An Introduction to Nmatrices: Part I

Many-valued Logics

What Do We Mean By "Logic"?

1. A formal language $\mathcal{L}$, based on which $\mathcal{L}$-formulas are constructed. *We denote by $F_{\mathcal{L}}$ the set of well-formed formulas of $\mathcal{L}$.*

2. A consequence relation $\vdash$ for $\mathcal{L}$.

A consequence relation (cr) for $\mathcal{L}$ is a binary relation $\vdash \subseteq 2^{F_{\mathcal{L}}} \times F_{\mathcal{L}}$, having the following properties:

strong reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
transitivity (cut): if $\Gamma \vdash \psi$ and $\Gamma, \psi \vdash \varphi$ then $\Gamma \vdash \varphi$.

Properties of Consequence Relations

- A cr $\vdash$ for $\mathcal{L}$ is structural if for every uniform $\mathcal{L}$-substitution σ and every Γ and ψ: if $\Gamma \vdash \psi$ then $\sigma[\Gamma] \vdash \sigma[\psi]$.
  Example: $p \wedge q \vdash q$ implies $\varphi \wedge \psi \vdash \psi$ for every $\varphi, \psi \in F_{\mathcal{L}}$.
- A cr $\vdash$ for $\mathcal{L}$ is consistent if there exist formulas φ and ψ, such that $\varphi \nvdash \psi$.
- A cr $\vdash$ for $\mathcal{L}$ is finitary if whenever $\Gamma \vdash \psi$, there exists some finite $\Gamma' \subseteq \Gamma$, such that $\Gamma' \vdash \psi$.
- A propositional logic is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a structural, consistent and finitary cr for $\mathcal{L}$.

Example of a Language

- The language $\mathcal{L}_{cl}$:
  - *Atomic formulas:* $p_1, p_2, ...,$
  - *Logical connectives:* $\neg$, $\wedge$, $\vee$, $\supset$
  - *Parentheses:* '(','')'
- The set of well-formed formulas $F_{cl} = F_{\mathcal{L}_{cl}}$:
  - *For any atomic formula* p, $p \in F_{cl}$.
  - *If* $A, B \in F_{cl}$, *then* $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B) \in F_{cl}$.

Ways of Defining Logics

- Semantically: $\Gamma \vdash_{\mathsf{S}} \psi$ if every “model” of Γ is a “model” of ψ in the semantics S.
- Syntactically: $\Gamma \vdash_{\mathsf{D}} \psi$ if ψ has a proof from Γ in the deduction system D.

Classical Truth-Tables

$\tilde{\neg}$	
t	**f**
f	**t**

$\tilde{\supset}$	**t**	**f**
t	**t**	**f**
f	**t**	**t**

$\tilde{\wedge}$	**t**	**f**
t	**t**	**f**
f	**f**	**f**

$\tilde{\vee}$	**t**	**f**
t	**t**	**t**
f	**t**	**f**

A classical valuation is a function $v : F_{cl} \longrightarrow \{t, f\}$, such that $v[\diamond(\psi_1, \ldots, \psi_n)] = \tilde{\diamond}[v[\psi_1], \ldots, v[\psi_n]]$ for any connective $\diamond \in \{\neg, \supset, \vee, \wedge\}$.

$$v[\mathsf{Rain}] = \mathbf{f} \quad v[\mathsf{CarStarts}] = \mathbf{t} \quad v[\mathsf{Trip}] = \mathbf{f}$$
$$v[\neg\mathsf{Rain}] = \mathbf{t} \quad v[\neg\mathsf{Rain} \wedge \mathsf{CarStarts}] = \mathbf{t}$$
$$v[(\neg\mathsf{Rain} \wedge \mathsf{CarStarts}) \supset \mathsf{Trip}] = \mathbf{f}$$

Semantic Way of Defining Classical Logic

- A classical valuation v is a model of an $\mathcal{L}$-formula ψ if $v[\psi] = \mathbf{t}$.
  v is a model of a theory Γ if v is a model of every $\psi \in \Gamma$.
- $\Gamma \vdash_{CPL} \psi$ if every classical model of Γ is a model of ψ.
- Example:

$$\{\mathsf{CarStarts} \supset \mathsf{Trip}, \neg\mathsf{Trip}\} \vdash_{CPL} \neg\mathsf{CarStarts}$$

Hilbert-style Proof Systems

- A Hilbert-style proof system for $\mathcal{L}$ consists of: (i) an (effective) set of $\mathcal{L}$-formulas called axioms, and (ii) an (effective) set of inference rules.

- A proof of ψ from Γ in a Hilbert-style system $\mathbf{H}$ is a finite sequence of $\mathcal{L}$-formulas, where the last formula is ψ, and each formula is: (i) an axiom of $\mathbf{H}$, (ii) a member of Γ, or (iii) is obtained from previous formulas in the sequence by applying some inference rule of $\mathbf{H}$. ψ is a theorem of $\mathbf{H}$ if ψ has a proof in $\mathbf{H}$ from $\emptyset$.

- *We denote $\Gamma \vdash_{\mathbf{H}} \psi$ if ψ has a proof from Γ in $\mathbf{H}$. $\vdash_{\mathbf{H}}$ is a finitary cr for any Hilbert-style system $\mathbf{H}$.*

- A system $\mathbf{H}$ for $\mathcal{L}$ is sound for a logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ if $\vdash_{\mathbf{H}} \subseteq \vdash$. $\mathbf{H}$ is complete for $\mathbf{L}$ if $\vdash \subseteq \vdash_{\mathbf{H}}$.

$\mathbf{HCL}^+$

- Axiom schemata:

  I1 $\varphi \supset (\psi \supset \varphi)$

  I2 $(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$

  I3 $((\psi \supset \varphi) \supset \psi) \supset \psi$

  C1 $\varphi \wedge \psi \supset \varphi$

  C2 $\varphi \wedge \psi \supset \psi$

  C3 $\varphi \supset (\psi \supset \varphi \wedge \psi)$

  D1 $\varphi \supset \varphi \vee \psi$

  D2 $\psi \supset \varphi \vee \psi$

  D3 $(\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$

- Inference Rule:

$$\frac{\psi \quad \psi \supset \varphi}{\varphi}\ MP$$

HCL

Obtained by adding to $\mathbf{HCL}^+$ (possibly without **D3**):

- Either the following axioms concerning negation:

  N1 $(\psi \supset \varphi) \supset (\psi \supset \neg\varphi) \supset \neg\psi$

  N2 $\neg\neg\varphi \supset \varphi$

- Or the following axioms concerning negation:

  N3 $\neg\varphi \vee \varphi$

  N4 $(\varphi \wedge \neg\varphi) \supset \psi$

Soundness and completeness theorem for CPL:

$$\Gamma \vdash_{HCL} \psi \;\Leftrightarrow\; \Gamma \vdash_{CPL} \psi$$

Gentzen-style Proof Systems

- Hilbert-style systems operate on $\mathcal{L}$-formulas. Gentzen-style systems operate on *sequents*.
- A sequent: an expression of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of $\mathcal{L}$-formulas.
- A standard Gentzen-type system for $\mathcal{L}$ consists of:
  1. Standard axioms: $\psi \Rightarrow \psi$.
  2. Structural Weakening and Cut rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Weakening) \qquad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}\ (Cut)$$

  3. Logical introduction rules for the connectives of $\mathcal{L}$.

Proofs in Gentzen-style Systems

- A proof of a sequent $\Gamma \Rightarrow \Delta$ from a set of sequents Θ in G is a finite sequence of sequents, where the last sequent is $\Gamma \Rightarrow \Delta$, and each sequent is: (i) an axiom of $\mathbf{G}$, (ii) a member of Θ, or (iii) is obtained from previous sequents in the sequence by applying some rule of G.

- $\Gamma \Rightarrow \Delta$ is provable in G if it has a proof from the empty set of sequents in G.

- $\Gamma \vdash_G \psi$ if there is some finite $\Gamma' \subseteq \Gamma$, such that $\Gamma' \Rightarrow \psi$ is provable in G.

- $\vdash_G$ is a finitary cr. If G is standard then $\vdash_G$ is also structural.

The System GCPL

$$\psi \Rightarrow \psi$$

$(Weakening)$ $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$ (Cut) $\dfrac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}$

$(\neg \Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$ $(\Rightarrow \neg)$ $\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$

$(\supset\Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta}$ $(\Rightarrow\supset)$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$

$(\wedge \Rightarrow)$ $\dfrac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$ $(\Rightarrow \wedge)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$

$(\vee \Rightarrow)$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$ $(\Rightarrow \vee)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$

Example: Proof of $\vdash_G \Rightarrow \neg(\psi \wedge \varphi) \supset \neg\psi \vee \neg\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \neg\psi, \varphi}\ (Weakening)}{\Rightarrow \neg\psi, \neg\varphi, \varphi}\ (\Rightarrow \neg)}{\Rightarrow \neg\psi \vee \neg\varphi, \psi}\ (\Rightarrow \vee) \qquad \dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \neg\varphi, \psi}\ (Weakening)}{\Rightarrow \neg\psi, \neg\varphi, \psi}\ (\Rightarrow \neg)}{\Rightarrow \neg\psi \vee \neg\varphi, \varphi}\ (\Rightarrow \vee)}{\dfrac{\Rightarrow \neg\psi \vee \neg\varphi, \psi \wedge \varphi}{\neg(\psi \wedge \varphi) \Rightarrow \neg\psi \vee \neg\varphi}\ (\neg \Rightarrow)}\ (\Rightarrow \wedge)}{\Rightarrow \neg(\psi \wedge \varphi) \supset \neg\psi \vee \neg\varphi}\ (\Rightarrow \supset)$$

Completeness and Cut-elimination

- A Gentzen-style System admits cut-elimination if whenever $\Gamma \Rightarrow \Delta$ is provable in G, $\Gamma \Rightarrow \Delta$ also has a cut-free proof in G.
- Completeness Theorem for Classical Logic:
  - $\Gamma \vdash_{GCPL} \psi$ iff $\Gamma \vdash_{CPL} \psi$
  - GCPL admits cut-elimination
- Important corollary - GCPL has the subformula property: *If $\Gamma \Rightarrow \Delta$ has a derivation in GCPL, then all the formulas in this derivation are subformulas of the formulas in $\Gamma \Rightarrow \Delta$.*

Basic Principles of Classical Logic

- Bivalence:
  Every proposition is either true or false (there are exactly two truth-values).
- Inconsistency Intolerance:
  A proposition and its negation cannot be both true.
- Truth-Functionality:
  The truth-value of a complex proposition is uniquely defined by the truth-values of its constituents.

Many-valued Logic - Motivation

- Sometimes incomplete information prevents us from telling if something is true or not.
- Łukasiewicz, "On Determinism", 1970: *If statements about future events are already true or false, then the future is as much determined as the past and differs from the past only in so far as it has not yet come to pass.*
- The idea: reject Bivalence by adding a third truth-value **I**, to be read as "possible".

Three-valued Łukasiewicz Logic

$\supset$	**f**	**I**	**t**
f	**t**	**t**	**t**
I	**I**	**t**	**t**
t	**f**	**I**	**t**

$\neg$	
f	**t**
I	**I**
t	**f**

- A legal valuation v is a Łuk-model of a formula ψ if $v[\psi] = \mathbf{t}$. v is a Łuk-model of a theory Γ if v is a Łuk-model of every $\psi \in \Gamma$.
- $\Gamma \vdash_{Luk} \psi$ if every Łuk-model of Γ is a Łuk-model of ψ.
- **Examples**:

  $\vdash_{Luk}$ **I1** $[\varphi \supset (\psi \supset \varphi)]$

  $\nvdash_{Luk}$ **I2** $[(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)]$

Kleene and McCarthy Logics

- Modelling parallel vs. sequential computation
- The third truth-value $\mathbf{I}$ - for "undefined"
- Negation is defined like in Łukasiewicz three-valued logic.
- The notion of a model of a formula and the associated cr are defined like in Łukasiewicz three-valued logic.

$\vee$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{I}$	$\mathbf{I}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

Kleene

$\vee$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{I}$	$\mathbf{I}$	$\mathbf{I}$	$\mathbf{I}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

McCarthy

J_3

- The third truth-value $\mathbf{I}$ - for "paradoxical"
- Implication is defined as follows:

$$a \tilde{\supset} b = \begin{cases} \mathbf{t} & \text{if } a = \mathbf{f} \\ b & \text{if } a \in \{\mathbf{t}, \mathbf{I}\} \end{cases}$$

- Other connectives are defined like in Kleene's logic.
- A legal valuation v is a J_3-model of a formula ψ if $v[\psi] \in \{\mathbf{t}, \mathbf{I}\}$.
  v is a J_3-model of a theory Γ if v is a J_3-model of every $\psi \in \Gamma$.
- $\Gamma \vdash_{J_3} \psi$ if every J_3-model of Γ is a J_3-model of ψ.
- **Examples**:
  $\vdash_{J_2}$ **I2** $[(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)]$
  $\nvdash_{J_3}$ **N1** $[(\psi \supset \varphi) \supset (\psi \supset \neg\varphi) \supset \neg\psi]$

Hilbert-type System for J_3

Obtained by adding to $\mathbf{HCL}^+$ the following axioms for negation:

N3 $\neg\psi \vee \psi$

NN $\neg\neg\psi \equiv \psi$

NC $\neg(\psi \wedge \varphi) \equiv \neg\psi \vee \neg\varphi$

ND $\neg(\psi \vee \varphi) \equiv \neg\psi \wedge \neg\varphi$

NI $\neg(\psi \supset \varphi) \equiv \psi \wedge \neg\varphi$

where $\psi \equiv \varphi \stackrel{\text{def}}{=} (\psi \supset \varphi) \wedge (\varphi \supset \psi)$.

Gentzen-type System for J_3

Obtained by deleting $(\neg \Rightarrow)$ from $\mathbf{GCPL}$ (From $\Gamma \Rightarrow \Delta, \varphi$ infer $\neg\varphi, \Gamma \Rightarrow \Delta$), and adding instead the following rules:

$$\frac{\Gamma, \psi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \supset \varphi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma \Rightarrow \neg(\psi \supset \varphi), \Delta}$$

$$\frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \varphi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \varphi)}$$

$$\frac{\Gamma, \neg\psi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \vee \varphi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \vee \varphi)}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \neg\neg\psi}$$

General Semantic Method for Defining Logics

- A denotational semantics for a language $\mathcal{L}$ is a pair $\mathsf{S} = \langle S, \models_S \rangle$, where S is a non-empty set of "valuations" and $\models_S \subset S \times F_{\mathcal{L}}$. A valuation is usually some mapping from $F_{\mathcal{L}}$ to some set of "truth values".

- For $v \in S$, v is a S-model of ψ if $v \models_S \psi$. v is an S-model of Γ if $v \models_S \psi$ for every $\psi \in \Gamma$.

- $\Gamma \vdash_{\mathsf{S}} \psi$ if every S-model of Γ is an S-model of ψ.

- For any denotational semantics $\mathsf{S} = \langle S, \models_{\mathsf{S}} \rangle$ for $\mathcal{L}$, $\vdash_{\mathsf{S}}$ is a cr. *However, $\langle \mathcal{L}, \vdash_{\mathsf{S}} \rangle$ may not be a logic.*

Many-valued Matrices as Denotational Semantics

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a matrix for a propositional language $\mathcal{L}$ if:

- $\mathcal{V}$ is a nonempty set of truth-values,
- $\emptyset \neq \mathcal{D} \subset \mathcal{V}$ (the set of designated truth-values),
- for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes an operation $\widetilde{\diamond} : \mathcal{V}^n \rightarrow \mathcal{V}$

Examples

- In classical logic: $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}$, while $\mathcal{D} = \{\mathbf{t}\}$.
- In the 3-valued logics of Łukasiewicz, Kleene, and McCarthy: $\mathcal{V} = \{\mathbf{t}, \mathbf{f}, \mathbf{I}\}$, while $\mathcal{D} = \{\mathbf{t}\}$.
- In J_3: $\mathcal{V} = \{\mathbf{t}, \mathbf{f}, \mathbf{I}\}$, while $\mathcal{D} = \{\mathbf{t}, \mathbf{I}\}$.

Valuations

- A valuation v in a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is any function from the set of $\mathcal{L}$-formulas to $\mathcal{V}$ such that:

$$v[\diamond(\psi_1, ..., \psi_n)] = \tilde{\diamond}[v[\psi_1], ..., v[\psi_n]]$$

- v is a model of an $\mathcal{L}$-formula ψ in $\mathcal{M}$, denoted by $v \models_{\mathcal{M}} \psi$, if $v[\psi] \in \mathcal{D}$.

- v is a model of a theory Γ in $\mathcal{M}$, denoted by $v \models_{\mathcal{M}} \Gamma$, if v is a model of every $\psi \in \Gamma$.

Consequence Relation Induced by a Matrix

- $\Gamma \vdash_{\mathcal{M}} \psi$ if for every valuation v in $\mathcal{M}$: $v \models_{\mathcal{M}} \Gamma$ implies $v \models_{\mathcal{M}} \psi$.
- Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be some logic.
  $\mathbf{L}$ is sound for a matrix $\mathcal{M}$ if $\vdash \subseteq \vdash_{\mathcal{M}}$.
  $\mathbf{L}$ is complete for $\mathcal{M}$ if $\vdash_{\mathcal{M}} \subseteq \vdash$.
  $\mathcal{M}$ is a characteristic matrix for $\mathbf{L}$ if $\mathbf{L}$ is sound and complete for $\mathcal{M}$.
- For any matrix $\mathcal{M}$ for $\mathcal{L}$, $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.
- *Converse direction: is every propositional logic induced by a matrix?*

Logics Induced by Matrices and their Families

- Let C be a family of matrices. We say that $\Gamma \vdash_C \psi$ if for every $\mathcal{M} \in C$: $\Gamma \vdash_{\mathcal{M}} \psi$.
- Every propositional logic is induced by some family of matrices.
- A set of $\mathcal{L}$-formulas (theory) Γ is $\vdash$-consistent if there exists some $\mathcal{L}$-formula ψ such that $\Gamma \nvdash \psi$.
- A logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is uniform if for every two theories Γ_1, Γ_2 and an $\mathcal{L}$-formula ψ: $\Gamma_1 \vdash \psi$ whenever $\Gamma_1, \Gamma_2 \vdash \psi$ and Γ_2 is a $\vdash$-consistent theory with no atoms in common with $\Gamma_1 \cup \{\psi\}$.
- Łos & Suszko: A finitary propositional logic has a characteristic matrix iff it is uniform.

Dunn-Belnap's Logic for Inconsistent and Incomplete Information

A framework for information collecting and processing:

Processor

Source 1: A is true, C is false

Source 2: A is false

Source 3: B is true

Dunn-Belnap's Logic: 4 truth-values

The truth-values which can be assigned to a formula A are subsets of $\{0, 1\}$:

- $\mathbf{t} = \{1\}$: *P has information that A is true, but no information that A is false.*
- $\mathbf{f} = \{0\}$: *P has information that A is false, but no information that A is true.*
- $\top = \{0, 1\}$: *P has both information that A is false and information that A is true.*
- $\bot = \emptyset$: *P has no information on A.*

Belnap's Logic: 4 truth-values

Processor

Source 1: A is true, C is false

Source 2: A is false

Source 3: B is true

$$v[A] = \top \quad v[B] = \mathbf{t} \quad v[C] = \mathbf{f} \quad v[D] = \bot$$

Belnap's Logic: 4 truth-values

Assumption 1: The sources provide information on atomic formulas, but not necessarily all of them.

Assumption 2: P respects the classical truth-tables in the following sense:

1. P ascribes 1 (0) to $\neg\varphi$ iff it ascribes 0 (1) to φ
2. P ascribes 1 to $\varphi \vee \psi$ iff it ascribes 1 to either φ or ψ
3. P ascribes 0 to $\varphi \vee \psi$ iff it ascribes 0 to both φ and ψ
4. P ascribes 1 to $\varphi \wedge \psi$ iff it ascribes 1 to both φ and ψ
5. P ascribes 0 to $\varphi \wedge \psi$ iff it ascribes 0 to either φ or ψ

"The processor ascribes $x \in \{0, 1\}$ to ψ": x is included in the subset of $\{0, 1\}$ which is assigned by the processor to ψ.

Dunn-Belnap's Logic

- $\mathcal{V} = \{\mathbf{t}, \mathbf{f}, \top, \bot\}$
- $\mathcal{D} = \{\mathbf{t}, \top\}$
- The truth-tables for the connectives:

$\vee$	$\bot$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\bot$	$\bot$	$\bot$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{f}$	$\bot$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$	$\top$

$\wedge$	$\bot$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\bot$	$\bot$	$\mathbf{f}$	$\bot$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\mathbf{t}$	$\bot$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\top$	$\mathbf{f}$	$\mathbf{f}$	$\top$	$\top$

$\neg$	
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$

Corresponding Gentzen-type System

Obtained by deleting from **GCPL** both of the negation rules (as well as the implication rules), and adding instead the following rules:

$$\frac{\Gamma, \neg\psi \Rightarrow \Delta \qquad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \varphi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \varphi)}$$

$$\frac{\Gamma, \neg\psi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \vee \varphi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi \qquad \Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \vee \varphi)}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \neg\neg\psi}$$

Fuzzy Logic: The Sorites Paradox

- 1 grain of wheat does not make a heap.
- If 1 grain of wheat does not make a heap then 2 grains of wheat do not.
- If 2 grains of wheat do not make a heap then 3 grains do not.
- ...
- If 9,999 grains of wheat do not make a heap then 10,000 do not.

10,000 grains of wheat do not make a heap???

Fuzzy Logics (L. Zadeh)

- Fuzzy Logic is an extension of Classical Logic.
- The idea: use infinitely many truth-values in [0,1]
- The interpretations of disjunction and conjunction are monotonic, and interpretations of all connectives agree with the classical ones on 0 and 1.

Gödel and Łukasiewicz Fuzzy Logics

$$\mathcal{V} = [0,1] \quad \mathcal{D} = \{1\} \quad \mathcal{O} = \{\widetilde{\vee}, \widetilde{\wedge}, \widetilde{\supset}, \widetilde{\neg}\}$$

$$a \widetilde{\vee} b = \mathsf{max}(a,b) \qquad a \widetilde{\wedge} b = \mathsf{min}(a,b) \qquad \widetilde{\neg} a = a \widetilde{\supset} 0$$

$$a \widetilde{\supset} b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases} \qquad a \widetilde{\supset} b = \begin{cases} 1 & \text{if } a \leq b \\ 1 - a + b & \text{if } a > b \end{cases}$$

(Gödel) (Łukasiewicz)

Models and cr in Fuzzy Logics

- A legal valuation v for a fuzzy logic $\mathbf{L}$ is an $\mathbf{L}$-model of a formula ψ if $v[\psi] = 1$. v is an $\mathbf{L}$-model of a theory Γ if v is an $\mathbf{L}$-model of any $\psi \in \Gamma$.
- $\Gamma \vdash_L \psi$ if every $\mathbf{L}$-model of Γ is an $\mathbf{L}$-model of ψ.
- **Examples:**
  $\vdash_L \psi \supset (\varphi \supset \psi)$
  $\nvdash_L \varphi \vee \neg\varphi$
  $\vdash_{\text{Łukasiewicz}} \psi \equiv \neg\neg\psi$, but $\nvdash_{\text{Gödel}} \psi \equiv \neg\neg\psi$
  $\vdash_{\text{Gödel}} \mathbf{I_2}$, but $\nvdash_{\text{Łukasiewicz}} \mathbf{I2}$
- Both Gödel and Łukasiewicz fuzzy logics are decidable.

Axioms for Gödel fuzzy logic

I1 $\varphi \supset (\psi \supset \varphi)$

I2 $(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$

C1 $\varphi \wedge \psi \supset \varphi$

C2 $\varphi \wedge \psi \supset \psi$

C3 $\varphi \supset (\psi \supset \varphi \wedge \psi)$

D1 $\varphi \supset \varphi \vee \psi$

D2 $\psi \supset \varphi \vee \psi$

D3 $(\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$

N1 $(\psi \supset \varphi) \supset (\psi \supset \neg\varphi) \supset \neg\psi$

N4 $(\varphi \wedge \neg\varphi) \supset \psi$

PL $(\varphi \supset \psi) \vee (\varphi \supset \psi)$

Gödel and Łukasiewicz Finite-valued Logics

For a given finite natural number n, the n-valued counterparts of the fuzzy logics of Gödel and Łukasiewicz are obtained by taking

$$\mathcal{V} = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$$

Then $\mathcal{D}$ and the interpretations of the connectives are defined analogously as for the infinite-valued logics.

Introduction to Nmatrices: Part II

Introducing Nmatrices

Lack of Modularity in Matrices

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

	$\neg$
t	**f**
f	**t**

		$\vee$
t	**t**	**t**
t	**f**	**t**
f	**t**	**t**
f	**f**	**f**

Lack of Modularity in Matrices

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$ reject LEM

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

	$\neg$
t	**f**
f	**???**

		$\vee$
t	**t**	**t**
t	**f**	**t**
f	**t**	**t**
f	**f**	**f**

Syntactic Underspecification $\Rightarrow$ Non-determinism

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

	$\neg$
t	{**f**}
f	{**t**,**f**}

		$\vee$
t	**t**	{**t**}
t	**f**	{**t**}
f	**t**	{**t**}
f	**f**	{**f**}

Introducing Non-determinism

- *Types of non-deterministic phenomena:*

  vagueness *incompleteness*

  uncertainty *imprecision*

  inconsistency

- *Non-deterministic phenomena are in conflict with the principle of truth-functionality.*

- Representation idea: non-deterministic evaluation (interpretation) of formulas.

  An example:

$\diamond$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{f}\}$

Linguistic Ambiguity

In many natural languages, “or” has both inclusive and exclusive meanings. For example, if a mathematician says:

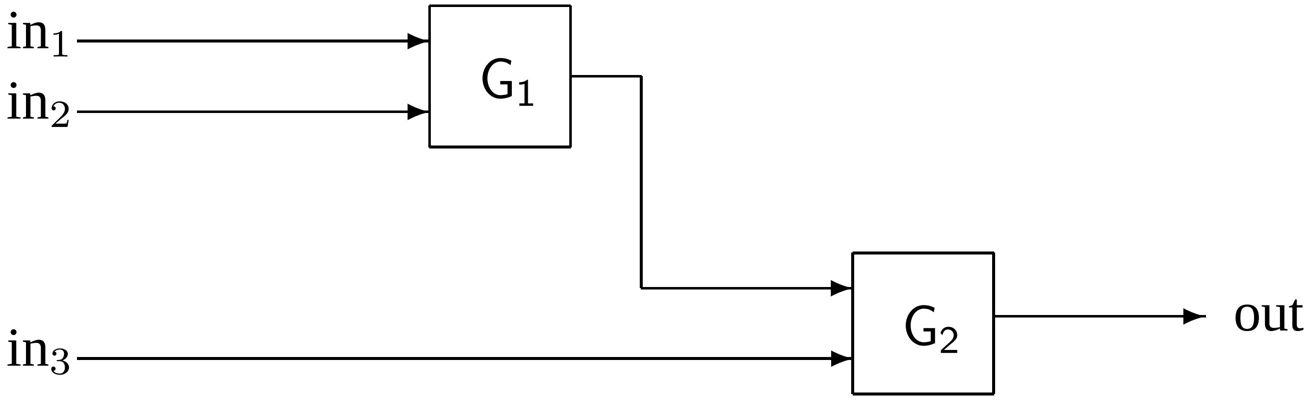
I shall either attack problem A or attack problem B

in many cases he may solve both problems, but in some situations he actually means *“but don’t expect me to solve both”*.

Capturing both meanings:

a	**b**	**a OR b**
t	**t**	$\{\mathbf{t}, \mathbf{f}\}$
t	**f**	$\{\mathbf{t}\}$
f	**t**	$\{\mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$

Circuit Example

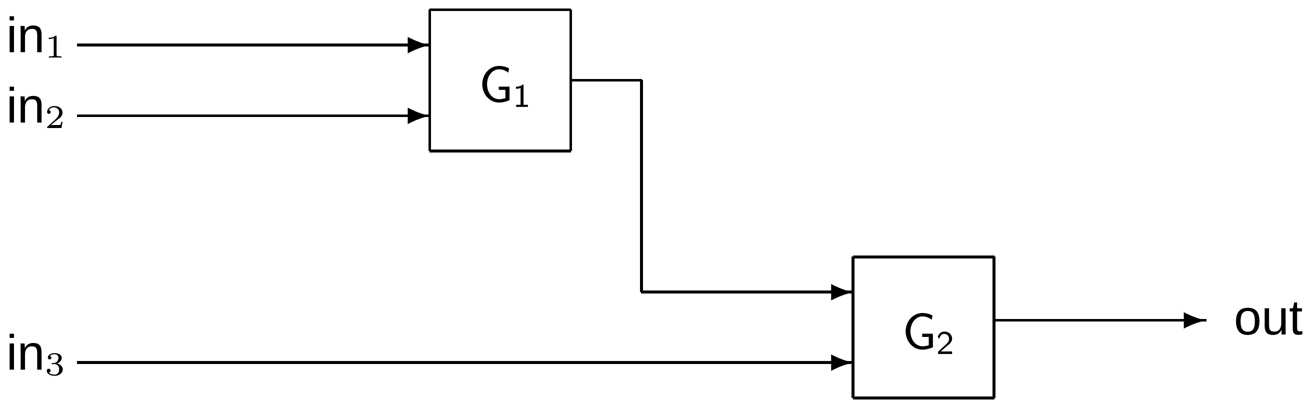


Here G_1 is an OR gate, and G_2 is an XOR gate

Unknown Boolean Functions

Suppose we know that each of the gates G_1 and G_2 is either an OR gate or an XOR gate, but we do not know which one exactly.

$\star$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t},\mathbf{f}\}$	$\{\mathbf{t}\}$
$\mathbf{f}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$

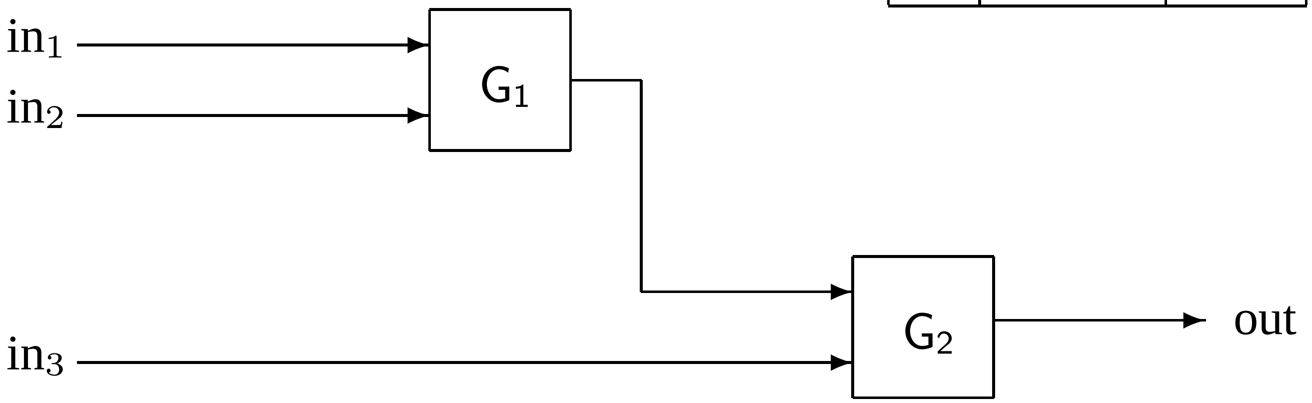


Here the interpretation of the non-deterministic "truth-table" is static.

Deviation from expected behavior

- *Problems in the manufacturing process*
- *Disturbing noise sources, temperature, etc.*
- *Adversary operations*

$\star$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{t}\}$
$\mathbf{f}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$



Now the interpretation of the non-deterministic “truth-table” is dynamic.

Evaluation with Unknown Computation Models

- We are sending a formula $A \vee B$ for evaluation to a distant computer, and we do not know whether it performs parallel or sequential computation.
- As parallel computation can be described using 3-valued Kleene connectives, and sequential computation — using 3-valued McCarthy connectives, to reason about the result of evaluation we must use some combination of both.

$\vee$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{I}\}$	$\{\mathbf{t}\}$
$\mathbf{I}$	$\{\mathbf{I}\}$	$\{\mathbf{I}\}$	$\{\mathbf{t}, \mathbf{I}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

Non-deterministic Matrices

A non-deterministic matrix (Nmatrix) for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$:

- $\mathcal{V}$ - the set of truth-values,
- $\mathcal{D}$ - the set of designated truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to P^+(\mathcal{V})$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

Ordinary matrices correspond to the case when each $\widetilde{\diamond}$ *is a function taking singleton values only* (*then it can be treated as a function* $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$).

Example 1: Two Truth-values

Let $\mathcal{L}$ — a language over $\{\vee, \wedge, \supset, \neg\}$, $\mathcal{V} = \{\mathbf{f}, \mathbf{t}\}, \mathcal{D} = \{\mathbf{t}\}$.

Suppose that $\vee$, $\wedge$ and $\supset$ are interpreted classically, while $\neg$ satisfies the law of excluded middle $\neg\varphi \vee \varphi$, but not necessarily the law of contradiction $\neg(\varphi \wedge \neg\varphi)$. This leads to the Nmatrix $\mathcal{M}^2 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where $\mathcal{O}$ is given by:

		$\tilde{\vee}$	$\tilde{\wedge}$	$\tilde{\supset}$
$\mathbf{t}$	$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\mathbf{f}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$
$\mathbf{f}$	$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$

	$\tilde{\neg}$
$\mathbf{t}$	$\{\mathbf{t}, \mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{t}\}$

Example 2: Three Truth-values

Two 3-valued Nmatrices with $\mathcal{V} = \{\mathbf{f}, \mathbf{I}, \mathbf{t}\}, \quad \mathcal{D} = \{\mathbf{I}, \mathbf{t}\}$

$$\mathcal{M}^3_L = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_L \rangle \qquad \mathcal{M}^3_S = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_S \rangle$$

Standard interpretations of disjunction, conjunction and implication:

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a = b = \mathbf{f} \end{cases} \qquad a \widetilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if either } a = \mathbf{f} \text{ or } b = \mathbf{f} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a = \mathbf{f} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a \in \mathcal{D} \text{ and } b = \mathbf{f} \end{cases}$$

Negation is interpreted more liberally in $\mathcal{M}^3_L$, and more strictly in $\mathcal{M}^3_S$:

$\mathcal{M}^3_L$:

	$\widetilde{\neg}$
$\mathbf{t}$	$\{\mathbf{f}\}$
$\mathbf{I}$	$\mathcal{V}$
$\mathbf{f}$	$\{\mathbf{t}\}$

$\mathcal{M}^3_S$:

	$\widetilde{\neg}$
$\mathbf{t}$	$\{\mathbf{f}\}$
$\mathbf{I}$	$\mathcal{D}$
$\mathbf{f}$	$\{\mathbf{t}\}$

Dynamic and Static Valuations

- An $\mathcal{M}$-legal dynamic valuation v is any function from $\mathcal{L}$-formulas to $\mathcal{V}$ which satisfies the condition:

$$v[\diamond(\psi_1, ..., \psi_n)] \in \tilde{\diamond}[v[\psi_1], ..., v[\psi_n]]$$

- An $\mathcal{M}$-legal static valuation is an $\mathcal{M}$-legal dynamic valuation v which satisfies also the following compositionality principle:

$$v[\diamond(\psi_1, ..., \psi_n)] = v[\diamond(\varphi_1, ..., \varphi_n)] \text{ if } v[\psi_i] = v[\varphi_i] \text{ for } 1 \leq i \leq n$$

- Unlike the static semantics, the dynamic semantics is non-truth-functional.

- *In the deterministic case there is no difference between static and dynamic valuations.*

Dynamic and Static Valuations Explained

Two possible ways to choose the truth-value of $\diamond(\psi_1, ..., \psi_n)$:

- Dynamic (online) computation: separately for every formula.
- Static (offline) choice: separately for every connective $\diamond$, but uniformly for all formulas $\diamond(\psi_1, ..., \psi_n)$.

As opposed to static valuations, dynamic valuations need not be truth-functional:

$$v_1[p_1] = v_1[p_2] = \mathbf{t}\ ;\ v_1[q_1] = v_1[q_2] = \mathbf{t};\ v_1[p_1 \star q_1] = v_1[p_2 \star q_2] = \mathbf{t}$$

$$v_2[p_1] = v_2[p_2] = \mathbf{t}\ ;\ v_2[q_1] = v_2[q_2] = \mathbf{t};\ v_2[p_1 \star q_1] = \mathbf{t}\ ;\ v_2[p_2 \star q_2] = \mathbf{f}$$

$\star$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{t}\}$
$\mathbf{f}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$

Non-deterministic Semantics

Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$.

- A valuation v in $\mathcal{M}$ is a model of a formula ψ if $v[\psi] \in \mathcal{D}$.
  A valuation v is a model of a set of formulas Γ if it is a model of every formula in Γ.
- $\Gamma \vdash^d_{\mathcal{M}} \psi$ if every dynamic model in $\mathcal{M}$ of Γ is a model of ψ.
- $\Gamma \vdash^s_{\mathcal{M}} \psi$ if every static model in $\mathcal{M}$ of Γ is a model of ψ.
- For every Nmatrix $\mathcal{M}$, both $\vdash^s_{\mathcal{M}}$ and $\vdash^d_{\mathcal{M}}$ define logics.
- A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is dynamically (statically) sound for an Nmatrix $\mathcal{M}$ for $\mathcal{L}$ if $\vdash_{\mathbf{L}} \subseteq \vdash^d_{\mathcal{M}}$ ($\vdash_{\mathbf{L}} \subseteq \vdash^s_{\mathcal{M}}$). $\mathbf{L}$ is dynamically (statically) complete for $\mathcal{M}$ if $\vdash^d_{\mathcal{M}} \subseteq \vdash_{\mathbf{L}}$ ($\vdash^s_{\mathcal{M}} \subseteq \vdash_{\mathbf{L}}$). $\mathcal{M}$ is dynamically (statically) characteristic for $\mathbf{L}$ if $\vdash^d_{\mathcal{M}} = \vdash_{\mathbf{L}}$ ($\vdash^s_{\mathcal{M}} = \vdash_{\mathbf{L}}$).

The Nmatrix $\mathcal{M}^2$ Revisited

		$\tilde{\vee}$	$\tilde{\wedge}$	$\tilde{\supset}$
$\mathbf{t}$	$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\mathbf{f}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$
$\mathbf{f}$	$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$

	$\tilde{\neg}$
$\mathbf{t}$	$\{\mathbf{t}, \mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{t}\}$

- Any dynamic valuation satisfies $\neg\psi \vee \psi$ but not necessarily $\psi \supset \neg\neg\psi$.
- Any static valuation satisfies both $\neg\psi \vee \psi$ and $\psi \supset \neg\neg\psi$ *(it admits only two interpretations of* $\neg$*: the classical one and* $\lambda x.\mathbf{t}$*).* However, it satisfies neither $\neg\neg\psi \supset \psi$ nor $(\varphi \wedge \neg\varphi) \supset \psi$ *(take* $\tilde{\neg} = \lambda x.\mathbf{t}$, $v[\varphi] = \mathbf{t}$, $v[\psi] = \mathbf{f}$*).*
- The dynamic semantics and the static semantics generate two different paraconsistent logics, known as $CLuN$ and CAR.

Reminder: The System GCPL

$$\psi \Rightarrow \psi$$

$$(Weakening) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \qquad (Cut) \quad \frac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}$$

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\supset\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\supset) \quad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$(\wedge \Rightarrow) \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow) \quad \frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad (\Rightarrow \vee) \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Reminder: The System HCL^+

- Axiom schemata:

I1 $\varphi \supset (\psi \supset \varphi)$

I2 $(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$

I3 $((\psi \supset \varphi) \supset \psi) \supset \psi$

C1 $\varphi \wedge \psi \supset \varphi$

C2 $\varphi \wedge \psi \supset \psi$

C3 $\varphi \supset (\psi \supset \varphi \wedge \psi)$

D1 $\varphi \supset \varphi \vee \psi$

D2 $\psi \supset \varphi \vee \psi$

D3 $(\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$

- Inference Rule:

$$\frac{\psi \quad \psi \supset \varphi}{\varphi} \; MP$$

The Logic $CLuN$

- A Gentzen-style system sound and complete for the dynamic semantics of $\mathcal{M}_2$ is obtained by deleting from GCPL the rule $(\neg \Rightarrow)$, and leaving the single negation rule $(\Rightarrow \neg)$, i.e.:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

- As the above rule corresponds to $\mathbf{N3}$: $\neg\varphi \vee \varphi$, a Hilbert-style system sound and complete for $\vdash^{d}_{\mathcal{M}_2}$ is obtained by adding $\mathbf{N3}$ to the system HCL$^+$ for positive classical logic.

- Since the above systems are respectively the standard Gentzen- and Hilbert-style systems used for the paraconsistent logic $CluN$ of Batens, the logic corresponding to the dynamic semantics of $\mathcal{M}_2$ is indeed $CluN$.

The Logic CAR

- A Hilbert-style system sound and complete for the static semantics of $\mathcal{M}_2$ is obtained by supplementing HCL$^+$ with **N3** $(\neg\varphi \vee \varphi)$ and the following weak form of **N4**: $(\varphi \wedge \neg\varphi) \supset \neg\psi$
- As the above system is equivalent to the original Hilbert-style system for the paraconsistent logic CAR of da Costa and Beziau, the logic corresponding to the static semantics of $\mathcal{M}_2$ is CAR.

$\mathcal{M}_L^3$ and $\mathcal{M}_S^3$ Revisited

$$\mathcal{V} = \{\mathbf{f}, \mathbf{I}, \mathbf{t}\}, \quad \mathcal{D} = \{\mathbf{I}, \mathbf{t}\}$$

$$\mathcal{M}_L^3 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_L \rangle \qquad \mathcal{M}_S^3 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_S \rangle$$

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a = b = \mathbf{f} \end{cases} \qquad a \widetilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if either } a = \mathbf{f} \text{ or } b = \mathbf{f} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a = \mathbf{f} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a \in \mathcal{D} \text{ and } b = \mathbf{f} \end{cases}$$

$\mathcal{M}_L^3$:

	$\widetilde{\neg}$
$\mathbf{t}$	$\{\mathbf{f}\}$
$\mathbf{I}$	$\mathcal{V}$
$\mathbf{f}$	$\{\mathbf{t}\}$

$\mathcal{M}_S^3$:

	$\widetilde{\neg}$
$\mathbf{t}$	$\{\mathbf{f}\}$
$\mathbf{I}$	$\mathcal{D}$
$\mathbf{f}$	$\{\mathbf{t}\}$

Proof System for Dynamic Semantics of $\mathcal{M}_L^3$ and $\mathcal{M}_S^3$

- Let the Gentzen-style system GC$_{min}$ be obtained by replacing the rule $(\neg \Rightarrow)$ in GCPL with the rule:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$$

  It can be shown that GC$_{min}$ is sound and complete for the dynamic semantics of both $\mathcal{M}_L^3$ and $\mathcal{M}_S^3$.

- Thus the dynamic semantics of the Nmatrices $\mathcal{M}_L^3$ and $\mathcal{M}_S^3$ generate the same logic.

The Logic $\mathbf{C_{min}}$

- As the negation rules

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Rightarrow \Delta, \neg\varphi}$$

  of GC_{min} translate to **N2** ($\neg\neg\varphi \supset \varphi$) and **N3** ($\neg\varphi \vee \varphi$), respectively, GC_{min} is equivalent to the system HC_{min} obtained by adding the above axiom schemes to HCL^{+}.

- The system HC_{min} represents the basic paraconsistent logic $\mathbf{C_{min}}$. *That logic was originally defined in terms of the proof systems only, without any semantics.*

An Application: Decidability of $\mathbf{C_{min}}$

- By the foregoing, the system HC$_{min}$ is sound and complete for the dynamic semantics of both $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$. In consequence, $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$ provide sound and complete semantics for the logic $\mathbf{C_{min}}$.

- This fact can be used for showing that some formulas are, or are not, theorems of $\mathbf{C_{min}}$ – by showing that they are, or resp. are not, valid in the dynamic semantics of $\mathcal{M}^3_L$ (or $\mathcal{M}^3_S$). Thus $\varphi \supset \neg\neg\varphi$ is not derivable in C$_{min}$, since $\not\vdash_{\mathcal{M}^3_L} \varphi \supset \neg\neg\varphi$.
  To see this, take $v[\varphi] = \mathbf{I}$ and $v[\neg\varphi] = \mathbf{t}$. Then v is a legal dynamic valuation in $\mathcal{M}^3_L$ and $v[\neg\neg\varphi] \in \tilde{\neg}[\mathbf{t}] = \{\mathbf{f}\}$, whence $v[\varphi \supset \neg\neg\varphi] \in \mathbf{I}\tilde{\supset}\mathbf{f} = \{\mathbf{f}\}$ and so $\not\vdash_{\mathcal{M}^3_L} \varphi \supset \neg\neg\varphi$.

- It follows that $\mathbf{C_{min}}$ is decidable.

Information Sources Logic: Generalization of Belnap's Logic

A processor P collects information from some sources, and assigns to a formula A the value:

- $\mathbf{t} = \{1\}$, if *P has information that A is true, but no information that A is false.*
- $\mathbf{f} = \{0\}$, if *P has information that A is false, but no information that A is true.*
- $\top = \{0, 1\}$, if *P has both information that A is false and information that A is true.*
- $\bot = \emptyset$, if *P has no information on A.*

Information Sources Logic

Assumption 1: The sources can provide information on any formula (also complex ones), but not necessarily on all of them,

Assumption 2: P respects the classical truth-tables, i.e.:

1. P ascribes 1(0) to $\neg\varphi$ iff it ascribes 0 (1) to φ
2. P ascribes 1 to $\varphi \vee \psi$ if it ascribes 1 to either φ or ψ
3. P ascribes 0 to $\varphi \vee \psi$ iff it ascribes 0 to both φ and ψ
4. P ascribes 1 to $\varphi \wedge \psi$ iff it ascribes 1 to both φ and ψ
5. P ascribes 0 to $\varphi \wedge \psi$ if it ascribes 0 to either φ or ψ

This framework generalizes Belnap's one by allowing the sources to give information on complex formulas too, and, in opposition to the former, cannot be captured using deterministic matrices.

Assignment of Values to Formulas by the Processor

The processor P assigns to a formula ψ a value $v[\psi]$ in the set

$$\bot = \emptyset \quad \mathbf{f} = \{0\} \quad \top = \{0, 1\} \quad \mathbf{t} = \{1\}$$

whereby $1 \in v[\psi]$ (resp. $0 \in v[\psi]$) iff either one of the sources tells P that ψ is true (resp. false), or P concludes the latter from the information it already has and from Rules 1-5 it obeys.

Thus if $v[\varphi] = v[\psi] = \mathbf{f} = \{0\}$, then surely $0 \in v[\varphi \vee \psi]$ by Rule 3. However, if some source says $\varphi \vee \psi$ is true, then P will ascribe 1 to $\varphi \vee \psi$, making $v[\varphi \vee \psi] = \{0, 1\} = \top$. Otherwise, we have $v[\varphi \vee \psi] = \{0\} = \mathbf{f}$. Summing up, $v[\varphi \vee \psi] \in \{\mathbf{f}, \top\}$ — which shows that the interpretation of $\vee$ must be non-deterministic.

Similarly, if $v[\varphi] = \bot$, then surely $0 \notin v[\varphi \vee \psi]$ by rule 3. Hence $v[\varphi \vee \psi] \in \{\mathbf{t}, \bot\}$ in this case. If in addition $v[\psi] \in \{\mathbf{t}, \top\}$ (i.e. $1 \in v[\psi]$) then rule 2 dictates that $v[\varphi \vee \psi] = \{\mathbf{t}\}$. Otherwise $v[\varphi \vee \psi] = \{\mathbf{t}, \bot\}$.

Nmatrix $\mathcal{M}_I^4$ for the Information Sources Logic

$\mathcal{V} = \{\mathbf{f}, \bot, \top, \mathbf{t}\}$ $\mathcal{D} = \{\mathbf{t}, \top\}$

$\widetilde{\neg}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{t}$	$\bot$	$\top$	$\mathbf{f}$

$\widetilde{\vee}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}, \top\}$	$\{\mathbf{t}, \bot\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\bot$	$\{\mathbf{t}, \bot\}$	$\{\mathbf{t}, \bot\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\top$	$\{\top\}$	$\{\mathbf{t}\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

$\widetilde{\wedge}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\bot$	$\{\mathbf{f}\}$	$\{\mathbf{f}, \bot\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}, \bot\}$
$\top$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\top\}$	$\{\top\}$
$\mathbf{t}$	$\{\mathbf{f}\}$	$\{\mathbf{f}, \bot\}$	$\{\top\}$	$\{\mathbf{t}, \top\}$

Gentzen System for the Information Sources Logic

Standard axioms, cut, weakening and the rules:

$$(\neg\neg\Rightarrow)\quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \qquad (\Rightarrow\neg\neg)\quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$$

$$(\Rightarrow\vee)\quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\neg\vee\Rightarrow)\quad \frac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta} \qquad (\Rightarrow\neg\vee)\quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi \qquad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$$

$$(\wedge\Rightarrow)\quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta,}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\Rightarrow\neg\wedge)\quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$$

Power of Dynamic Semantics

- Let $\mathcal{M}$ be a 2-valued Nmatrix which has at least one proper non-deterministic operation. There is no finite family of finite ordinary matrices C, such that $\vdash^d_{\mathcal{M}} = \vdash_C$.
- If in addition $\mathcal{M}$ includes the classical implication, then there is even no finite family of finite ordinary matrices C such that $\vdash^d_{\mathcal{M}} \psi$ iff $\vdash_C \psi$.

Expressive Power of Static Semantics

- For every (finite) Nmatrix $\mathcal{M}$ there is a (finite) family C of matrices such that $\vdash^{s}_{\mathcal{M}} = \vdash_{C}$.
- Thus only the expressive power of the dynamic semantics based on Nmatrices is stronger than that of ordinary matrices.

Accordingly, in the rest of this course our main focus will be on the dynamic semantics, and we shall write simply $\vdash_{\mathcal{M}}$ instead of $\vdash^{d}_{\mathcal{M}}$.

Analycity, Decidability and Compactness

An obvious, yet crucial fact: any partial $\mathcal{M}$-legal valuation defined on a set closed under subformulas can be extended to a full $\mathcal{M}$-legal valuation.

If $\mathcal{M}$ is finite then this entails that $\vdash_{\mathcal{M}}$ is:

- Decidable
- Finitary (the compactness theorem obtains)

Refinements

Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}$.

- $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$ if:
  1. $\mathcal{V}_1 \subseteq \mathcal{V}_2$
  2. $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$
  3. $\widetilde{\diamond}_{\mathcal{M}_1}[x_1, \ldots, x_n] \subseteq \widetilde{\diamond}_{\mathcal{M}_2}[x_1, \ldots, x_n]$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $x_1, \ldots, x_n \in \mathcal{V}_1$.
- If $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

Examples of Refinements

- $\mathcal{M}_S^3$ is a refinement of $\mathcal{M}_L^3$.
- The classical two-valued matrix is a refinement of $\mathcal{M}_S^3$.
- The classical two-valued matrix is also a refinement of $\mathcal{M}^2$.

$\mathcal{M}_L^3$:

	$\tilde{\neg}$
t	{**f**}
I	$\mathcal{V}$
f	{**t**}

$\mathcal{M}_S^3$:

	$\tilde{\neg}$
t	{**f**}
I	$\mathcal{D}$
f	{**t**}

$\mathcal{M}^2$:

	$\tilde{\neg}$
t	{**t**, **f**}
f	{**t**}

Expansions

Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. Assume that F is a function that assigns to each $x \in \mathcal{V}$ a nonempty set $F[x]$ so that $F[x_1] \cap F[x_2] = \emptyset$ if $x_1 \neq x_2$. The F-expansion of $\mathcal{M}$ is the following Nmatrix $\mathcal{M}_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F \rangle$:

- $\mathcal{V}_F = \bigcup_{x \in \mathcal{V}} F[x]$
- $\mathcal{D}_F = \bigcup_{x \in \mathcal{D}} F[x]$
- $\widetilde{\diamond}_{\mathcal{M}_F}[y_1, \ldots, y_n] = \bigcup_{z \in \widetilde{\diamond}_{\mathcal{M}}[x_1, \ldots, x_n]} F[z]$ whenever $\diamond$ is an n-ary connectives of $\mathcal{L}$, and $x_i \in \mathcal{V}$, $y_i \in F[x_i]$ for every $1 \leq i \leq n$.

If $\mathcal{M}_1$ is an expansion of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} = \vdash_{\mathcal{M}_1}$.

Example

Let $F[\mathbf{t}] = \{\mathbf{t}, \mathbf{I}\}$ and $F[\mathbf{f}] = \{\mathbf{f}\}$. The $\{\vee, \wedge, \supset\}$-part of $\mathcal{M}_L^3$ (or $\mathcal{M}_S^3$) is obtained as the F-expansion of the positive part of the classical two-valued matrix.

$$\mathcal{V} = \{\mathbf{f}, \mathbf{I}, \mathbf{t}\}, \quad \mathcal{D} = \{\mathbf{I}, \mathbf{t}\}$$

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a = b = \mathbf{f} \end{cases} \qquad a \widetilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if either } a = \mathbf{f} \text{ or } b = \mathbf{f} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a = \mathbf{f} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a \in \mathcal{D} \text{ and } b = \mathbf{f} \end{cases}$$

Introduction to Nmatrices: Part III

Example: Non-deterministic Semantics for Logics of Formal (In)consistency

Problem: Inconsistent Information in Databases

There are two main approaches to this problem:

1. Belief revision: eliminating contradictions.
   Make an inconsistent theory consistent by revising it.

2. Paraconsistent logics: reasoning in the presence of contradictions.

Paraconsistent Logics

- In classical logic (and most other logics), the explosive non-contradiction principle

$$\varphi, \neg\varphi \vdash \psi$$

  allows us to derive any formula out of a contradiction. This makes any inconsistent theory trivial, and so no sensible reasoning can take place in the presence of contradictions.

- Paraconsistent logics do allow non-trivial inconsistent theories.

The Brazilian School of da Costa

The explosive character of contradictions is controlled using the following two ideas:

- There are two types of formulas: “normal” (“consistent”) and “abnormal” (“inconsistent”). Contradictions should be explosive only for “normal” formulas.
- The consistency of a formula φ is expressed by a formula $\circ\varphi$ of the language.

An Example: da Costa's System C_1

Obtained by:

- Taking $\circ\varphi = \neg(\varphi \wedge \neg\varphi)$
- Adding to some Hilbert-style system for positive classical logic (e.g. $\mathbf{HCL}^+$) the following axioms concerning negation:

  N2: $\neg\neg\varphi \supset \varphi$

  N3: $\neg\varphi \vee \varphi$

  $\mathbf{a}_\wedge$**:** $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$

  $\mathbf{a}_\vee$**:** $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi)$

  $\mathbf{a}_\supset$**:** $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \supset \psi)$

  and *either* of the following two axioms:

  N$\circ$1: $\circ\varphi \supset (\psi \supset \varphi) \supset (\psi \supset \neg\varphi) \supset \neg\psi$

  N$\circ$4: $(\circ\varphi \wedge \varphi \wedge \neg\varphi) \supset \psi$

$\mathbf{HCL}^{+}$

I1 $\varphi \supset (\psi \supset \varphi)$

I2 $(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$

I3 $((\psi \supset \varphi) \supset \psi) \supset \psi$

C1 $\varphi \wedge \psi \supset \varphi$

C2 $\varphi \wedge \psi \supset \psi$

C3 $\varphi \supset (\psi \supset \varphi \wedge \psi)$

D1 $\varphi \supset \varphi \vee \psi$

D2 $\psi \supset \varphi \vee \psi$

D3 $(\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$

MP $\varphi, \varphi \supset \psi \vdash \psi$

Logics of Formal (In)Consistency - LFIs

- Internalize the meta-theoretical notions of consistency and inconsistency by adding to the language a new unary connective $\circ$. The intended meaning of $\circ\varphi$ is "φ is consistent".
- Control the explosive character of contradictions:

$$\varphi, \neg\varphi, \circ\varphi \vdash \psi$$

but in general

$$\varphi, \neg\varphi \nvdash \psi$$

The Basic Logic of Formal (In)Consistency

Language: $\mathcal{L}_{\mathrm{C}} = \{\wedge, \vee, \supset, \neg, \circ\}$.
Logic: $\mathbf{B}$ is the minimal logic in $\mathcal{L}_{\mathrm{C}}$ which extends classical positive logic and satisfies the following two conditions:

(t) $\vdash \neg\varphi \vee \varphi$

(b) $\circ\varphi, \neg\varphi, \varphi \vdash \psi$

Corresponding Hilbert-type System: Add the following two axioms to $\mathbf{HCL}^{+}$:

(N3) $\neg\varphi \vee \varphi$

(N∘4) $(\circ\varphi \wedge \varphi \wedge \neg\varphi) \supset \psi$

Extensions of $\mathbf{B}$

(c) $\neg\neg\varphi \supset \varphi$

(e) $\varphi \supset \neg\neg\varphi$

(i$_1$) $\neg\circ\varphi \supset \varphi$ **(i$_2$)** $\neg\circ\varphi \supset \neg\varphi$

(k$_1$) $\circ\varphi \vee \varphi$ **(k$_2$)** $\circ\varphi \vee \neg\varphi$

(l) $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$

(a$_\sharp$) $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi\sharp\psi)$ for $\sharp \in \{\wedge, \vee, \supset\}$

(o$_\sharp$) $(\circ\varphi \vee \circ\psi) \supset \circ(\varphi\sharp\psi)$ for $\sharp \in \{\wedge, \vee, \supset\}$

An example: C_1 is Equivalent to **Bcila**

Reminder: Non-deterministic Matrices

A non-deterministic matrix $\mathcal{M}$ for $\mathcal{L}$ is a tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$:

- $\mathcal{V}$ - the set of truth-values,
- $\mathcal{D}$ - the set of designated truth-values ($\emptyset \neq D \subset V$),
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

5-valued Semantics for **B** - Intuition

- The idea: include all the relevant data concerning a formula ψ in the truth-value assigned to ψ:
  1. The truth/falsity of ψ
  2. The truth/falsity of $\neg\psi$
  3. The truth/falsity of $\circ\psi$
- This leads to the use of elements from $\{0,1\}^3$ as truth-values, where the intended meaning of $v[\psi] = \langle x, y, z\rangle$ is as follows:

  $x = 1$ iff $v[\psi] \in \mathcal{D}$

  $y = 1$ iff $v[\neg\psi] \in \mathcal{D}$

  $z = 1$ iff $v[\circ\psi] \in \mathcal{D}$

5-valued Semantics for **B** - Intuition

However, the axioms of **B** rule out some of the truth-values:

- $\varphi \vee \neg\varphi$ rules out $\langle 0,0,1\rangle$ and $\langle 0,0,0\rangle$.
- $(\circ\varphi \wedge \varphi \wedge \neg\varphi) \supset \psi$ rules out $\langle 1,1,1\rangle$.

We are left with the following five truth-values:

$$t = \langle 1,0,1\rangle,\ t_I = \langle 1,0,0\rangle,\ I = \langle 1,1,0\rangle,\ f = \langle 0,1,1\rangle,\ f_I = \langle 0,1,0\rangle$$

Note that since the first component of a truth-value assigned to a formula should indicate whether that formula is true, the designated truth-values should be those whose first component is 1: $\mathcal{D} = \{t, t_I, I\}$.

5-valued Semantics for **B**

$\mathcal{M}_5 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined by:

$\mathcal{V} = \{t, t_I, I, f_I, f\}$, $\mathcal{D} = \{t, I, t_I\}$, and for $\mathcal{F} = \mathcal{V} - \mathcal{D}$:

$$a \tilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \end{cases} \qquad a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } a \in \{I, f, f_I\} \\ \mathcal{F} & \text{if } a \in \{t, t_I\} \end{cases} \qquad \tilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \{t, f\} \\ \mathcal{F} & \text{if } a \in \{I, t_I, f_I\} \end{cases}$$

$$t = \langle 1, 0, 1 \rangle \quad t_I = \langle 1, 0, 0 \rangle \quad I = \langle 1, 1, 0 \rangle \quad f = \langle 0, 1, 1 \rangle \quad f_I = \langle 0, 1, 0 \rangle$$

Reminder: Refinements

Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}$.

- $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$ if:
  1. $\mathcal{V}_1 \subseteq \mathcal{V}_2$
  2. $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$
  3. $\widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n) \subseteq \widetilde{\diamond}_{\mathcal{M}_2}(x_1, \ldots, x_n)$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $x_1, \ldots, x_n, y \in \mathcal{V}_1$.
- If $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

Effects of the Axioms

$(\mathbf{c})\ \neg\neg\varphi \supset \varphi$

Cond(c): if $x \in \{f, f_I\}$ then $\tilde{\neg} x \subseteq \{t, t_I\}$

$\mathcal{M}_5$:

$\tilde{\neg}$	**f**	$\mathbf{f_I}$	**I**	**t**	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$

$\tilde{\circ}$	**f**	$\mathbf{f_I}$	**I**	**t**	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$

$\mathcal{M}_5\mathbf{c}$:

$\tilde{\neg}$	**f**	$\mathbf{f_I}$	**I**	**t**	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$

$t = \langle 1, 0, 1\rangle \quad t_I = \langle 1, 0, 0\rangle \quad I = \langle 1, 1, 0\rangle \quad f = \langle 0, 1, 1\rangle \quad f_I = \langle 0, 1, 0\rangle$

Effects of the Axioms

$$(\mathbf{c})\ \neg\neg\varphi \supset \varphi \quad (\mathbf{i_1})\ \neg\circ\varphi \supset \varphi$$

Cond(c): if $x \in \{f, f_I\}$ then $\tilde{\neg}x \subseteq \{t, t_I\}$
Cond(i_1): f_I should be deleted, and $\tilde{\circ}f \subseteq \{t, t_I\}$

$\mathcal{M}_5$:

$\tilde{\neg}$	**f**	$\mathbf{f_I}$	**I**	**t**	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$

$\tilde{\circ}$	**f**	$\mathbf{f_I}$	**I**	**t**	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$

$\mathcal{M}_5\mathbf{ci_1}$:

$\tilde{\neg}$	**f**	**I**	**t**	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{f\}$	$\{f\}$

$\tilde{\circ}$	**f**	**I**	**t**	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{f\}$	$\{I, t, t_I\}$	$\{f\}$

Effects of the Axioms

$(\mathbf{c})\ \neg\neg\varphi \supset \varphi \quad (\mathbf{i_1})\ \neg\circ\varphi \supset \varphi \quad (\mathbf{i_2})\ \neg\circ\varphi \supset \neg\varphi$

Cond(c): if $x \in \{f, f_I\}$ then $\tilde{\neg}x \subseteq \{t, t_I\}$
Cond(i$_1$): f_I should be deleted, and $\tilde{\circ}f \subseteq \{t, t_I\}$
Cond(i$_2$): t_I should be deleted, and $\tilde{\circ}t = \{t\}$

$\mathcal{M}_5$:

$\tilde{\neg}$	**f**	$\mathbf{f_I}$	**I**	**t**	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$

$\tilde{\circ}$	**f**	$\mathbf{f_I}$	**I**	**t**	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$

$\mathcal{M}_5\mathbf{ci_1i_2}$ ($\mathcal{M}_5\mathbf{ci}$):

$\tilde{\neg}$	**f**	**I**	**t**
	$\{t\}$	$\{I, t\}$	$\{f\}$

$\tilde{\circ}$	**f**	**I**	**t**
	$\{t\}$	$\{f\}$	$\{t\}$

Effects of the Axioms (except for Axiom (l))

Cond(c) : if $x \in \{f, f_I\}$ then $\tilde{\neg} x \subseteq \{t, t_I\}$

Cond(e): $\tilde{\neg} I = \{I\}$

Cond(i$_1$) : f_I should be deleted, and $\tilde{\circ} f \subseteq \{t, t_I\}$

Cond(i$_2$) : t_I should be deleted, and $\tilde{\circ} t = \{t\}$

Cond(k$_1$) : f_I should be deleted.

Cond(k$_1$) : t_I should be deleted.

Cond(a$_\sharp$) : if $a, b \in \{t, f\}$, then $a \tilde{\sharp} b \subseteq \{t, f\}$

Cond(o$_\sharp$) : if $a \in \{t, f\}$ or $b \in \{t, f\}$, then $a \tilde{\sharp} b \subseteq \{t, f\}$

For $\mathbf{X} \subseteq Ax$, $\mathcal{M}_5(\mathbf{X})$ is the weakest refinement of $\mathcal{M}_5$, in which the conditions of the schemata from $\mathbf{X}$ are satisfied.

Some Applications

- Axiom $\mathbf{k}_j$ follows in $\mathbf{B}$ from Axiom $\mathbf{i}_j$ ($j = 1, 2$).
- 1. $\vdash_{Bia} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$

  2. $\nvdash_{Bcie} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$
- $\mathbf{BX}$ is decidable for every $\mathbf{X} \subseteq Ax$.
- Let $\mathbf{L}$ be a logic in a language which includes $\{\neg, \wedge, \vee, \supset\}$. If $\mathbf{Bcioe}$ is an extension of $\mathbf{L}$ then two formulas in $\{\neg, \wedge, \vee, \supset\}$ are logically indistinguishable in $\mathbf{L}$ iff they are identical.

  (Two formulas A and B are called *logically indistinguishable* in $\mathbf{L}$ if $\varphi(A) \vdash_{\mathbf{L}} \varphi(B)$ and $\varphi(B) \vdash_{\mathbf{L}} \varphi(A)$ for every formula $\varphi(p)$ in the language of $\mathbf{L}$.)

No Improvements Possible

Theorem. Let $\mathcal{L}$ be either $\{\neg, \wedge, \vee, \supset\}$ or $\mathcal{L}_C$, and let $\mathbf{L}$ be a logic in $\mathcal{L}$. Assume that the set of valid formulas of $\mathbf{L}$ includes that of positive classical logic, and is included in that of **Bcioe**. Then $\mathbf{L}$ does not have a finite weakly characteristic matrix.

Theorem. Let $\mathbf{L}$ be as above. Then $\mathbf{L}$ does not have a weakly characteristic two-valued Nmatrix.

Axiom (l): $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$

- The validity of $(\mathbf{l})$ means that whenever $\circ\varphi$ is “false”, so is $\neg(\varphi \wedge \neg\varphi)$. *Thus conjunction of an “inconsistent” formula with its negation should be distinguishable from other types of conjunctions.*
- Enforcing a unique connection between the truth-value of an “inconsistent” formula and its negation requires a supply of infinitely many truth-values corresponding to “inconsistent” formulas. *What truth-values should we use?*

Reminder: Expansions

Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. Assume that F is a function that assigns to each $x \in \mathcal{V}$ a nonempty set $F[x]$ so that $F[x_1] \cap F[x_2] = \emptyset$ if $x_1 \neq x_2$. The F-expansion of $\mathcal{M}$ is the following Nmatrix $\mathcal{M}_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F \rangle$:

- $\mathcal{V}_F = \bigcup_{x \in \mathcal{V}} F[x]$
- $\mathcal{D}_F = \bigcup_{x \in \mathcal{D}} F[x]$
- $\widetilde{\diamond}_{\mathcal{M}_F}[y_1, \ldots, y_n] = \bigcup_{z \in \widetilde{\diamond}_{\mathcal{M}}[x_1, \ldots, x_n]} F[z]$ whenever $\diamond$ is an n-ary connectives of $\mathcal{L}$, and $x_i \in \mathcal{V}$, $y_i \in F[x_i]$ for every $1 \leq i \leq n$.

If $\mathcal{M}_1$ is an expansion of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} = \vdash_{\mathcal{M}_1}$.

Axiom (l): $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$

Observation: the axioms $(\mathbf{k_1})$ and $(\mathbf{k_2})$ are theorems of $\mathbf{Bl}$.

Accordingly, our Nmatrix for $\mathbf{Bl}$ *will be a refinement of an expansion of the Nmatrix for* $\mathcal{M}_5(\{(k_1),(k_2)\})$. *Recall that in this Nmatrix* $\mathcal{V}=\{t,I,f\}$, $\mathcal{D}=\{t,I\}$, *and for* $\mathcal{F}=\mathcal{V}-\mathcal{D}$:

$$a\widetilde{\wedge}b = \begin{cases} \mathcal{D} & \text{if } a\in\mathcal{D} \text{ and } b\in\mathcal{D} \\ \mathcal{F} & \text{if } a\in\mathcal{F} \text{ or } b\in\mathcal{F} \end{cases} \qquad a\widetilde{\supset}b = \begin{cases} \mathcal{D} & \text{if } a\in\mathcal{F} \text{ or } b\in\mathcal{D} \\ \mathcal{F} & \text{if } a\in\mathcal{D} \text{ and } b\in\mathcal{F} \end{cases}$$

$$\widetilde{\neg}a = \begin{cases} \mathcal{D} & \text{if } a\in\{I,f,f_I\} \\ \mathcal{F} & \text{if } a\in\{t,t_I\} \end{cases} \qquad \widetilde{\circ}a = \begin{cases} \mathcal{D} & \text{if } a\in\{t,f\} \\ \mathcal{F} & \text{if } a\in\{I,t_I,f_I\} \end{cases}$$

The Nmatrix $\mathcal{M}_{\mathbf{Bl}} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$

The idea: making infinitely many copies of the truth-values t, I

$$\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\} \quad \mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}$$

$\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$, $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{F} = \{f\}$, $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{if } a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases} \qquad \tilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases} \qquad a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

Effects of the Axioms (in the Presence of (l))

Cond(c) : $\tilde{\neg} f \subseteq \mathcal{T}$

Cond(e): $\tilde{\neg}(I_i^j) = \{I_i^{j+1}\}$

Cond(i$_1$) : $\tilde{\circ} f \subseteq \mathcal{T}$

Cond(i$_2$) : If $a \in \mathcal{T}$ then $\tilde{\circ} a \subseteq \mathcal{T}$

Cond(a$_\sharp$) : if $a, b \in \mathcal{T} \cup \mathcal{F}$, then $a \tilde{\sharp} b \subseteq \mathcal{T} \cup \mathcal{F}$

Cond(o$_\sharp$) : if $a \in \mathcal{T} \cup \mathcal{F}$ or $b \in \mathcal{T} \cup \mathcal{F}$, then $a \tilde{\sharp} b \subseteq \mathcal{T} \cup \mathcal{F}$

For X $\subseteq Ax$, $\mathcal{M}_{\mathrm{BlX}}$ is the weakest refinement of $\mathcal{M}_{\mathrm{Bl}}$, in which the conditions of the schemata from X are satisfied.

$\mathbf{BlX}$ is Decidable

To check whether a given formula φ is provable in $\mathbf{BlX}$ (where $\mathbf{X} \subseteq Ax$), it suffices to check all legal partial valuations v in $\mathcal{M}_{\mathbf{BlX}}$ which assign to subformulas of φ values in

$$\{f\} \cup \{t_i^j \mid 0 \leq i \leq n(\varphi), 0 \leq j \leq k(\varphi)\} \cup \{I_i^j \mid 0 \leq i \leq n(\varphi), 0 \leq j \leq k(\varphi)\}$$

where $n(\varphi)$ is the number of subformulas of φ which do not begin with $\neg$, and $k(\varphi)$ is the maximal number of consecutive negation symbols occurring within φ. This is a finite process.

Analycity

An obvious, yet crucial fact: if $\mathcal{M}$ is an Nmatrix, then any partial $\mathcal{M}$-legal assignment which is defined on a set closed under subformulas can be extended to a full $\mathcal{M}$-legal assignment.

If $\mathcal{M}$ is finite, this entails that $\vdash_{\mathcal{M}}$ is:

- Decidable
- Finitary (the compactness theorem obtains)

Example: Semantics for $\mathbf{C_1}$

da Costa's system $\mathbf{C_1}$ is decidable, and its semantics is as follows:

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \text{ and } b \notin \mathcal{I} \\ \mathcal{T} & \text{if } b \in \mathcal{T} \text{ and } a \notin \mathcal{I} \\ \mathcal{D} & \text{otherwise} \end{cases} \qquad a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{T} \text{ and } b \in \mathcal{T} \\ \mathcal{T} & \text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

No Improvements Possible

Theorem. No logic between **Bl** and **Blcieo** can have a finite characteristic Nmatrix.

Corollary. da Costa's system $\mathbf{C_1}$ has no finite characteristic Nmatrix.

Introduction to Nmatrices: Part IV

Proof Systems

Reminder: What is a Propositional Logic?

A propositional logic is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a formal propositional language, and $\vdash$ is a structural, consistent and finitary consequence relation for $\mathcal{L}$.

A consequence relation (cr) for $\mathcal{L}$ is a binary relation $\vdash \subseteq 2^{F_\mathcal{L}} \times F_\mathcal{L}$, having the following properties:

strong reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.

monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.

transitivity (cut): if $\Gamma \vdash \psi$ and $\Gamma, \psi \vdash \varphi$ then $\Gamma \vdash \varphi$.

Reminder: What is a Propositional Logic?

A cr $\vdash$ for $\mathcal{L}$ is:

- structural if for every uniform $\mathcal{L}$-substitution σ and every Γ and ψ: if $\Gamma \vdash \psi$ then $\sigma[\Gamma] \vdash \sigma[\psi]$.
- consistent if there exist formulas φ and ψ such that $\varphi \nvdash \psi$.
- finitary if whenever $\Gamma \vdash \psi$, there exists some finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \psi$.

Reminder: Gentzen-type Systems

- *A Gentzen-type system G* is an axiomatic system which manipulates *sequents* of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of formulas.
- Intuitive meaning of $\Gamma \Rightarrow \Delta$: either one of the formulas in Γ is false, or one of the formulas in Δ is true.
- The associated cr is:
  $\Gamma \vdash_{\mathbf{G}} \psi$ iff $\Gamma' \Rightarrow \psi$ is a theorem of $\mathbf{G}$ for some finite $\Gamma' \subseteq \Gamma$.
- A Gentzen-type system $\mathbf{G}$ is called *standard* if:
  1. Its set of axioms includes the standard axioms:

$$\psi \Rightarrow \psi$$

  2. It has *weakening* and *cut* as its structural rules.

The System GCPL

$$\psi \Rightarrow \psi$$

$$(Weak)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \qquad (Cut)\quad \frac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}$$

$$(\neg \Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg)\quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\supset\Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\supset)\quad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$(\wedge \Rightarrow)\quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow)\quad \frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad (\Rightarrow \vee)\quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

What is a Canonical Rule?

- An "ideal" logical rule: an introduction rule for *exactly one connective*, on *exactly one side of a sequent*.
- In its formulation: *exactly one occurrence* of the introduced connective, no other occurrences of other connectives.
- The rule should also be *pure* (i.e. context-independent): no side conditions limiting its application.
- Its active formulas: *immediate subformulas* of its principal formula.

What is a Canonical Rule?

Stage 1.

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

Stage 2.

$$\frac{\psi, \varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi}$$

Stage 3.

$$\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \ ; \ \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

Canonical Systems

- *A sequent*: an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are *finite* sets of $\mathcal{L}$-formulas.
- *A clause*: a sequent consisting of atomic formulas.
- A *canonical rule* has one of the forms:

$$\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} / \diamond (p_1, ..., p_n) \Rightarrow$$

$$\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} / \Rightarrow \diamond(p_1, ..., p_n)$$

where $m \geq 0$ *and for all* $1 \leq i \leq m$*:* $\Pi_i \Rightarrow \Sigma_i$ *is a clause over* $\{p_1, ..., p_n\}$.

Canonical Rules

Application of a canonical rule of the form
$\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \diamond (p_1, ..., p_n) \Rightarrow$:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \diamond(\psi_1, ..., \psi_n) \Rightarrow \Delta}$$

where Π_i^* and Σ_i^* are obtained from Π_i and Σ_i respectively by substituting ψ_j for p_j for all $1 \leq j \leq n$, and Γ, Δ are any finite sets of formulas (the context).

Example 1

Conjunction rules:

$$\{p_1, p_2 \Rightarrow\} \;/\; p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \;;\; \Rightarrow p_2\} \;/\; \Rightarrow p_1 \wedge p_2$$

Their applications:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

Example 2

Implication rules:

$$\{p_1 \Rightarrow p_2\} \ / \ \Rightarrow p_1 \supset p_2 \qquad \{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \supset p_2 \Rightarrow$$

Their applications:

$$\frac{\Gamma, \psi \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$$

Example 3

Semi-implication rules:

$$\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \leadsto p_2 \Rightarrow \qquad \{\Rightarrow p_2\} \ / \ \Rightarrow p_1 \leadsto p_2$$

Their applications:

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \leadsto \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \leadsto \varphi}$$

Example 4

"Tonk" rules:

$$\{p_2 \Rightarrow\} \;/\; p_1 T p_2 \Rightarrow \qquad \{\Rightarrow p_1\} \;/\; \Rightarrow p_1 T p_2$$

Their applications:

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi T \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi T \psi}$$

What Sets of Rules are Acceptable?

- A standard Gentzen-type system is canonical if each of its logical (i.e. non-structural) rules is canonical.
- If G is a canonical system, then $\vdash_G$ is a structural and finitary cr. But is it a logic? i.e., is it also consistent?

Coherence

- A canonical calculus G is *coherent* if for every pair of rules $\Theta_1 / \Rightarrow \diamond(p_1, ..., p_n)$ and $\Theta_2 / \diamond(p_1, ..., p_n) \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically unsatisfiable (and so inconsistent, i.e., the empty sequent can be derived from it using only cuts)
- For a canonical calculus G, $\vdash_G$ is a logic iff G is coherent.

Coherent Calculi:

$$\{p_1, p_2 \Rightarrow\} \;/\; p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \;;\; \Rightarrow p_2\} \;/\; \Rightarrow p_1 \wedge p_2$$

$$\{p_1 \Rightarrow p_2\} \;/\; \Rightarrow p_1 \supset p_2 \qquad \{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; p_1 \supset p_2 \Rightarrow$$

$$\{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; p_1 \leadsto p_2 \Rightarrow \qquad \{\Rightarrow p_2\} \;/\; \Rightarrow p_1 \leadsto p_2$$

$$\{p_1 \Rightarrow\} \;/\; \Rightarrow \neg p_1 \qquad \{\Rightarrow p_1\} \;/\; \neg p_1 \Rightarrow$$

Non-coherent: “Tonk”!

$$\{p_2 \Rightarrow\} \;/\; p_1 T p_2 \Rightarrow \qquad \{\Rightarrow p_1\} \;/\; \Rightarrow p_1 T p_2$$

From these rules, we can derive $p \Rightarrow q$ for any p, q:

$$\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow pTq} \qquad \dfrac{q \Rightarrow q}{pTq \Rightarrow q}}{p \Rightarrow q}$$

Every Coherent Calculus Has a Characteristic 2Nmatrix

Consider the canonical calculus G_0 over the language $\{\wedge, \leadsto\}$ with no canonical rules whatsoever:

ψ	φ	$\psi\wedge\varphi$
t	t	$\{$**t,f**$\}$
t	f	$\{$**t,f**$\}$
f	t	$\{$**t,f**$\}$
f	f	$\{$**t,f**$\}$

ψ	φ	$\psi\leadsto\varphi$
t	t	$\{$**t,f**$\}$
t	f	$\{$**t,f**$\}$
f	t	$\{$**t,f**$\}$
f	f	$\{$**t,f**$\}$

Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule $\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$, which can be split in $\{p_1 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$ and $\{p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$

ψ	φ	$\psi \wedge \varphi$
t	t	$\{\mathbf{t},\mathbf{f}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{f}\}$
f	f	$\{\mathbf{f}\}$

ψ	φ	$\psi \rightsquigarrow \varphi$
t	t	$\{\mathbf{t},\mathbf{f}\}$
t	f	$\{\mathbf{t},\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t},\mathbf{f}\}$

Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule $\{\Rightarrow p_1; \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$

ψ	φ	$\psi \wedge \varphi$
t	t	$\{\mathbf{t}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{f}\}$
f	f	$\{\mathbf{f}\}$

ψ	φ	$\psi \rightsquigarrow \varphi$
t	t	$\{\mathbf{t},\mathbf{f}\}$
t	f	$\{\mathbf{t},\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t},\mathbf{f}\}$

Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule $\{\Rightarrow p_2\} / \Rightarrow p_1 \leadsto p_2$

ψ	φ	$\psi \wedge \varphi$
t	t	{**t**}
t	f	{**f**}
f	t	{**f**}
f	f	{**f**}

ψ	φ	$\psi \leadsto \varphi$
t	t	{**t**}
t	f	{**t,f**}
f	t	{**t**}
f	f	{**t,f**}

Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule $\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\}/p_1 \rightsquigarrow p_2 \Rightarrow$

ψ	φ	$\psi \wedge \varphi$
t	t	$\{\mathbf{t}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{f}\}$
f	f	$\{\mathbf{f}\}$

ψ	φ	$\psi \rightsquigarrow \varphi$
t	t	$\{\mathbf{t}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{t}\}$
f	f	$\{\mathbf{t,f}\}$

Every 2Nmatrix Has a Corresponding Coherent Calculus

p_1	p_2	$p_1 \circ p_2$
t	t	$\{\mathbf{f}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t}\}$

$$\{\Rightarrow p_1 \; ; \; \Rightarrow p_2\} \; / \; p_1 \circ p_2 \Rightarrow$$

$$\{\Rightarrow p_1 \; ; \; p_2 \Rightarrow\} \; / \; p_1 \circ p_2 \Rightarrow$$

$$\{p_1 \Rightarrow \; ; \; p_2 \Rightarrow\} \; / \; \Rightarrow p_1 \circ p_2$$

Exact Correspondence

Definition: A Gentzen-type system G admits *strong cut-elimination* if whenever a sequent is provable in G from a set of sequents Θ, it also has a proof in which all cut formulas occur in Θ. G admits *cut-elimination* if every sequent provable in G has a cut-free proof.

Theorem: If G is a canonical calculus, then the following statements are equivalent:

1. $\vdash_G$ is consistent (and so it is a logic).
2. G is coherent.
3. G has a characteristic 2Nmatrix.
4. G admits strong cut-elimination.
5. G admits cut-elimination.

Signed Formulas: the Two-valued Case

An alternative formulation of Gentzen-type calculi:

- instead of $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k$, write:

$$\{\mathbf{f}:\varphi_1, \ldots, \mathbf{f}:\varphi_n, \mathbf{t}:\psi_1, \ldots, \mathbf{t}:\psi_k\}$$

  $\mathbf{f}:\varphi$ and $\mathbf{t}:\psi$ are called signed formulas.

- Examples how the rules are reformulated:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

  are rewritten as

$$\frac{\{\mathbf{f}:\psi, \mathbf{f}:\varphi\} \cup \Omega}{\{\mathbf{f}:\psi \wedge \varphi\} \cup \Omega} \qquad \frac{\{\mathbf{t}:\psi\} \cup \Omega \quad \{\mathbf{t}:\varphi\} \cup \Omega}{\{\mathbf{t}:\psi \wedge \varphi\} \cup \Omega}$$

  where $\Omega = \{\mathbf{f}:\varphi \mid \varphi \in \Gamma\} \cup \{\mathbf{t}:\psi \mid \psi \in \Delta\}$

Signed Formulas: the General Case

Let $\mathcal{V}$ be a finite set of signs.

- *A signed formula*: an expression of the form $a : \psi$, where ψ is a formula and $a \in \mathcal{V}$.
- *A sequent*: a finite set of signed formulas.
- *A clause*: a sequent consisting of atomic signed formulas.
- A valuation v *satisfies* a signed formula $a : \psi$ if $v[\psi] = a$.
- v *satisfies* a set of signed formulas Ω if it satisfies some signed formula in Ω. *Sequents are interpreted disjunctively.*
- v *satisfies* a set of sequents Θ if it satisfies all sequents in Θ.

Signed Calculi

Let $\mathcal{L}$ be a propositional language and $\mathcal{V}$ a finite set of signs.
Denote: $\mathcal{L}_{\mathcal{V}} = \{a : \psi \mid a \in \mathcal{V}, \psi \in \mathcal{F}_{\mathcal{L}}\}$.

- A *standard axiom* for $\mathcal{L}_{\mathcal{V}}$ is a sequent of the form $\{a : \psi \mid a \in \mathcal{V}\}$.
- The *cut and weakening* rules for $\mathcal{L}_{\mathcal{V}}$:

$$\frac{\Omega \cup \{a : \psi \mid a \in A_1\} \quad \Omega \cup \{a : \psi \mid a \in A_2\}}{\Omega \cup \{a : \psi \mid a \in A_1 \cap A_2\}}$$

$$\frac{\Omega}{\Omega, a : \psi}$$

  where $A_1, A_2 \subseteq \mathcal{V}$ and $a \in \mathcal{V}$.

- Standard signed calculi are defined exactly like standard Gentzen-type calculi.

Semantics for Signed Calculi

- We use Nmatrices of the form $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for calculi over a set of signs $\mathcal{V}$.

- Given a family of sets of signed formulas Θ and a set of signed formulas Ω, $\Theta \vdash_{\mathcal{M}} \Omega$ if each $\mathcal{M}$-legal valuation satisfying (all elements of) Θ satisfies also Ω (i.e. it satisfies some signed formula of Ω).

- The connection with the cr between formulas induced by $\mathcal{M}$:
  $\Gamma \vdash_{\mathcal{M}} \psi$ iff $\vdash_{\mathcal{M}} \{d : \psi \mid d \in \mathcal{D}\} \cup \{n : \varphi \mid n \in \mathcal{V} - \mathcal{D}, \varphi \in \Gamma\}$

Signed Calculus Generated by an Nmatrix

Given an n-valued Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, let $SC_{\mathcal{M}}$ be the following standard signed calculus:

Axioms: The standard axioms $\{a : \psi \mid a \in \mathcal{V}\}$.

Structural inference rules: Weakening, Cut.

Logical inference rules: For every m-ary connective $\diamond \in \mathcal{O}$ and any logical values $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_k \in \mathcal{V}$ such that $\widetilde{\diamond}(a_1, \ldots, a_m) = \{b_1, \ldots, b_k\}$, $SC_{\mathcal{M}}$ contains the rule:

$$\frac{\Omega, a_1 : \psi_1 \quad \ldots \quad \Omega, a_m : \psi_m}{\Omega, b_1 : \diamond(\psi_1, \ldots, \psi_m), \ldots, b_k : \diamond(\psi_1, \ldots, \psi_m)}$$

- $SC_{\mathcal{M}}$ is sound and complete for $\vdash_{\mathcal{M}}$.
- If $\vdash_{\mathcal{M}} \Omega$ then Ω has a cut-free proof in $SC_{\mathcal{M}}$.

Example: McCarthy-Kleene Nmatrix

$\mathcal{M}_{MK} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, with $\mathcal{V} = \{\mathbf{f}, \mathbf{I}, \mathbf{t}\}, \mathcal{D} = \{\mathbf{t}\}, \mathcal{O} = \{\widetilde{\neg}, \widetilde{\vee}\}$, where:

$\widetilde{\vee}$	$\mathbf{f}$	$\mathbf{I}$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{I}\}$	$\{\mathbf{t}\}$
$\mathbf{I}$	$\{\mathbf{I}\}$	$\{\mathbf{I}\}$	$\{\mathbf{t}, \mathbf{I}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

$\widetilde{\neg}$	
$\mathbf{f}$	$\mathbf{t}$
$\mathbf{I}$	$\mathbf{I}$
$\mathbf{t}$	$\mathbf{f}$

Example 1: Signed Calculus for $\mathcal{M}_{MK}$

$$\frac{\Omega, \mathbf{f} : \varphi}{\Omega, \mathbf{t} : \neg\varphi} \qquad \frac{\Omega, \mathbf{I} : \varphi}{\Omega, \mathbf{I} : \neg\varphi} \qquad \frac{\Omega, \mathbf{t} : \varphi}{\Omega, \mathbf{f} : \neg\varphi}$$

$$\frac{\Omega, \mathbf{f} : \varphi \quad \Omega, a : \psi}{\Omega, a : \varphi \vee \psi}(a = \mathbf{f}, \mathbf{I}, \mathbf{t}) \qquad \frac{\Omega, \mathbf{t} : \varphi}{\Omega, \mathbf{t} : \varphi \vee \psi}$$

$$\frac{\Omega, \mathbf{I} : \varphi \quad \Omega, \mathbf{I} : \psi, \mathbf{f} : \psi}{\Omega, \mathbf{I} : \varphi \vee \psi} \qquad \frac{\Omega, \mathbf{I} : \varphi \quad \Omega, \mathbf{t} : \psi}{\Omega, \mathbf{I} : \varphi \vee \psi, \mathbf{t} : \varphi \vee \psi}$$

Canonical Signed Rules

- $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}$:

$$\{p_1 \Rightarrow p_2\} \;/\; \Rightarrow p_1 \supset p_2$$

$$\{\Rightarrow p_1 \;,\; p_2 \Rightarrow\} \;/\; p_1 \supset p_2 \Rightarrow$$

$$\{\ \{\mathbf{f} : p_1, \mathbf{t} : p_2\}\ \} \;/\; \{\mathbf{t}\} : p_1 \supset p_2$$

$$\{\{\mathbf{t} : p_1\} \;,\; \{\ \mathbf{f} : p_2\}\} \;/\; \{\mathbf{f}\} : p_1 \supset p_2$$

- $\mathcal{V} = \{a, b, c\}$:

$$\{\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}\} \;/\; \{a, c\} : \circ(p_1, p_2, p_3)$$

$$\{\ \{c : p_2\} \;,\; \{a : p_3, b : p_3\} \;,\; \{c : p_1\}\} \;/\; \{b, c\} : \circ(p_1, p_2, p_3)$$

Canonical Signed Rules

- A *signed canonical rule* for an n-ary connective $\diamond$:

$$\{\Sigma_1, ..., \Sigma_m\} \ / \ S : \diamond(p_1, \ldots, p_n)$$

where $S \subset \mathcal{V}$, $m \geq 0$ and for every $1 \leq j \leq m$: Σ_j is a clause consisting of atomic signed formulas of the form $a : p_k$, where $a \in \mathcal{V}$ and $1 \leq k \leq n$.

- An *application* of a rule $\{\Sigma_1, ..., \Sigma_m\} \ / \ S : \diamond(p_1, \ldots, p_n)$:

$$\frac{\Omega \cup \Sigma_1^* \quad \ldots \quad \Omega \cup \Sigma_m^*}{\Omega \cup S : \diamond(\psi_1, ..., \psi_n)}$$

where $\psi_1, ..., \psi_n$ are $\mathcal{L}$-formulas, Ω is a sequent, for all $1 \leq i \leq m$: Σ_i^* is obtained from Σ_i by replacing p_j by ψ_j for every $1 \leq j \leq n$, and $S : \diamond(\psi_1, ..., \psi_n) = \{a : \diamond(\psi_1, ..., \psi_n) \mid a \in S\}$.

Example 1

Standard conjunction rules:

$$\{\{\mathbf{f}:p_1,\mathbf{f}:p_2\}\}\ /\ \{\mathbf{f}\}:p_1\wedge p_2 \qquad \{\{\mathbf{t}:p_1\},\{\mathbf{t}:p_2\}\}\ /\ \{\mathbf{t}\}:p_1\wedge p_2$$

Their applications:

$$\frac{\Omega\cup\{\mathbf{f}:\psi_1,\mathbf{f}:\psi_2\}}{\Omega\cup\{\mathbf{f}:\psi_1\wedge\psi_2\}} \qquad \frac{\Omega\cup\{\mathbf{t}:\psi_1\} \quad \Omega\cup\{\mathbf{t}:\psi_2\}}{\Omega\cup\{\mathbf{t}:\psi_1\wedge\psi_2\}}$$

Example 2

Two canonical rules for disjunction in $\mathcal{M}_{MK}$:

$$\{\{\mathbf{t} : p_1\}\}/\mathbf{t} : p_1 \vee p_2 \qquad \{\{\mathbf{I} : p_1\}, \{\ \mathbf{t} : p_2\}\}/\{\mathbf{I}, \mathbf{t}\} : p_1 \vee p_2$$

Their applications:

$$\frac{\Omega, \mathbf{t} : \varphi}{\Omega, \mathbf{t} : \varphi \vee \psi} \qquad \frac{\Omega, \mathbf{I} : \varphi \quad \Omega, \mathbf{t} : \psi}{\Omega, \mathbf{I} : \varphi \vee \psi, \mathbf{t} : \varphi \vee \psi}$$

Coherence

- Example for $\mathcal{V} = \{a, b, c\}$:

$$\{\{a : p_1\}\ ,\ \{c : p_2\}\}\ /\ \{a, b\} : p_1 \diamond p_2$$

$$\{\{a : p_1\}\ ,\ \{c : p_2\}\}\ /\ \{b, c\} : p_1 \diamond p_2$$

$$\{\{a : p_1\}\ ,\ \{c : p_2\}\}\ /\ \{a, c\} : p_1 \diamond p_2$$

It is not enough to check only pairs of rules.

- A canonical signed calculus G is *coherent* if $\Theta_1 \cup ... \cup \Theta_m$ is unsatisfiable (and so inconsistent, i.e., the empty sequent can be derived from it using only cuts) whenever $\Theta_1/S_1 : \diamond(p_1, \ldots, p_n), ..., \Theta_m/S_m : \diamond(p_1, \ldots, p_n)$ is a set of rules of G such that $S_1 \cap ... \cap S_m = \emptyset$.

Examples

Non-coherent:

$$\{\{a : p_1\}, \{b : p_2\}\} \ / \ \{a, b\} : \circ(p_1, p_2, p_3)$$

$$\{\{a : p_2, c : p_3\}\} \ / \ \{c\} : \circ(p_1, p_2, p_3)$$

The set $\{\{a : p_1\}, \{b : p_2\}, \{a : p_2, c : p_3\}\}$ is satisfiable.

Coherent:

1. $\{\{\mathbf{f} : p_1, \mathbf{f} : p_2\}\} \ / \ \{\mathbf{f}\} : p_1 \wedge p_2 \quad \{\{\mathbf{t} : p_1\}, \{\mathbf{t} : p_2\}\} \ / \ \{\mathbf{t}\} : p_1 \wedge p_2$

$$\dfrac{\dfrac{\{\mathbf{t} : p_1\} \quad \{\mathbf{f} : p_1, \mathbf{f} : p_2\}}{\{\mathbf{f} : p_2\}}\ \mathbf{cut} \qquad \{\mathbf{t} : p_2\}}{\emptyset}\ \mathbf{cut}$$

2. $\{\{\mathbf{f} : p_1\}, \{\mathbf{f} : p_2\}\} \ / \ \{\mathbf{f}\} : p_1 \vee p_2$
   $\{\{\mathbf{I} : p_1\}, \{\mathbf{t} : p_2\}\} \ / \ \{\mathbf{I}, \mathbf{t}\} : p_1 \vee p_2$

Signed Calculus for an Nmatrix Revisited

The sound and complete signed calculus $SC_{\mathcal{M}}$ for an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ defined above is canonical. It can be described as consisting of:

- The standard axioms and structural rules
- A rule of the form

$$\{\{a_1 : p_1\}, \dots, \{a_m : p_m\}\}/\{b_1, \dots, b_k\} : \diamond(p_1, \dots, p_m)$$

  for every m-ary connective $\diamond \in \mathcal{O}$ and any $a_1, a_2, \dots, a_m$, $b_1, b_2, \dots, b_k \in \mathcal{V}$ such that $\widetilde{\diamond}(a_1, \dots, a_m)$ $=\{b_1, \dots, b_k\}$.

Since $SC_{\mathcal{M}}$ is also coherent, this shows that every finite Nmatrix has a corresponding coherent canonical calculus.

“Reading off” the Semantics from Canonical Rules

Let $\mathcal{V} = \{t, f, \top, \bot\}$. Let G be a calculus with no rules for $\circ$.

$\circ$	t	f	$\top$	$\bot$
t	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$
f	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$
$\top$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$
$\bot$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$

“Reading off” the Semantics from Canonical Rules

Add the rule

$$\{\{f : p_1, f : p_2\}\} \ / \ \{\bot, f\} : p_1 \circ p_2$$

$\circ$	t	f	$\top$	$\bot$
t	$\mathcal{V}$	$\{\bot, \mathbf{f}\}$	$\mathcal{V}$	$\mathcal{V}$
f	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$
$\top$	$\mathcal{V}$	$\{\bot, \mathbf{f}\}$	$\mathcal{V}$	$\mathcal{V}$
$\bot$	$\mathcal{V}$	$\{\bot, \mathbf{f}\}$	$\mathcal{V}$	$\mathcal{V}$

“Reading off” the Semantics from Canonical Rules

Add the rule

$$\{\{t : p_1, \top : p_1\}\} \,/\, \{f\} : p_1 \circ p_2$$

$\circ$	t	f	$\top$	$\bot$
t	$\{\mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
f	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$
$\top$	$\{\mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\bot$	$\mathcal{V}$	$\{\bot, \mathbf{f}\}$	$\mathcal{V}$	$\mathcal{V}$

Another Exact Correspondence

A Gentzen system G admits:

- *analytic cut-elimination* if whenever $\vdash_G \Omega$, Ω has a proof in G where each cut formula is a subformula of some formula in Ω,
- *strong analytic cut-elimination* if whenever $\Theta \vdash_G \Omega$, Ω has a proof from Θ in G where each cut formula is a subformula of some formula in $\Theta \cup \{\Omega\}$.

For any canonical calculus G, the following are equivalent:

1. G is coherent.
2. G has a strongly characteristic Nmatrix.
3. G admits strong analytic cut-elimination.
4. G admits analytic cut-elimination.

Introduction to Nmatrices: Part V

Nmatrices for Languages with Quantifiers

Reminder: First-order Languages

A first-order language L includes:

- *A set of variables* $x_1, x_2, ...,$
- *Parentheses, logical connectives (e.g.* $\wedge, \vee, \supset, \neg$*) and quantifiers (e.g.,* $\forall$ *and* $\exists$*)*
- *The signature of L:*
  - *a (non-empty) set of predicate symbols*
  - *a set of constants*
  - *a set of function symbols*

Unary Quantifiers in Deterministic Matrices

A unary quantifier $\mathcal{Q}$ is usually interpreted by $\tilde{\mathcal{Q}} : P^+(\mathcal{V}) \to \mathcal{V}$.

Examples:

- Truth-values: $\{f, t\}$ (or $\{0, 1\}$)

H	$\tilde{\forall}$[H]
{t}	t
{t,f}	f
{f}	f

H	$\tilde{\exists}$[H]
{t}	t
{t,f}	t
{f}	f

- Truth-values: $\{0, \frac{1}{n-1}, \frac{2}{n-1}, ..., 1\}$ $(n > 1)$

$$\tilde{\forall} = \text{min} \qquad \tilde{\exists} = \text{max}$$

Matrices with Unary Quantifiers

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a (deterministic) matrix for a language L with unary quantifiers if:

1. $\mathcal{V}$ is a nonempty set of truth-values,
2. $\emptyset \neq \mathcal{D} \subset \mathcal{V}$ is a set of designated truth-values,
3. for every n-ary connective $\diamond$ of L, $\mathcal{O}$ includes an operation $\widetilde{\diamond} : \mathcal{V}^n \rightarrow \mathcal{V}$,
4. for every unary quantifier $\mathcal{Q}$ of L, $\mathcal{O}$ includes an operation $\widetilde{\mathcal{Q}} : P^+(\mathcal{V}) \rightarrow \mathcal{V}$.

L-structures

An L-structure for a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a pair $S = \langle D, I \rangle$, where D is a non-empty domain and I satisfies the following conditions:

- For every constant c of L: $I[c] \in D$.
- For an n-ary predicate symbol p of L: $I[p] : D^n \to \mathcal{V}$.
- For every n-ary function symbol f of L: $I[f] : D^n \to D$.

I is extended to interpret closed terms of L as follows:

$$I[f(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[f][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$$

Examples

Let L have the signature: $\mathbf{0} : \iota \quad \oplus : \iota^2 \to \iota \quad \doteq : \iota^2 \to o$.

1. $\mathcal{M}_2$ is the classical 2-valued matrix. Let $S_1 = \langle D_1, I_1 \rangle$, where $D_1 = \mathbb{N}$,

$$I_1[\mathbf{0}] = 0 \quad I_1[\oplus] = +$$

$$I_1[\doteq][n_1, n_2] = t \Leftrightarrow n_1 = n_2$$

2. K is the Kleene 3-valued matrix (where $\mathcal{V} = \{t, I, f\}$). Let $S_2 = \langle D_2, I_2 \rangle$, where $D_2 = \mathbb{N} \cup \{\mathbf{u}\}$, $I_2[\mathbf{0}] = 0$, and

$$I_2[\oplus][n_1, n_2] = \begin{cases} n_1 + n_2 & if\ n_1, n_2 \in \mathbb{N} \\ \mathbf{u} & otherwise \end{cases}$$

$$I_2[\doteq][n_1, n_2] = \begin{cases} t & if\ n_1 = n_2\ and\ n_1, n_2 \in \mathbb{N} \\ f & if\ n_1 \neq n_2\ and\ n_1, n_2 \in \mathbb{N} \\ I & otherwise \end{cases}$$

Matrices: Objectual Quantification

- *A variable is thought of as ranging over a set of objects from the domain, and assignments map variables to elements of the domain.*

- Given an L-structure $S = \langle D, I \rangle$, a *variable assignment* G in S is any function mapping the variables of L to D.
  G is extended to terms: $G[c] = I[c]$ and
  $G[f(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[f][G[\mathbf{t}_1], ..., G[\mathbf{t}_n]]$.

- Given S and G, the valuation $v_{S,G} : F_L \to \mathcal{V}$ is defined by:
  - $v_{S,G}[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][G[\mathbf{t}_1], ..., G[\mathbf{t}_n]]$.
  - $v_{S,G}[\diamond(\psi_1, ..., \psi_n)] = \tilde{\diamond}[v_{S,G}[\psi_1], ..., v_{S,G}[\psi_n]]$.
  - $v_{S,G}[\mathcal{Q}x\psi] = \tilde{\mathcal{Q}}[\{v_{S,G\{x:=a\}}[\psi] \mid a \in D\}]$.
    where $G\{x := a\}$ *is the variable assignment which coincides with G except for assigining* $a \in D$ *to* x.

Matrices: Substitutional Quantification

- *In classical first-order substitutional semantics, a universally quantified sentence is true iff each of its substitution instances is true.*

- Assumption: every element of the domain has a name.
  Given an L-structure $S = \langle D, I \rangle$, extend the language with the set of individual constants $\{\overline{a} \mid a \in D\}$ interpreted as the corresponding domain elements: $I[\overline{a}] = a$. Denote the resulting language $L(D)$.

- The valuation $v_S : F^{\text{cl}}_{L(D)} \to \mathcal{V}$ is defined as follows:
  - $v_S[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$
  - $v_S[\diamond(\psi_1, ..., \psi_n)] = \tilde{\diamond}[v_S[\psi_1], ..., v_S[\psi_n]]$
  - $v_S[\mathcal{Q}x\psi] = \tilde{\mathcal{Q}}[\{v_S[\psi\{\overline{a}/x\}] \mid a \in D\}]$

Example 1

Let $\mathcal{M}_2$ be the classical two-valued matrix.

$$S = \langle \{1, 2, 3\}, I \rangle$$

$$I[\overline{1}] = 1 \qquad I[\overline{2}] = 2 \qquad I[\overline{3}] = 3$$

$$I[p][1] = I[p][2] = t \qquad I[p][3] = f$$

$$v_S[p(\overline{1})] = I[p][I[\overline{1}]] = I[p][1] = t$$

$$v_S[\forall x p(x)] = \tilde{\forall}[\{v_S[p(\overline{a})] \mid a \in D\}] = \tilde{\forall}[\{t, f\}] = f$$

$$v_S[\exists x p(x)] = \tilde{\exists}[\{v_S[p(\overline{a})] \mid a \in D\}] = \tilde{\exists}[\{t, f\}] = t$$

Example 2

Let $\mathcal{V} = \{0, \frac{1}{2}, 1\}$, $\tilde{\forall} =$ min and $\tilde{\exists} =$ max.

$$S = \langle \mathbb{N} \cup \{\mathbf{u}\}, I \rangle$$

$$I[Zero][0] = 1 \quad I[Zero][\mathbf{u}] = \frac{1}{2} \quad I[Zero][n] = 0 \ \mathit{for}\ n \in \mathbb{N} - \{0, \mathbf{u}\}$$

$$v_S[Zero(\overline{0})] = I[Zero][I[\overline{0}]] = I[Zero][0] = 1$$

$$v_S[\forall x Zero(x)] = \tilde{\forall}[\{v_S[p(\overline{a})] \mid a \in D\}] = \text{min}\{0, \frac{1}{2}, 1\} = 0$$

$$v_S[\exists x Zero(x)] = \tilde{\exists}[\{v_S[p(\overline{a})] \mid a \in D\}] = \text{max}\{0, \frac{1}{2}, 1\} = 1$$

Nmatrices with Unary Quantifiers

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a non-deterministic matrix (Nmatrix) for a language L with unary quantifiers if:

1. $\mathcal{V}$ is a nonempty set of truth-values,
2. $\emptyset \neq \mathcal{D} \subset \mathcal{V}$ is a set of designated truth-values,
3. for every n-ary connective $\diamond$ of L, $\mathcal{O}$ includes an operation $\widetilde{\diamond} : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$,
4. for every unary quantifier $\mathcal{Q}$ of L, $\mathcal{O}$ includes an operation $\widetilde{\mathcal{Q}} : P^+(\mathcal{V}) \rightarrow P^+(\mathcal{V})$.

Example

Consider the two-valued Nmatrix $\mathcal{M}_1 = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ for a language L over $\{Q, \forall, \neg\}$, where $\mathcal{O}$ contains the following operations:

H	$\tilde{Q}$[H]
{t}	{t}
{t,f}	{t,f}
{f}	{f}

H	$\tilde{\forall}$[H]
{t}	{t}
{t,f}	{f}
{f}	{f}

a	$\neg a$
t	{t,f}
f	{t}

L-structures

An L-structure for an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a pair $S = \langle D, I \rangle$, where D is a non-empty domain and I satisfies the following conditions:

- For every constant c of L: $I[c] \in D$.
- For an n-ary predicate symbol p of L: $I[p] : D^n \to \mathcal{V}$.
- For every n-ary function symbol f of L: $I[f] : D^n \to D$.

I is extended to interpret closed terms of L as follows:

$$I[f(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[f][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$$

Nmatrices: Objectual Quantification

Given an L-structure $S = \langle D, I \rangle$, a *variable assignment* G in S is any function mapping the variables of L to D.

$v_{S,G} : F_L \rightarrow \mathcal{V}$ is a valuation in an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ if:

- $v_{S,G}[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][G[\mathbf{t}_1], ..., G[\mathbf{t}_n]]$.
- $v_{S,G}[\diamond(\psi_1, ..., \psi_n)] \in \tilde{\diamond}[v_{S,G}[\psi_1], ..., v_{S,G}[\psi_n]]$.
- $v_{S,G}[\mathcal{Q}x\psi] \in \tilde{\mathcal{Q}}[\underbrace{\{v_{S,G[x:=a]}[\psi]}_{???} \mid a \in D\}]$.

A Better Option: Substitutional Quantification

Reminder: For $S = \langle D, I \rangle$, the language extended by individual constants is denoted by $L(D)$

Let $S = \langle D, I \rangle$ be an L-structure. A valuation in an Nmatrix $\mathcal{M}$ for L is a function v from sentences of $L(D)$ to $\mathcal{V}$, satisfying:

- $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$
- $v[\diamond(\psi_1, \ldots, \psi_n)] \in \widetilde{\diamond}[v[\psi_1], \ldots, v[\psi_n]]$
- $v[\mathcal{Q}x\psi] \in \widetilde{\mathcal{Q}}[\{v[\psi\{\overline{a}/x\}] \mid a \in D\}]$

The Problem of α-Equivalence

- $\psi \equiv_\alpha \psi'$ if ψ can be obtained from ψ' by renaming bound variables.
- Problem: two α-equivalent sentences are not necessarily assigned the same truth-value.
- Example:

H	$\tilde{\forall}$[H]
{t}	{t}
{t,f}	{f}
{f}	{f}

a	$\neg a$
t	{t,f}
f	{t}

Let $S = \langle \{1,2\}, I \rangle$, $I[p][1] = I[p][2] = t$.

Consider: $\neg \forall x p(x)$ and $\neg \forall y p(y)$

Definition of a Non-deterministic Valuation - Corrected

Let $S = \langle D, I \rangle$ be an L-structure. A valuation in an Nmatrix $\mathcal{M}$ for L is a function v from closed sentences of $L(D)$ to $\mathcal{V}$ satisfying:

- $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$.
- $v[\diamond(\psi_1, \ldots, \psi_n)] \in \widetilde{\diamond}[v[\psi_1], \ldots, v[\psi_n]]$.
- $v[\mathcal{Q}x\psi] \in \widetilde{\mathcal{Q}}[\{v[\psi\{\overline{a}/x\}] \mid a \in D\}]$.
- If $\psi_1 \equiv_\alpha \psi_2$, then $v[\psi_1] = v[\psi_2]$.

Other Problems to Handle

- Terms denoting the same objects cannot be used interchangeably.
- Void quantification for first-order quantifiers $\forall$ and $\exists$.
- Example:

H	$\tilde{\forall}$[H]
{t}	{t}
{t,f}	{f}
{f}	{f}

a	$\neg a$
t	{t,f}
f	{t}

Let $S = \langle \{1,2\}, I \rangle$, $I[p][1] = I[p][2] = t$ *and* $I[c] = I[d] = 1$. Consider:

(i) $\neg p(c)$ and $\neg p(d)$, (ii) $\neg\forall x p(c)$ and $\neg p(c)$.

- Solution: add appropriate congruence relations. For instance, $A \sim_{void} QxA$ if $x \notin Fv(A)$.

Analycity

- An Nmatrix $\mathcal{M}$ for L is analytic if for every L-structure S and every partial $\mathcal{M}$-legal S-valuation v_p defined on a set of L-sentences closed under subformulas: v_p can be extended to a full $\mathcal{M}$-legal valuation.
- Analycity is not guaranteed anymore when congruence relations are involved.
- Some good cases:
  - Analycity for $\equiv_\alpha$ is always guaranteed.
  - Denote $\varphi_1 \sim^{dc} \varphi_2$ if φ_2 can be obtained from φ_1 by renaming bound variables and deleting/adding void quantifiers. Analycity for $\sim^{dc}$ is guaranteed iff $a \in \tilde{\mathcal{Q}}_{\mathcal{M}}(\{a\})$ for every quantifier $\mathcal{Q}$ of L.

Application: First-order LFIs

Language: $\mathcal{L}_{\rm QC} = \{\wedge, \vee, \supset, \neg, \circ, \forall, \exists\}$.

Logic: $\mathbf{QB}$ is obtained by adding the following axioms to some standard Hilbert-type system for classical positive FOL (e.g, $\mathbf{HFOL}^+$):

(N3) $\neg\varphi \vee \varphi$

(N∘4) $(\circ\varphi \wedge \varphi \wedge \neg\varphi) \supset \psi$

(DC) $\varphi_1 \supset \varphi_2$ whenever $\varphi_1 \sim^{dc} \varphi_2$.

$\varphi_1 \sim^{dc} \varphi_2$ if φ_2 can be obtained from φ_1 by renaming bound variables and deleting/adding void quantifiers.

$\mathbf{HFOL}^{+}$

Add to $\mathbf{HCL}^{+}$ the following axioms and rules:

$$\forall x\psi \to \psi\{\mathbf{t}/x\} \qquad \psi\{\mathbf{t}/x\} \to \exists x\psi$$

$$\frac{(\varphi \to \psi)}{(\varphi \to \forall x\psi)} \qquad \frac{(\psi \to \varphi)}{(\exists x\psi \to \varphi)}$$

where $\mathbf{t}$ is free for x in ψ and x is not free in φ.

Extensions of QB

(c) $\neg\neg\varphi \supset \varphi$

(e) $\varphi \supset \neg\neg\varphi$

(i$_1$) $\neg\circ\varphi \supset \varphi$ **(i$_2$)** $\neg\circ\varphi \supset \neg\varphi$

(l) $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$

(a$_\sharp$) $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi\sharp\psi)$ for $\sharp \in \{\wedge, \vee, \supset\}$

(o$_\sharp$) $(\circ\varphi \vee \circ\psi) \supset \circ(\varphi\sharp\psi)$ for $\sharp \in \{\wedge, \vee, \supset\}$

(a$_Q$) $\forall x \circ\varphi \supset \circ(\mathcal{Q}x\varphi)$ for $\mathcal{Q} \in \{\forall, \exists\}$

(o$_Q$) $\exists x \circ\varphi \supset \circ(\mathcal{Q}x\varphi)$ for $\mathcal{Q} \in \{\forall, \exists\}$

Example: da-Costa's original C_1^* is equivalent to **QBcila**.

5-valued Semantics for QB - Reminder

- The idea: include all the relevant data concerning a sentence ψ in the truth-value:
  1. The truth/falsity of ψ
  2. The truth/falsity of $\neg\psi$
  3. The truth/falsity of $\circ\psi$
- This leads to the use of elements from $\{0,1\}^3$ as truth-values, where the intended meaning of $v[\psi] = \langle x, y, z\rangle$ is as follows:

  $x = 1$ iff $v[\psi] \in \mathcal{D}$

  $y = 1$ iff $v[\neg\psi] \in \mathcal{D}$

  $z = 1$ iff $v[\circ\psi] \in \mathcal{D}$

5-valued Semantics for QB - Reminder

However, the axioms of **QB** rule out some of the truth-values:

- $\varphi \vee \neg\varphi$ rules out $\langle 0, 0, 1\rangle$ and $\langle 0, 0, 0\rangle$.
- $(\circ\varphi \wedge \varphi \wedge \neg\varphi) \supset \psi$ rules out $\langle 1, 1, 1\rangle$.

We are left with the following five truth-values:

$$t = \langle 1, 0, 1\rangle,\ t_I = \langle 1, 0, 0\rangle,\ I = \langle 1, 1, 0\rangle,\ f = \langle 0, 1, 1\rangle,\ f_I = \langle 0, 1, 0\rangle$$

The designated truth-values are those whose first component is 1: $\mathcal{D} = \{t, t_I, I\}$.

5-valued Semantics for QB

The Nmatrix $\mathcal{QM}_5 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined by:

$\mathcal{V} = \{t, t_I, I, f_I, f\}$, $\mathcal{D} = \{t, I, t_I\}$, and for $\mathcal{F} = \{f, f_I\}$:

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \end{cases} \quad a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } \ a \in \{I, f, f_I\} \\ \mathcal{F} & \text{if } \ a \in \{t, t_I\} \end{cases} \quad \widetilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \{t, f\} \\ \mathcal{F} & \text{if } a \in \{I, t_I, f_I\} \end{cases}$$

$$\widetilde{\forall}[H] = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases} \quad \widetilde{\exists}[H] = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

Effects of Axioms (Except for $(\mathbf{l})$)

Cond(c) : if $x \in \{f, f_I\}$ then $\tilde{\neg} x \subseteq \{t, t_I\}$

Cond(e): $\tilde{\neg} I = \{I\}$

Cond($\mathbf{i}_1$) : f_I should be deleted, and $\tilde{\circ} f \subseteq \{t, t_I\}$

Cond($\mathbf{i}_2$) : t_I should be deleted, and $\tilde{\circ} t = \{t\}$

Cond($\mathbf{a}_\sharp$) : if $a, b \in \{t, f\}$, then $a \tilde{\sharp} b \subseteq \{t, f\}$

Cond($\mathbf{o}_\sharp$) : if $a \in \{t, f\}$ or $b \in \{t, f\}$, then $a \tilde{\sharp} b \subseteq \{t, f\}$

Cond($\mathbf{a}_Q$) : for every $H \subseteq \{t, f\}$, $\tilde{\mathcal{Q}}[H] \subseteq \{t, f\}$

Cond($\mathbf{o}_Q$) : if $H \cap \{t, f\} \neq \emptyset$ then $\tilde{\mathcal{Q}}[H] \subseteq \{t, f\}$

For $\mathbf{X} \subseteq Ax$, $\mathcal{QM}_5(\mathbf{X})$ is the weakest refinement of $\mathcal{QM}_5$, in which the conditions of the schemata from $\mathbf{X}$ are satisfied.

Effects of $(\mathbf{a})$ and $(\mathbf{o})$

Cond(a$_Q$) : for every $H \subseteq \{t, f\}$, $\tilde{Q}[H] \subseteq \{t, f\}$

Cond(o$_Q$) : if $H \cap \{t, f\} \neq \emptyset$ then $\tilde{Q}[H] \subseteq \{t, f\}$

$\mathbf{QB} + (\mathbf{i})$:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t, I\}$	$\{t, I\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t, I\}$
$\{t, I\}$	$\{t, I\}$	$\{t, I\}$
$\{f, I\}$	$\{f\}$	$\{t, I\}$
$\{t, f, I\}$	$\{f\}$	$\{t, I\}$

$\mathbf{QB} + (\mathbf{i}) + (\mathbf{a})$:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t\}$
$\{t, I\}$	$\{t, I\}$	$\{t, I\}$
$\{f, I\}$	$\{f\}$	$\{t, I\}$
$\{t, f, I\}$	$\{f\}$	$\{t, I\}$

$\mathbf{QB} + (\mathbf{i}) + (\mathbf{o})$:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t\}$
$\{t, I\}$	$\{t\}$	$\{t\}$
$\{f, I\}$	$\{f\}$	$\{t\}$
$\{t, f, I\}$	$\{f\}$	$\{t\}$

The Nmatrix $\mathcal{M}_{C_1^*}$

$\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}, \mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$ $, \mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}$ $, \mathcal{F} = \{f\}, \mathcal{D} = \mathcal{T} \cup \mathcal{I}.$

$$a \tilde{\supset} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \text{ and } b \notin \mathcal{I} \\ \mathcal{T} & \text{if } b \in \mathcal{T} \text{ and } a \notin \mathcal{I} \\ \mathcal{D} & \text{otherwise} \end{cases} \qquad a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{T} \text{ and } b \in \mathcal{T} \\ \mathcal{T} & \text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases}$$

$$\tilde{\forall}[H] = \begin{cases} \mathcal{T} & \text{if } H \subseteq \mathcal{T} \\ \mathcal{D} & \text{if } H \subseteq \mathcal{D} \text{ and } H \cap \mathcal{I} \neq \emptyset \\ \mathcal{F} & f \in H \end{cases} \qquad \tilde{\exists}[H] = \begin{cases} \mathcal{T} & \text{if } H \subseteq \mathcal{T} \cup \mathcal{F} \text{ and } H \cap \mathcal{T} \neq \emptyset \\ \mathcal{D} & \text{if } H \cap \mathcal{I} \neq \emptyset \\ \mathcal{F} & H = \{f\} \end{cases}$$

Note that $\mathcal{M}_{C_1^*}$ is analytic.

Application: $\neg\exists x \neg p(x) \nvdash_{C_1^*} \forall x p(x)$

- A rather complex syntactic proof of da Costa (1974).
- A much easier semantic proof: refutation using $\mathcal{M}_{C_1^*}$.

$$S = \langle \{a, b\}, I \rangle$$

$$I[p][a] = I_0^0 \quad I[p][b] = f$$

Next define a partial valuation v on the set of subformulas of $\{\neg\exists x \neg p(x), \forall x p(x)\}$ as follows:

$$v[p(\overline{a})] = I_0^0 \quad v[p(\overline{b})] = f \quad v[\neg p(\overline{a})] = I_0^1 \quad v[\neg p(\overline{b})] = t_0^0$$

$$v[\exists x \neg p(x)] = I_0^1 \quad v[\neg\exists x \neg p(x)] = t_0^2 \quad v[\forall x p(x)] = f$$

v is $\mathcal{M}_{C_1^*}$-legal, and (by the analycity of $\mathcal{M}_{C_1^*}$) it can be extended to a full $\mathcal{M}_{C_1^*}$-legal valuation.

Another Application: Canonical Systems with Quantifiers

Universal quantification rules:

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}$$

where z is a variable free for w in A, z is not free in $\Gamma \cup \Delta \cup \{\forall w A\}$, and $\mathbf{t}$ is any term free for w in A.

$$\Downarrow$$

$$\frac{A\{\mathbf{t}/w\} \Rightarrow}{\forall w\, A \Rightarrow} \qquad \frac{\Rightarrow A\{z/w\}}{\Rightarrow \forall w\, A}$$

$$\Downarrow$$

$$\{p(c) \Rightarrow\}/\forall w\, p(w) \Rightarrow \qquad \{\Rightarrow p(y)\}/ \Rightarrow \forall w\, p(w)$$

An eigenvariable is marked by a variable, and a term is marked by a constant.

Coherence

- A canonical calculus G is coherent if for every two canonical rules of G of the form $\Theta_1 / \Rightarrow A$ and $\Theta_2 / A \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent.
- *The coherence of a canonical calculus G is decidable.*
- Examples:

Coherent:

$$\{p(c) \Rightarrow\} \;/\; \forall x\, p(x) \Rightarrow \qquad \{\Rightarrow p(y)\} \;/\; \Rightarrow \forall x\, p(x)$$

Non-coherent:

$$\{\Rightarrow p(c)\} \;/\; \Rightarrow \mathcal{Q}xp(x) \qquad \{p(d) \Rightarrow\} \;/\; \mathcal{Q}xp(x) \Rightarrow$$

Correspondence Theorem

The following statements concerning a canonical system G with unary quantifiers are equivalent:

1. G is coherent.
2. G has a characteristic 2Nmatrix.
3. G admits cut elimination.
4. G admits strong cut elimination.

More General Quantifiers

- A natural step: n-ary quantifiers:

  If $\mathcal{Q}$ is an n-ary quantifier, then $\mathcal{Q}x(\psi_1, ..., \psi_n)$ is a formula.

- Examples:

  1. *Unary quantifiers:* $\forall$, $\exists$.
  2. *Binary quantifiers: bounded universal and existential quantifiers* $\overline{\forall}$ *and* $\overline{\exists}$, *where:*
     - $\overline{\forall}(\psi_1, \psi_2)$ means $\forall x(\psi_1 \to \psi_2)$.
     - $\overline{\exists}(\psi_1, \psi_2)$ means $\exists x(\psi_1 \wedge \psi_2)$.

Nmatrices with n-ary quantifiers

- An n-ary quantifier $\mathcal{Q}$ in an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is interpreted by a function $\tilde{\mathcal{Q}} : P^+(\mathcal{V}^n) \to P^+(\mathcal{V})$.
- Example: for every $\mathcal{E} \in P^+(\{t, f\}^2)$:

$$\tilde{\tilde{\forall}}[\mathcal{E}] = \begin{cases} \{f\} & \text{if } \langle t, f \rangle \in \mathcal{E} \\ \{t\} & \text{otherwise} \end{cases} \qquad \tilde{\tilde{\exists}}[\mathcal{E}] = \begin{cases} \{t\} & \text{if } \langle t, t \rangle \in \mathcal{E} \\ \{f\} & \text{otherwise} \end{cases}$$

The definition of an $\mathcal{M}$-valuation v is now modified as follows:

$$v[\mathcal{Q}x(\psi_1, ..., \psi_n)] \in \tilde{\mathcal{Q}}_{\mathcal{M}}[\{\langle v[\psi_1\{\overline{a}/x\}], ..., v[\psi_n\{\overline{a}/x\}]\rangle \mid a \in D\}]$$

$\mathbf{H}$	$\tilde{\tilde{\forall}}[\mathbf{H}]$	$\tilde{\tilde{\exists}}[\mathbf{H}]$	$\tilde{\mathcal{Q}}_2[\mathbf{H}]$
$\{\langle \mathbf{t}, \mathbf{t} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{f}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{f}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{f}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{f}, \mathbf{f} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{t} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$