \{f\}$ is first translated into: if $\varphi \Rightarrow$ is true and $\psi \Rightarrow$ is true then $\varphi \vee \psi \Rightarrow$ is true. By adding context we get the rule:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$$

On the other hand the entry $f \vee \top = \{t, \top\}$ is first translated into: if $\varphi \Rightarrow$ is true, and $\Rightarrow \psi$ is true, and $\Rightarrow \neg\psi$ is true, then $\Rightarrow \varphi \vee \psi$ is true. By adding context we get the rule:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

In a similar way we can derive the rules which correspond to the other seven entries of the truth table for $\vee$. The streamlining principles can then be used to combine all the rules except the one that corresponds to the entry $f \vee f = \{f\}$ (which are exactly the rules that have $\Gamma \Rightarrow \Delta, \varphi \vee \psi$ as their conclusion) into the single rule:

$$\frac{\Gamma, \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Instead of deriving first eight different rules and then combining them, we can directly derive the last rule by observing that taken together, the relevant eight entries mean that if either $v(\varphi) \in \{t, \top\}$ or $v(\psi) \in \{t, \top\}$ then $v(\varphi \vee \psi) \in \{t, \top\}$. This directly translates into: if $\Rightarrow \varphi, \psi$ is true then $\Rightarrow \varphi \vee \psi$ is true, and by adding context we get the last rule.

Using this method we get the following systems for $\vdash_{\mathcal{M}_0^3}$ and $\vdash_{\mathcal{M}^3}$:

DEFINITION 31 (The system $\mathbf{G_k}$).

Axioms of $\mathbf{G_k}$: $\psi \Rightarrow \psi$

Rules of $\mathbf{G_k}$:   Cut, Weakening, and the following logical rules:

$$(\wedge \Rightarrow) \quad \frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \wedge \phi \Rightarrow \Delta} \qquad\qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \wedge \phi}$$

$$(\vee \Rightarrow) \quad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \vee \phi \Rightarrow \Delta} \qquad\qquad (\Rightarrow \vee) \quad \frac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \vee \phi}$$

$$(\supset \Rightarrow) \quad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \supset \phi \Rightarrow \Delta} \qquad\qquad (\Rightarrow \supset) \quad \frac{\Gamma, \psi \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \psi \supset \phi, \Delta}$$

$$(\Rightarrow \neg) \quad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$(\circ \Rightarrow) \quad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma, \circ\psi \Rightarrow \Delta} \qquad\qquad (\Rightarrow \circ) \quad \frac{\Gamma, \psi, \neg\psi \Rightarrow \Delta}{\Gamma \Rightarrow \circ\psi, \Delta}$$

DEFINITION 32 (The system $\mathbf{G_0}$). $\mathbf{G_0}$ is the system obtained from $\mathbf{G_k}$ by deleting the rule ($\Rightarrow \circ$).

Now from the results of [Avron *et al.*, 2006] we directly get the following facts (first proved in [Avron, 2005a]):

PROPOSITION 33.

1. $\mathcal{M}_0^3$ *is a characteristic Nmatrix for* $\mathbf{G_0}$.

2. $\mathcal{M}^3$ *is a characteristic Nmatrix for* $\mathbf{G_k}$.

3. *Both* $\mathbf{G_0}$ *and* $\mathbf{G_k}$ *enjoy cut-admissibility.*

EXAMPLE 34. Below we provide a proof that $(\mathbf{l}) \Rightarrow (\mathbf{k_2})$ in $\mathbf{G}_0$, where $(\mathbf{l})$ is the famous axiom $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$, which is implicit in da Costa's historical system C_1.[4] The proof of $(\mathbf{l}) \Rightarrow (\mathbf{k_1})$ is similar.

$$\cfrac{\cfrac{\cfrac{\cfrac{\varphi, \neg\varphi \Rightarrow \circ\varphi, \neg\varphi}{\varphi \wedge \neg\varphi \Rightarrow \circ\varphi, \neg\varphi} \ (\wedge \Rightarrow)}{\Rightarrow \neg(\varphi \wedge \neg\varphi), \circ\varphi, \neg\varphi} \ (\Rightarrow \neg) \qquad \circ\varphi \Rightarrow \circ\varphi, \neg\varphi}{\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi \Rightarrow \circ\varphi, \neg\varphi} \ (\supset\Rightarrow)}{\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi \Rightarrow \circ\varphi \vee \neg\varphi} \ (\Rightarrow \vee)$$

REMARK 35. It is easy to see that with the exception of the rule $(\Rightarrow \neg)$ (which has no $(\neg \Rightarrow)$ counterpart), all the other rules of $\mathbf{G_k}$ are invertible in $\mathbf{G_k}$. This is due to the fact that with the exception of negation, each connective of $\mathcal{L}_C$ has in $\mathbf{G_k}$ two rules which are dual to each other (see Remark 3).

It is also interesting to note that by adding to $\mathbf{Bk}$ the classical $(\neg \Rightarrow)$ rule (the dual of $(\Rightarrow \neg)$) we get a system which is sound and complete for the classical two valued logic, where $\circ$ is interpreted by the function $\lambda x \in \{t, f\}.t$.

The method we have just used for $\mathcal{M}_0^3$ can be applied to each of its simple refinement separately. In this way we can obtain a cut-free Gentzen-type formulation for each of the LFIs we have considered above. However, it would be much easier to do this in a modular way, by translating the semantic effect of each extra axiom into rules (and using the streamlining principles to simplify the results). The results of this process are again given in Tables 1 and 2.

EXAMPLE 36. To see how the Gentzen-type rules from Tables 1, 2 are derived, consider again the schema $(\mathbf{o_\vee^1})$ (see Example 22). By Table 2, the validity of this schema is equivalent to the combination of the following two conditions: (i) $t \vee x = \{t\}$ and (ii) $f \vee t = f \vee \top = \{t\}$. Now (i) can be reformulated as follows: if $\neg\varphi \Rightarrow$ is true, then $\neg(\varphi \vee \psi) \Rightarrow$ is true. By adding context, we obtain:

$$\frac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$$

In turn, (ii) can be reformulated as follows: if $\varphi \Rightarrow$ and $\Rightarrow \psi$ are true, then so is $\neg(\varphi \vee \psi) \Rightarrow$. Again, by adding context we get the following rule:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$$

Taken together, these two Gentzen-type rules correspond to the schema $(\mathbf{o_\vee^1})$.

DEFINITION 37. For each $ax \in \mathbf{A}$, the set of Gentzen-type rules $\mathsf{R}(ax)$, corresponding to ax, is defined as in Tables 1,2. For $A \subseteq \mathbf{A}$, $\mathbf{G}[A]$ is the Gentzen-type system obtained by adding to $\mathbf{G_k}$ the set of rules $\mathsf{R}(ax)$ for every $ax \in A$.

EXAMPLE 38. Figure 1 below contains three examples of cut-free proofs in three of our systems: $\mathbf{Bka_\wedge}$, $\mathbf{Bko_\wedge^1}$, and the basic system $\mathbf{Bk}$.

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[4]See the last section for a discussion why this axiom could not be treated using the method presented in this paper.

A proof of $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$ in $\mathbf{Bka_\wedge}$:

$$\cfrac{\cfrac{\varphi, \psi, \neg(\varphi \wedge \psi) \Rightarrow \varphi, \psi \ (\Rightarrow \wedge)}{\cfrac{\varphi \wedge \psi, \neg(\varphi \wedge \psi), \varphi, \psi \ (\circ \Rightarrow)}{\Rightarrow \circ(\varphi \wedge \psi), \varphi, \psi}} \qquad \cfrac{\cfrac{\neg(\varphi \wedge \psi), \varphi, \psi \Rightarrow \neg\varphi, \psi \ (\Rightarrow \wedge)}{\cfrac{\neg(\varphi \wedge \psi), \varphi \wedge \psi, \neg\psi, \psi \ (\Rightarrow \circ)}{\Rightarrow \circ(\varphi \wedge \psi), \varphi, \neg\psi, \psi}} \qquad \cfrac{\cfrac{\neg(\varphi \wedge \psi), \varphi, \psi \Rightarrow \neg\varphi, \psi, \varphi, \psi, \psi \ (\mathbf{a_\wedge})}{\cfrac{\neg(\varphi \wedge \psi), \varphi, \varphi \wedge \psi, \neg\psi \ (\vee \Rightarrow)}{\cfrac{\neg(\varphi \wedge \psi), \varphi, \neg\psi \ (\Rightarrow \circ)}{\Rightarrow \circ(\varphi \wedge \psi), \varphi, \neg\psi} }}}}{\cfrac{\Rightarrow \circ(\varphi \wedge \psi), \varphi}{\cfrac{\circ\psi, \circ\varphi \Rightarrow \circ(\varphi \wedge \psi)}{\cfrac{\circ\varphi, \circ\psi \Rightarrow \circ(\varphi \wedge \psi)}{\Rightarrow (\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)}}}}$$

A proof of $\circ\varphi \supset \circ(\varphi \wedge \psi)$ in $\mathbf{Bko^1_\wedge}$:

$$\cfrac{\cfrac{\varphi, \psi, \neg\varphi \Rightarrow \neg\varphi, \varphi, \psi, \neg\varphi, \psi \ (\mathbf{o^1_\wedge})}{\cfrac{\neg(\varphi \wedge \psi), \varphi \Rightarrow \neg\varphi \ (\Rightarrow \vee)}{\cfrac{\neg(\varphi \wedge \psi), \varphi, \varphi \wedge \psi, \neg\varphi \ (\Rightarrow \circ)}{\Rightarrow \circ(\varphi \wedge \psi), \neg\varphi, \varphi}}}}{\cfrac{\circ\varphi \Rightarrow \circ(\varphi \wedge \psi) \ (\supset \Rightarrow)}{\Rightarrow \circ\varphi \supset \circ(\varphi \wedge \psi)}}$$

A proof of $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi) \Rightarrow \vdash (\varphi \vee \psi) \supset (\neg\varphi \vee \neg\psi)$ in $\mathbf{Bk}$:

$$\cfrac{\cdots}{\cfrac{\cdots}{\cdots}}$$

Figure 1. Selected Examples of Proofs

Note that the first example in Figure 1 shows that the axiom $\mathbf{a}_\wedge$ indeed has a cut-free proof in $\mathbf{Bka}_\wedge$, even though the Gentzen-type rule $(\mathbf{a}_\wedge)$ that corresponds to this axiom does not even mention the connective $\circ$. The second example does the same for the axiom $\mathbf{o}_\wedge^1$ and the system $\mathbf{Bko}_\wedge^1$. Finally, the third example shows that the axiom which corresponds to the semantic condition $t \vee t = \{t\}$ follows in the basic system $\mathbf{Bk}$ from the axiom $\mathbf{a}_\wedge$.

THEOREM 39. *Let $A \subseteq \mathbf{A}$.*

1. $\mathcal{M}^3[A]$ *is a characteristic Nmatrix for* $\mathbf{G_k}[A]$.

2. $\mathbf{G_k}[A]$ *enjoys cut-admissibility.*

Proof. It is easy to see that $\mathbf{G_k}[A]$ is the calculus obtained for $\mathcal{M}^3[A]$ by the algorithm of [Avron *et al.*, 2006]. Thus the theorem follows from Proposition 26 and the results of [Avron *et al.*, 2006]. ■

REMARK 40. It is important to note that all the Gentzen-type rules we employ in this paper have the following uniform form:

1. Each of them introduces exactly one formula in its conclusion, on exactly one of its two sides;

2. The formula which is introduced is either of the form $\diamond(\psi_1, \ldots, \psi_n)$ or $\neg \diamond (\psi_1, \ldots, \psi_n)$, where $\diamond$ is a primitive n-ary connective of the language;

3. Let $\diamond(\psi_1, \ldots, \psi_n)$ be the formula mentioned in the previous item. Then the principal formulas in the premises of the rule are all taken from the set $\{\psi_1, \ldots, \psi_n, \neg\psi_1, \ldots, \neg\psi_n\}$;

4. There are no restrictions on the side formulas of the rule (i.e., every context is legitimate).

We call rules of this form *quasi-canonical*, because they provide a natural generalization of the class of canonical rules ([Avron and Lev, 2001; Avron and Lev, 2005; Avron and Zamansky, 2011]) — the type of rules that are used in standard Gentzen-type systems for classical logic (the difference is that canonical rules allow to introduce in the conclusion only formulas of the form $\diamond(\psi_1, \ldots, \psi_n)$, and the principal formulas in their premises are then taken just from the set $\{\psi_1, \ldots, \psi_n\}$). It should be noted that quasi-canonical Gentzen-type systems like those presented here (i.e. Gentzen-type systems in which all rules are either structural or quasi-canonical) have already been used extensively in the proof theory of non-classical logics, and even in calculi for classical logic in which one-sided sequents are employed. As far as we know, this cannot be said about any Gentzen-type formulation of a C-system that has been suggested before!

3.3   Other C-systems for Which The Method Applies

Large as it already is, there are plenty more C-systems for which the method described above can be used for developing corresponding quasi-canonical, cut-free Gentzen-type systems:

- From Remark 25 it easily follows that we can handle by our method any extension of $\mathbf{B}$ by a subset of the set $\{(\mathbf{c}), (\mathbf{e}), \mathbf{i_1}, \mathbf{i_2}, \mathbf{k_1}, \mathbf{k_2}\}$. For this we translate first the conditions $\mathsf{C}(\mathbf{k_1})$ and $\mathsf{C}(\mathbf{k_2})$ from Definition 20 into the following rules:

$$(\mathsf{R}(\mathbf{k_1})) \quad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \circ\psi} \qquad\qquad (\mathsf{R}(\mathbf{k_2})) \quad \frac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \circ\psi}$$

  Then we develop cut-free Gentzen type systems for this family of logics by modularly using these two rules together with the rules which are associated in Table 1 with the other four.[5]

- [Avron, 2007a] treats many other classical tautologies involving $\neg$ (but not $\circ$) in exactly the same way the axioms of $\mathbf{A}$ have been treated here: it modularly associates with each of them a semantic condition on $\mathcal{M}_0^3$ and a translation of it to a quasi-canonical Gentzen-type rule. The list of axioms treated there which are not in $\mathbf{A}$ includes the following:[6]

$(\neg \supset \Rightarrow)_1\textbf{:} \quad \neg(\varphi \supset \psi) \supset \varphi$

$(\neg \supset \Rightarrow)_2\textbf{:} \quad \neg(\varphi \supset \psi) \supset \neg\psi$

$(\Rightarrow \neg \supset)\textbf{:} \quad (\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$

$(\neg\vee \Rightarrow)_1\textbf{:} \quad \neg(\varphi \vee \psi) \supset \neg\varphi$

$(\neg\vee \Rightarrow)_2\textbf{:} \quad \neg(\varphi \vee \psi) \supset \neg\psi$

$(\Rightarrow \neg\vee)\textbf{:} \quad (\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$

$(\neg\wedge \Rightarrow)\textbf{:} \quad \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$

$(\Rightarrow \neg\wedge)_1\textbf{:} \quad \neg\varphi \supset \neg(\varphi \wedge \psi)$

$(\Rightarrow \neg\wedge)_2\textbf{:} \quad \neg\psi \supset \neg(\varphi \wedge \psi)$

  Using the rules given in [Avron, 2007a] for these axioms (the identities of which are hinted in the names given to these axioms), it is possible to provide in a modular way a cut-free quasi-canonical Gentzen-type system to every logic which is obtained by adding some subset of the above set of axioms to any of the systems treated here so far (including those mentioned in the previous item of this subsection). One should note however that not all of the resulting systems are paraconsistent (although all of them are included in classical logic if we interpret $\circ$ by $\lambda x \in \{t, f\}.t$). However, it is very easy to determine which does and which does not by checking if the set of corresponding semantic conditions involves no conflict. An example of a case in which such a conflict arises is when we try to add $(\Rightarrow \neg\wedge)_2$ to $\mathbf{Bko}_\wedge^1$. The semantic condition that corresponds

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[5] Since $\mathbf{i_1}$ entails $\mathbf{k_1}$ in $\mathbf{B}$, and $\mathbf{i_2}$ entails $\mathbf{k_2}$, there are exactly sixteen logics in this family which are neither extensions of $\mathbf{Bk}$ nor identical to $\mathbf{B}$: the extensions of either $\mathbf{Bk_1}$, $\mathbf{Bk_2}$, $\mathbf{Bi_1}$ or $\mathbf{Bi_2}$ by some subset of $\{(\mathbf{c}), (\mathbf{e})\}$. One of these systems is the system $\mathbf{Bc}$, which was provided cut-free Gentzen-type system in [Gentilini, 2011] (where it is called $\mathbf{bC}$).

[6] The axiom $(\neg\wedge \Rightarrow)$ is equivalent in $\mathbf{Bk}$, but not in $\mathbf{B}$, to $\mathbf{a_\wedge}$. All other axioms in this list are independent over $\mathbf{Bk}$ from the union of $\mathbf{A}$ with the set of the other axioms in the list.

to $(\Rightarrow \neg\wedge)_2$ is: $t \wedge \top = \top \wedge \top = \{\top\}$. On the other hand the semantic condition that corresponds to $\mathbf{Bko}^1_\wedge$ is $t \wedge \top = t \wedge t = \{t\}$. There is an obvious conflict in the case of $t \wedge \top$. This conflict is resolved only by deleting $\top$ from the set of available truth-values. This implies that the system under discussion is not paraconsistent (and so it is equivalent to classical logic, since it is easy to see that any extension by axioms of $\mathbf{Bk}$ which is not paraconsistent is equivalent to classical logic).

4 Conclusions and Further Research

In this paper we provide a *uniform* way to *systematically* construct analytic calculi for a large family of thousands LFIs, each having a semantic characterization in terms of a three-valued Nmatrix. We believe that these results will help in producing efficient tools for automated reasoning with inconsistency, eventually making Logics of Formal (In)consistency a more appealing formalism for reasoning under uncertainty.

The most immediate directions for further research include:

- Extending the method to LFIs like $\mathbf{Ba}$, which have finite-valued non-deterministic semantics with more than three truth-values. The easiest natural step here would be to try to adapt the method given here to the use of the five-valued semantics for extensions of $\mathbf{B}$ with axioms from $\mathbf{A} \cup \{(\mathbf{k_1}),(\mathbf{k_2})\}$, given in [Avron, 2007b].

- There are two axioms that are included in some of the most important C-systems, but cannot be handled in the framework developed in this paper. These are the following famous axioms from [Carnielli and Marcos, 2002; Carnielli *et al.*, 2007]:

$$(\mathbf{l})\ \ \neg(\varphi \wedge \neg\varphi) \supset \circ\varphi \qquad (\mathbf{d})\ \ \neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$$

Systems with these axioms include, for instance, da Costa's historic system C_1, which can be shown to be equivalent to $\mathbf{Bk}[\{(\mathbf{a}),(\mathbf{c}),(\mathbf{i}),(\mathbf{l})\}]$.

It is quite obvious how to translate $(\mathbf{l})$ (say) into a Gentzen-type rule: simply substitute in $(\circ \Rightarrow)$ the formula $\neg(\psi \wedge \neg\psi)$ for $\circ\psi$ (doing the same for $(\Rightarrow \circ)$ results in a derivable rule of $\mathbf{B}$). In the cases in which cut-free systems for logics with $(\mathbf{l})$ have already been given in the literature (like in [Béziau, 1993; Gentilini, 2011]) this simple procedure leads to systems which are equivalent to those given before. However, it is not yet clear whether this will be true in general, and whether the crucial modularity of our framework is preserved by this procedure.

Now the main obstacle in extending the method of this paper to systems with $(\mathbf{l})$ or $(\mathbf{d})$ is that such systems have no semantic characterization in terms of finite-valued Nmatrices (this was shown in [Avron, 2007b]). However, they do have infinitely-valued such characterizations (which still suffice for guaranteeing their decidability). It is not clear yet whether these characterizations can be used for the same purposes the three-valued framework has been used here.

- It is clear that for building LFI-based theorem provers for real-life applications, the results of this paper need to be extended to the first-order case. To the best of our knowledge, currently there are no known analytic systems available on the first-order level. In [Avron and Zamansky, 2007] finite non-deterministic semantics were provided for first-order LFIs, which could be exploited along the lines of the approach presented in this paper.

- As we noted in Remark 40, all the sequent systems presented in this paper are what we called there "quasi-canonical". Now there exists a quite well-developed theory of *canonical systems* ([Avron and Lev, 2005; Avron and Zamansky, 2011]). Thus it is known that such systems have semantic characterizations in terms of two-valued Nmatrices, and that there is a strong connections between their semantics, their being non-trivial, and the admissibility of the cut rule in them. There is also a strong connection between the determinism of their semantics, and their having the properties of invertibility and axiom-expansion ([Avron *et al.*, 2009]). An interesting research direction would be to develop a similar theory of quasi-canonical systems.

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