Non-deterministic Semantics as a Proof-Theoretical Tool

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The Big Picture

- Our goals:
  - Characterization of important syntactic properties of calculi: *cut-admissibility, the subformula property, invertibility of rules,...*
  - Understanding the dependencies between them.
- Our tool: non-deterministic semantics.
- Our case study: canonical labelled calculi.

Cut-Admissibility

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$$\vdash_{\mathbf{G}} s \qquad \Longrightarrow \qquad \vdash_{\mathbf{G}-(cut)} s$$

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Can we semantically characterize $\vdash_{\mathbf{G}-(cut)}$?

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Can we semantically characterize $\vdash_{\mathbf{G}-(cut)}$?

For example, what is the semantics of the logic induced by $\mathbf{LK} - (cut)$?

What is a logic?

1. A formal language $\mathcal{L}$, based on which $\mathcal{L}$-formulas are constructed.
2. A relation $\vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, satisfying:

Reflexivity: if $\psi \in \mathcal{T}$ then $\mathcal{T} \vdash \psi$.
Monotonicity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.
Transitivity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \varphi$ then $\mathcal{T}, \mathcal{T}' \vdash \varphi$.

How are logics defined by sequent calculi?

- Sequent calculi can induce logics in two possible ways:

  v: $\mathcal{T} \vdash^{v}_{\mathbf{G}} \varphi \quad \Longleftrightarrow \quad \{ \Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{G}} \Rightarrow \varphi$

  t: $\mathcal{T} \vdash^{t}_{\mathbf{G}} \varphi \quad \Longleftrightarrow \quad \vdash_{\mathbf{G}} \Gamma \Rightarrow \varphi$ *for some finite* $\Gamma \subseteq \mathcal{T}$

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  t: $\mathcal{T} \vdash^{t}_{\mathbf{G}} \varphi \iff \vdash_{\mathbf{G}} \Gamma \Rightarrow \varphi$ *for some finite* $\Gamma \subseteq \mathcal{T}$

Lemma

For any sequent calculus $\mathbf{G}$, $\vdash^{v}_{\mathbf{G}}$ *is a logic.*

But if $\mathbf{G}$ does not include cut, $\vdash^{t}_{\mathbf{G}}$ is not necessarily a logic!

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$$\vdash_{\mathbf{G}} s \qquad \Longrightarrow \qquad \vdash_{\mathbf{G}-(cut)} s$$

Can we semantically characterize the logic $\vdash^{v}_{\mathbf{LK}-(cut)}$?

- $\vdash^{v}_{\mathbf{LK}}$ and $\vdash^{v}_{\mathbf{LK}-(cut)}$ are different logics:

$$\Rightarrow p_1 \supset p_2 \vdash_{\mathbf{LK}} \Rightarrow p_1 \supset (p_3 \supset p_2)$$

$$\Rightarrow p_1 \supset p_2 \nvdash_{\mathbf{LK}-(cut)} \Rightarrow p_1 \supset (p_3 \supset p_2)$$

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$$\Rightarrow p_1 \supset p_2 \vdash_{\mathbf{LK}} \Rightarrow p_1 \supset (p_3 \supset p_2)$$

$$\Rightarrow p_1 \supset p_2 \not\vdash_{\mathbf{LK}-(cut)} \Rightarrow p_1 \supset (p_3 \supset p_2)$$

Classical Logic

The Matrix $\mathbf{M_{LK}}$

- Truth-values: $\{\mathrm{T}, \mathrm{F}\}$
- An $\mathbf{M_{LK}}$-valuation is a *model* of a sequent $\Gamma \Rightarrow \Delta$ iff $v(\psi) = \mathrm{F}$ for some $\psi \in \Gamma$ or $v(\psi) = \mathrm{T}$ for some $\psi \in \Delta$.
- Truth-tables:

$\widetilde{\supset}$	T	F
T	T	F
F	T	T

$\widetilde{\wedge}$	T	F
T	T	F
F	F	F

Soundness and Completeness

$\Omega \vdash_{\mathbf{LK}} s$ iff every $\mathbf{M_{LK}}$-valuation which is a model of every sequent in Ω is also a model of s.

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(Trivial) Observation

Every $\mathbf{M_{LK}}$-valuation v is either a model of $\Rightarrow \varphi$ or of $\varphi \Rightarrow$, but not both!

The semantics for $\vdash^{v}_{\mathbf{LK}-(cut)}$

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- Why not both? Because of cut: $$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$
- Discarding cut makes this option possible.

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- Discarding cut makes this option possible.
- New truth-values: $\{\{\mathrm{T}\}, \{\mathrm{F}\}, \{\mathrm{T}, \mathrm{F}\}\}$
- New definition of model: a valuation is a *model* of a sequent $\Gamma \Rightarrow \Delta$ iff $\mathrm{F} \in v(\psi)$ for some $\psi \in \Gamma$ or $\mathrm{T} \in v(\psi)$ for some $\psi \in \Delta$.
  - For example: $v(\varphi) = \{\mathrm{T}, \mathrm{F}\}$ iff v is a model of both $\Rightarrow \varphi$ and $\varphi \Rightarrow$.

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    - For example: $v(\varphi) = \{\mathrm{T}, \mathrm{F}\}$ iff v is a model of both $\Rightarrow \varphi$ and $\varphi \Rightarrow$.
- But no new truth-tables!

Theorem

(Lahav, 2012) $\vdash^{v}_{\mathbf{LK}-(cut)}$ *does not have a finite characteristic matrix.*

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- Our case study: canonical labelled calculi.

Non-deterministic Semantics - Motivation

- Principle of Truth-Functionality (PTF): the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas.
- *Non-deterministic phenomena in possible conflict with PTF:*
  vagueness *incompleteness*
  uncertainty *imprecision*
  inconsistency
- Relaxing PTF: non-deterministic evaluation of formulas.

$\diamond$	T	F
T	$\{T\}$	$\{T, F\}$
F	$\{T, F\}$	$\{F\}$

Intuition for Introducing Non-determinism

Consider a fully structural calculus with the following rules:

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

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	$\neg$
T	F
F	T

		$\vee$
T	T	T
T	F	T
F	T	T
F	F	F

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	$\neg$
T	F
F	?

		$\vee$
T	T	T
T	F	T
F	T	T
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$$\frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

	$\neg$
T	{F}
F	{T,F}

		$\vee$
T	T	{T}
T	F	{T}
F	T	{T}
F	F	{T, F}

Many-valued Matrices

A (deterministic) matrix M for $\mathcal{L}$ consists of:

- $\mathcal{V}$ - the set of truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

An M-valuation $v : Frm_{\mathcal{L}} \to \mathcal{V}$ satisfies:

$$v(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

Non-deterministic Matrices [Avron and Lev, 2001]

A non-deterministic matrix **M** for $\mathcal{L}$ consists of:

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Example: The Paraconsistent Logic CLuN of Batens

$\mathcal{L}$ — a language over $\{\vee, \wedge, \supset, \neg\}$, $\mathcal{V} = \{\mathrm{F}, \mathrm{T}\}, \mathcal{D} = \{\mathrm{T}\}$.
$\vee, \wedge$ and $\supset$ are interpreted classically, while $\neg$ satisfies the law of excluded middle $\neg\varphi \vee \varphi$, but not the law of contradiction $\neg(\varphi \wedge \neg\varphi)$.
$\mathbf{M}^2 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where $\mathcal{O}$ is given by:

		$\widetilde{\vee}$	$\widetilde{\wedge}$	$\widetilde{\supset}$
T	T	{T}	{T}	{T}
T	F	{T}	{F}	{F}
F	T	{T}	{F}	{T}
F	F	{F}	{F}	{T}

	$\widetilde{\neg}$
T	{T, F}
F	{T}

Key property of Nmatrices:

- Analyticity: any partial **M**-valuation can be extended to a full **M**-valuation.
- Consequence: decidability (in the finite case).

What is the semantics of $\vdash^{v}_{\mathbf{LK}-(cut)}$?

- We start with the simplest system: identity axiom + weakening (no logical rules, no cut)
- Truth-values: $\{\{\mathrm{T}\}, \{\mathrm{F}\}, \{\mathrm{T}, \mathrm{F}\}\}$

What is the semantics of $\vdash^{v}_{\mathbf{LK}-(cut)}$?

- We start with the simplest system: identity axiom + weakening (no logical rules, no cut)
- Truth-values: $\{\{T\}, \{F\}, \{T, F\}\}$

The corresponding Nmatrix:

$\widetilde{\wedge}$	$\{T\}$	$\{F\}$	$\{T, F\}$
$\{T\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$
$\{F\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$
$\{T, F\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$

What is the semantics of $\vdash^{v}_{\mathbf{LK}-(cut)}$?

Adding the rule:

$$(\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow \Delta, \psi \qquad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

The corresponding Nmatrix:

$\widetilde{\wedge}$	$\{T\}$	$\{F\}$	$\{T, F\}$
$\{T\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$
$\{F\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$
$\{T, F\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$

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$\{T\}$	$\{\{T\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{T, F\}\}$
$\{F\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$
$\{T, F\}$	$\{\{T\}, \{T, F\}\}$	$\{\{T\}, \{F\}, \{T, F\}\}$	$\{\{T\}, \{T, F\}\}$

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The corresponding Nmatrix:

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$\{T\}$	$\{\{T\},\{T,F\}\}$	$\{\{T\},\{F\},\{T,F\}\}$	$\{\{T\},\{T,F\}\}$
$\{F\}$	$\{\{T\},\{F\},\{T,F\}\}$	$\{\{T\},\{F\},\{T,F\}\}$	$\{\{T\},\{F\},\{T,F\}\}$
$\{T,F\}$	$\{\{T\},\{T,F\}\}$	$\{\{T\},\{F\},\{T,F\}\}$	$\{\{T\},\{T,F\}\}$

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The corresponding Nmatrix:

$\widetilde{\wedge}$	$\{T\}$	$\{F\}$	$\{T, F\}$
$\{T\}$	$\{\{T\}, \{T, F\}\}$	$\{\{F\}, \{T, F\}\}$	$\{\{T, F\}\}$
$\{F\}$	$\{\{F\}, \{T, F\}\}$	$\{\{F\}, \{T, F\}\}$	$\{\{F\}, \{T, F\}\}$
$\{T, F\}$	$\{\{T, F\}\}$	$\{\{F\}, \{T, F\}\}$	$\{\{T, F\}\}$

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The corresponding Nmatrix:

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$\{F\}$	$\{\{F\}, \{T, F\}\}$	$\{\{F\}, \{T, F\}\}$	$\{\{F\}, \{T, F\}\}$
$\{T, F\}$	$\{\{T, F\}\}$	$\{\{F\}, \{T, F\}\}$	$\{\{T, F\}\}$

Recall: *An valuation is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $f \in v(\psi)$ for some $\psi \in \Gamma$ or $T \in v(\psi)$ for some $\psi \in \Delta$.*

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$\widetilde{\wedge}$	$\{T\}$	$\{F\}$	$\{T, F\}$
$\{T\}$	$\{\{T\}, \{T, F\}\}$	$\{\{F\}, \{T, F\}\}$	$\{\{T, F\}\}$
$\{F\}$	$\{\{F\}, \{T, F\}\}$	$\{\{F\}, \{T, F\}\}$	$\{\{F\}, \{T, F\}\}$
$\{T, F\}$	$\{\{T, F\}\}$	$\{\{F\}, \{T, F\}\}$	$\{\{T, F\}\}$

Recall: *An valuation is a model of a sequent* $\Gamma \Rightarrow \Delta$ *iff* $f \in v(\psi)$ *for some* $\psi \in \Gamma$ *or* $T \in v(\psi)$ *for some* $\psi \in \Delta$.

Soundness and Completeness

$\Omega \vdash_{\mathbf{LK}-(cut)} s$ iff every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation which is a model of every sequent in Ω is also a model of s.

$\hookrightarrow$ New formulation of results of Schütte (1960) and Girard (1987).

Application: Semantic Proof of Cut-Admissibility in **LK**

Cut-Admissibility in **LK**

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Cut-Admissibility in **LK**

$$\vdash_{\mathbf{LK}} s \qquad \Longrightarrow \qquad \vdash_{\mathbf{LK}-(cut)} s$$

- Reduces to proving that for every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation which is not a model of some sequent s, there exists an $\mathbf{M}_{\mathbf{LK}}$-valuation which is not a model of s.
- Proof by induction on the build-up of formulas.

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- **Our case study: canonical labelled calculi.**

What is a Canonical Rule?

- An *"ideal" logical rule*: an introduction rule for *exactly one connective*, on *exactly one side of a sequent*.
- In its formulation: *exactly one occurrence* of the introduced connective, no other occurrences of other connectives.
- Its active formulas: *immediate subformulas* of its principal formula.

Examples of Canonical Rules

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

Example 1

Let $\mathbf{G}_1$ be a fully structural calculus with the following rules:

$$\{\Rightarrow \psi_1 \ ; \ \Rightarrow \psi_2\} \ / \ \psi_1 \diamond \psi_2 \Rightarrow \qquad \{\psi_1 \Rightarrow \ ; \ \psi_2 \Rightarrow\} \ / \ \Rightarrow \psi_1 \diamond \psi_2$$

a	b	$\diamond(a, b)$
T	T	{F}
T	F	{T,F}
F	T	{T,F}
F	F	{T}

Example 2

Let $\mathbf{G}_2$ be a fully structural calculus with the following rules:

$$\{\psi_2 \Rightarrow\} \;/\; \psi_1 \circ \psi_2 \Rightarrow \qquad \{\Rightarrow \psi_1\} \;/\; \Rightarrow \psi_1 \circ \psi_2$$

a	b	$\circ(a, b)$
T	T	{T}
T	F	∅????
F	T	{T,F}
F	F	{F}

Non-deterministic Matrices

A non-deterministic matrix for $\mathcal{L}$ consists of:

- $\mathcal{T}$ - the set of truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

Non-deterministic Partial Matrices

A non-deterministic partial matrix for $\mathcal{L}$ consists of:

- $\mathcal{T}$ - the set of truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to P(\mathcal{V})$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

A PNmatrix is proper if it includes no "empty spots".

Key property of Nmatrices:

- Analyticity: any partial **M**-valuation can be extended to a full **M**-valuation.
- Consequence: decidability (in the finite case).

Key property of PNmatrices:

- Weak Analyticity: it is decidable whether a partial **M**-valuation can be extended to a full **M**-valuation.
- Consequence: decidability (in the finite case).

The two-sided case: a direct correspondence

Theorem

If **G** *is a (two-sided) canonical calculus, then the following statements are equivalent:*

1. **G** *has a characteristic proper two-valued PNmatrix.*
2. **G** *enjoys strong cut-admissibility.*
3. **G** *enjoys the subformula property.*

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1. **G** *has a characteristic proper two-valued PNmatrix.*
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- The Subformula Property: Whenever $\Omega \vdash_{\mathbf{G}} s$, there is a derivation of s from Ω in **G** consisting solely of $\mathcal{E}$-sequents (i.e. sequents consisting solely of formulas from $\mathcal{E}$).
- Strong Cut-Admissibility Whenever $\Omega \vdash_{\mathbf{G}} s$, there is a derivation of s from Ω in **G** in which cuts are allowed only on formulas from Ω.

Labelled Calculi

$$\psi_1, \psi_2 \Rightarrow \psi_3, \psi_4, \psi_5 \quad \Rightarrow \quad \{\mathrm{F} : \psi_1, \mathrm{F} : \psi_2, \mathrm{T} : \psi_3, \mathrm{T} : \psi_4, \mathrm{T} : \psi_5\}$$

- A finite set of labels Ł.
- A labelled formula: $a : \psi$ for $a \in$ Ł
- A sequent: a finite set of labelled formulas.
- *Canonical labelled calculi* have in addition to weakening two types of rules: primitive rules and canonical introduction rules.

Primitive Rules

$$\frac{(L_1 : \psi) \cup s \quad \ldots \quad (L_n : \psi) \cup s}{(L : \psi) \cup s \cup \ldots \cup s}$$

Notation: we write $(\{a, b, c\} : \psi)$ *instead of* $\{a : \psi, b : \psi, c : \psi\}$.

Examples:

$$\frac{\{\mathrm{F} : \psi\} \cup s \quad \{\mathrm{T} : \psi\} \cup s}{s}$$

$$\frac{s}{(\{\mathrm{T}, \mathrm{F}\} : \psi) \cup s}$$

$$\frac{(\{a\} : \psi) \cup s \quad (\{b\} : \psi) \cup s}{(\{c, d\} : \psi) \cup s}$$

Canonical Introduction Rules

$$\frac{\{\mathrm{T} : \psi_1\} \cup s \quad \{\mathrm{T} : \psi_2\} \cup s}{\{\mathrm{T} : \psi_1 \wedge \psi_2\} \cup s}$$

$$\frac{\{\mathrm{F} : \psi_1, \mathrm{F} : \psi_2\} \cup s}{\{\mathrm{F} : \psi_1 \wedge \psi_2\} \cup s}$$

$$\frac{\{a : \psi_1, b : \psi_2\} \cup s \quad \{c : \psi_2, a : \psi_3, b : \psi_3\} \cup s}{(\{a, b\} : \circ(\psi_1, \psi_2, \psi_3) \cup s}$$

Semantics for Canonical Labelled Calculi

- Possible truth-values in the two-sided case: $\{\emptyset, \{\mathrm{F}\}, \{\mathrm{T}\}, \{\mathrm{T}, \mathrm{F}\}\}$.
- Possible truth-values in the labelled case: $P(Ł)$.
- A valuation v is a model of a sequent Ω if for some labelled formula $a : \psi$ in Ω, $a \in v(\psi)$.
- Primitive rules determine the actual set of truth-values.
- Introduction rules determine the truth-tables of the logical connectives.

Example

Start with the calculus over Ł $= \{a, b, c\}$ including only weakening.

$$\text{Vals} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

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Now we add the primitive rules:

$$\frac{s}{(\{a, b\} : \psi) \cup s} \qquad \frac{\{a : \psi\} \cup s \quad \{b : \psi\} \cup s \quad \{c : \psi\} \cup s}{s}$$

$$\text{Vals} = \{\{b\}, \{a\}, \{a, b\}\}$$

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$$\mathsf{Vals} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Now we add the primitive rules:

$$\frac{s}{(\{a, b\} : \psi) \cup s} \qquad \frac{\{a : \psi\} \cup s \quad \{b : \psi\} \cup s \quad \{c : \psi\} \cup s}{s}$$

$$\mathsf{Vals} = \{\{b\}, \{a\}, \{a, b\}\}$$

The corresponding PNmatrix:

$\widetilde{\wedge}$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$
$\{b\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$
$\{a, b\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$

Example

Adding the introduction rule:

$$\frac{\{a : \psi_1\} \cup s \quad \{a : \psi_2\} \cup s}{\{a : \psi_1 \wedge \psi_2\} \cup s}$$

The corresponding PNmatrix:

$\widetilde{\wedge}$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$
$\{b\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$
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$\{a, b\}$	$\{\{a\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{a, b\}\}$

Example

Adding the introduction rule:

$$\frac{\{b : \psi_1, b : \psi_2\} \cup s}{\{b : \psi_1 \wedge \psi_2\} \cup s}$$

The corresponding PNmatrix:

$\widetilde{\wedge}$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{\{a\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{a, b\}\}$
$\{b\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	$\{\{a\}, \{b\}, \{a, b\}\}$
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$\{b\}$	$\{\{b\},\{a,b\}\}$	$\{\{b\},\{a,b\}\}$	$\{\{b\},\{a,b\}\}$
$\{a,b\}$	$\{\{a,b\}\}$	$\{\{b\},\{a,b\}\}$	$\{\{a,b\}\}$

Example

Adding the introduction rule:

$$\frac{\{b : \psi_1\} \cup s \quad \{b : \psi_2\} \cup s}{\{c : \psi_1 \wedge \psi_2\} \cup s}$$

The corresponding PNmatrix:

$\widetilde{\wedge}$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{\{a\}, \{a, b\}\}$	$\{\{b\}, \{a, b\}\}$	$\{\{a, b\}\}$
$\{b\}$	$\{\{b\}, \{a, b\}\}$	$\emptyset$	$\emptyset$
$\{a, b\}$	$\{\{a, b\}\}$	$\emptyset$	$\emptyset$

All Labelled Calculi are Decidable

Theorem

Every canonical labelled calculus has a characteristic (finite) PNmatrix.

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Corollary

Any logic induced by canonical labelled calculus is decidable.

Application: characterization of syntactic properties

The Subformula Property

Whenever $\Omega \vdash_{\mathbf{G}} s$, there is a derivation of s from Ω in **G** consisting solely of $\mathcal{E}$-sequents (i.e. sequents consisting solely of formulas from $\mathcal{E}$).

Application: characterization of syntactic properties

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Whenever $\Omega \vdash_{\mathbf{G}} s$, there is a derivation of s from Ω in **G** consisting solely of $\mathcal{E}$-sequents (i.e. sequents consisting solely of formulas from $\mathcal{E}$).

Strong Cut-Admissibility

Whenever $\Omega \vdash_{\mathbf{G}} s$, there is a derivation of s from Ω in **G** in which cuts are allowed only on formulas from Ω.

We call cut any primitive rule of the form $\dfrac{(L_1 : \psi) \quad \ldots (L_n : \psi)}{s}$

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Are these properties equivalent?

The subformula property $\not\Rightarrow$ strong cut-admissibility

$$Ł = \{a, b, c\}$$

G has the following cuts:

$$\frac{\{a : \psi\} \cup s \quad \{b : \psi\} \cup s}{s} \quad \frac{\{a : \psi\} \cup s \quad \{c : \psi\} \cup s}{s} \quad \frac{\{b : \psi\} \cup s \quad \{c : \psi\} \cup s}{s}$$

and the following introduction rules:

$$\frac{(\{a, b\} : \psi) \cup s}{\{a : \star\psi\} \cup s} \quad \frac{(\{b, c\} : \psi) \cup s}{\{a : \star\psi\} \cup s}$$

Then we can derive:

$$\dfrac{\dfrac{\{a : \psi\}}{\{a, b\} : \star\psi} \qquad \dfrac{\{a : \psi\}}{\{b, c\} : \star\psi}}{\{b : \star\psi\}}\ cut$$

But $\{b : \star\psi\}$ has no derivation from $\{a : \psi\}$ with cuts only on ψ.

Solution: harmless primitive rules

- The problem can be solved by adding the primitive rule (which does not affect the semantics of the calculus):

$$\frac{(\{a, b\} : \psi) \cup s \quad (\{b, c\} : \psi) \cup s}{\{b : \psi\} \cup s} \; pr$$

Then we have a (cut-free!) derivation:

$$\frac{\dfrac{\{a : \psi\}}{\{a, b\} : \star\psi} \qquad \dfrac{\{a : \psi\}}{\{b, c\} : \star\psi}}{\{b : \star\psi\}} \; pr$$

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Then we have a (cut-free!) derivation:

$$\frac{\dfrac{\{a:\psi\}}{\{a,b\}:\star\psi} \qquad \dfrac{\{a:\psi\}}{\{b,c\}:\star\psi}}{\{b:\star\psi\}}\ pr$$

- The addition of all such harmless primitive rules leads to a cut-saturated calculus.

Theorem

For every labelled canonical calculus **G** *an equivalent cut-saturated* $\mathbf{G}'$ *can be constructed.*

Finally: a semantic characterization

Theorem

Let **G** *be a cut-saturated canonical labelled calculus. Then the following statements are equivalent:*

1. **G** *has a proper characteristic PNmatrix.*
2. **G** *enjoys strong cut-admissibility.*
3. **G** *enjoys the subformula property.*

The Big Picture

- Our goals:
  - Characterization of important syntactic properties of calculi.
  - Understanding the dependencies between them.
- Our tool: non-deterministic semantics.
- Our case study: canonical labelled calculi.

Summary

- The techniques can be applied to many families of proof systems: *single-conclusioned canonical calculi, basic systems, canonical Gödel hypersequent systems and more.*
- Future research directions:
  - First-order case
  - Extension to calculi with less restrictive primitive and introduction rules.
  - Substructural logics...