A Tutorial on Canonical Gentzen-type Systems

Arnon Avron and Anna Zamansky

School of Computer Science
Tel-Aviv University

What is a Propositional Logic?

Scott consequence relation (scr) between sets of formulas:

strong reflexivity: if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$.

monotonicity: if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$.

transitivity (*cut*): if $\Gamma \vdash \psi, \Delta$ and $\Gamma, \psi \vdash \Delta$ then $\Gamma \vdash \Delta$.

Tarskian consequence relation (tcr) between sets of formulas and formulas:

strong reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.

monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.

transitivity (*cut*): if $\Gamma \vdash \psi$ and $\Gamma, \psi \vdash \varphi$ then $\Gamma \vdash \varphi$.

What is a Propositional Logic?

- A tcr or scr $\vdash$ for $\mathcal{L}$ is structural if for every uniform $\mathcal{L}$-substitution σ and every Γ and Δ: if $\Gamma \vdash \Delta$ then $\sigma[\Gamma] \vdash \sigma[\Delta]$.
- A tcr or scr $\vdash$ for $\mathcal{L}$ is consistent if there exist non-empty Γ and Δ, such that $\Gamma \not\vdash \Delta$.
- A tcr or scr $\vdash$ for $\mathcal{L}$ is finitary if whenever $\Gamma \vdash \Delta$, there exist finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, such that $\Gamma' \vdash \Delta'$.
- A propositional logic is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a tcr or scr for $\mathcal{L}$ which is structural, consistent and finitary.

Gentzen-type Systems

- *A Gentzen-type system G* is an axiomatic system which manipulates *sequents* of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of formulas.

- A Gentzen-type system $\mathbf{G}$ is called *standard* if:
  1. Its set of axioms includes the standard axioms:
  $$\psi \Rightarrow \psi$$
  2. It has among its rules the standard structural rules: *permutation*, *contraction*, *weakening* and *cut*.

- The associated scr: $\Gamma \vdash_{\mathbf{G}} \Delta$ iff $\Gamma' \Rightarrow \Delta'$ is a theorem of $\mathbf{G}$ for some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$.

The System GCPL

$$(\neg\Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg)\quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\supset\Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\supset)\quad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$(\wedge\Rightarrow)\quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee\Rightarrow)\quad \frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad (\Rightarrow\vee)\quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Useful Notions of Cut Elimination

- A Gentzen-type system G admits *cut-elimination* if whenever a sequent is provable in G, it also has a cut-free proof in G.
- A cut is called a Θ-*cut* if the cut formula occurs in Θ.
- A cut is called Θ-*analytic* if the cut formula is a subformula of some formula occurring in Θ. A proof is Θ-*analytic* if all cuts in it are Θ-analytic.
- A Gentzen-type system G admits *strong cut-elimination* if whenever a sequent is provable in G from a set of sequents Θ, it also has a proof in which all cuts are Θ-cuts.
- G admits *strong analytic cut-elimination* if whenever $\Theta \vdash_G \Omega$, Ω has in G a $\Theta \cup \{\Omega\}$-analytic proof from Θ.

"Ideal" Logical Rules

- Gentzen's vision of a "well-behaved" rule:

  "...The introductions represent, as it were, the 'definitions' of the symbols concerned..."

  G. Gentzen, "Investigations into Logical Deduction".

- Thesis: the meaning of the connective is given by its introduction rules. *(Strongly challenged by Prior by using his famous "Tonk" connective...)*

What is a Canonical Rule?

- An “ideal” logical rule: an introduction rule or an elimination rule for *exactly one connective*.
- In its formulation: *exactly one occurrence* of the introduced connective, no other occurrences of other connectives.
- The rule should also be *pure* (i.e. context-independent): no side conditions limiting its application.
- Its active formulas: *immediate subformulas* of its principal formula.

What is a Canonical Rule?

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

$$\frac{\psi, \varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi}$$

$$\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \; ; \; \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

Canonical Systems: The Multiple-conclusion Case

- *A sequent*: an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are *finite* sets of $\mathcal{L}$-formulas.
- *A clause*: a sequent consisting of atomic formulas.
- A *canonical rule* has one of the forms:

$$\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} / \diamond (p_1, ..., p_n) \Rightarrow$$

$$\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} / \Rightarrow \diamond (p_1, ..., p_n)$$

where $m \geq 0$ *and for all* $1 \leq i \leq m$: $\Pi_i \Rightarrow \Sigma_i$ *is a clause over* $\{p_1, ..., p_n\}$.

Canonical Rules

Application of a canonical rule of the form
$\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \diamond (p_1, ..., p_n) \Rightarrow$:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \diamond(\psi_1, ..., \psi_n) \Rightarrow \Delta}$$

where Π_i^* *and* Σ_i^* *are obtained from* Π_i *and* Σ_i *respectively by substituting* ψ_j *for* p_j *for all* $1 \leq j \leq n$*, and* Γ, Δ *are any finite sets of formulas.*

Example 1

Conjunction rules:

$$\{p_1, p_2 \Rightarrow\} \ / \ p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \ ; \ \Rightarrow p_2\} \ / \ \Rightarrow p_1 \wedge p_2$$

Their applications:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

Example 2

Implication rules:

$$\{p_1 \Rightarrow p_2\} \;/\; \Rightarrow p_1 \supset p_2 \qquad \{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; p_1 \supset p_2 \Rightarrow$$

Their applications:

$$\frac{\Gamma, \psi \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$$

Example 3

Semi-implication rules:

$$\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} \ / \ p_1 \leadsto p_2 \Rightarrow \qquad \{\Rightarrow p_2\} \ / \ \Rightarrow p_1 \leadsto p_2$$

Their applications:

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \leadsto \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \leadsto \varphi}$$

Example 4

"Tonk" rules:

$$\{p_2 \Rightarrow\} \ / \ p_1 T p_2 \Rightarrow \qquad \{\Rightarrow p_1\} \ / \ \Rightarrow p_1 T p_2$$

Their applications:

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi T \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi T \psi}$$

What Sets of Rules are Acceptable?

- A standard Gentzen-type system is canonical if each of its logical rules is canonical.
- If G is a canonical calculus, then $\vdash_G$ is a structural and finitary scr. But is it a logic?

Coherence - an Example

$$\{p_1, p_2 \Rightarrow\} \;/\; p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \;;\; \Rightarrow p_2\} \;/\; \Rightarrow p_1 \wedge p_2$$

These two classical rules form a coherent pair, where:

$$S_1 = \{p_1, p_2 \Rightarrow\} \qquad S_2 = \{\Rightarrow p_1 \;;\; \Rightarrow p_2\}$$

$$S_1 \cup S_2 = \{p_1, p_2 \Rightarrow \;;\; \Rightarrow p_1 \;;\; \Rightarrow p_2\}$$

$$\{\Rightarrow p_1\}/p_1 \circ p_2 \Rightarrow \qquad \{\Rightarrow p_1 \;;\; \Rightarrow p_2\}/ \Rightarrow p_1 \circ p_2$$

These two rules form an incoherent pair, where:

$$S_1 = \{\Rightarrow p_1\} \qquad S_2 = \{\Rightarrow p_1 \;;\; \Rightarrow p_2\}$$

$$S_1 \cup S_2 = \{\Rightarrow p_1 \;;\; \Rightarrow p_2\}$$

Coherence

- A canonical calculus G is *coherent* if for every pair of rules $\Theta_1 / \Rightarrow \diamond(p_1, ..., p_n)$ and $\Theta_2 / \diamond(p_1, ..., p_n) \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent (and so the empty set can be derived from it using cuts).

- For a canonical calculus G, $\vdash_G$ is a logic iff G is coherent.

Coherent Calculi:

$$\{p_1, p_2 \Rightarrow\} \,/\, p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \,;\, \Rightarrow p_2\} \,/\, \Rightarrow p_1 \wedge p_2$$

$$\{p_1 \Rightarrow p_2\} \,/\, \Rightarrow p_1 \supset p_2 \qquad \{\Rightarrow p_1 \,;\, p_2 \Rightarrow\} \,/\, p_1 \supset p_2 \Rightarrow$$

$$\{\Rightarrow p_1 \,;\, p_2 \Rightarrow\} \,/\, p_1 \leadsto p_2 \Rightarrow \qquad \{\Rightarrow p_2\} \,/\, \Rightarrow p_1 \leadsto p_2$$

$$\{p_1 \Rightarrow\} \,/\, \Rightarrow \neg p_1 \qquad \{\Rightarrow p_1\} \,/\, \neg p_1 \Rightarrow$$

Non-coherent: “Tonk”!

$$\{p_2 \Rightarrow\} \ / \ p_1 T p_2 \Rightarrow \qquad \{\Rightarrow p_1\} \ / \ \Rightarrow p_1 T p_2$$

This is what is wrong with “Tonk”: these rules do not define a logic!

Splitting Canonical Rules

$$\frac{p_1, p_2 \Rightarrow}{p_1 \wedge p_2 \Rightarrow}$$

$$\Downarrow$$

$$\frac{p_1 \Rightarrow}{p_1 \wedge p_2 \Rightarrow} \qquad \frac{p_2 \Rightarrow}{p_1 \wedge p_2 \Rightarrow}$$

"Reading off" the Semantics from Canonical Rules

$$\frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi} \qquad \frac{\varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\psi \Rightarrow}{\psi \wedge \varphi \Rightarrow}$$

$$\frac{\Rightarrow \psi \quad \varphi \Rightarrow}{\psi \supset \varphi \Rightarrow} \qquad \frac{\Rightarrow \varphi}{\Rightarrow \psi \supset \varphi} \qquad \frac{\psi \Rightarrow}{\Rightarrow \psi \supset \varphi}$$

ψ	φ	$\psi \wedge \varphi$
t	t	t
t	f	f
f	t	f
f	f	f

ψ	φ	$\psi \supset \varphi$
t	t	t
t	f	f
f	t	t
f	f	t

How to Deal with Underspecification

$$\frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi} \qquad \frac{\varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\psi \Rightarrow}{\psi \wedge \varphi \Rightarrow}$$

$$\frac{\Rightarrow \psi \quad \varphi \Rightarrow}{\psi \supset \varphi \Rightarrow} \qquad \frac{\Rightarrow \varphi}{\Rightarrow \psi \supset \varphi} \qquad \frac{\psi \Rightarrow}{\Rightarrow \psi \supset \varphi}$$

ψ	φ	$\psi \wedge \varphi$
t	t	?
t	f	f
f	t	f
f	f	f

ψ	φ	$\psi \supset \varphi$
t	t	?
t	f	f
f	t	?
f	f	?

Non-deterministic Matrices

$$\frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi} \qquad \frac{\varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\psi \Rightarrow}{\psi \wedge \varphi \Rightarrow}$$

$$\frac{\Rightarrow \psi \quad \varphi \Rightarrow}{\psi \supset \varphi \Rightarrow} \qquad \frac{\Rightarrow \varphi}{\Rightarrow \psi \supset \varphi} \qquad \frac{\psi \Rightarrow}{\Rightarrow \psi \supset \varphi}$$

ψ	φ	$\psi \wedge \varphi$
t	t	{**t**,**f**}
t	f	{**f**}
f	t	{**f**}
f	f	{**f**}

ψ	φ	$\psi \supset \varphi$
t	t	{**t**,**f**}
t	f	{**f**}
f	t	{**t**,**f**}
f	f	{**t**,**f**}

Non-deterministic Matrices

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a non-deterministic matrix (Nmatrix) for $\mathcal{L}$ if:

- $\mathcal{V}$ is a nonempty set of truth-values.
- $\emptyset \neq \mathcal{D} \subset \mathcal{V}$ is a set of designated truth-values.
- For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes an operation $\widetilde{\diamond} : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$

Any ordinary (deterministic) matrix can be identified with an Nmatrix whose functions in $\mathcal{O}$ always return singletons.

Non-deterministic Matrices

Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$.

- An *$\mathcal{M}$-valuation* is a function v from $\mathcal{L}$-formulas to $\mathcal{V}$, such that:

$$v[\diamond(\psi_1 \ldots \psi_n)] \in \widetilde{\diamond}[v[\psi_1] \ldots v[\psi_n]]$$

- An $\mathcal{M}$-valuation v *satisfies* a formula ψ if $v[\psi] \in \mathcal{D}$. v *satisfies* a set of formulas Γ if it satisfies every formula in Γ.

- $\Gamma \vdash_{\mathcal{M}} \Delta$ if for every $\mathcal{M}$-valuation v which satisfies Γ, v satisfies some formula in Δ.

- $\mathcal{M}$ is a *characteristic Nmatrix* for a calculus G if $\vdash_G = \vdash_{\mathcal{M}}$.

Analycity

An obvious, yet crucial fact: any partial $\mathcal{M}$-legal assignment defined on a set closed under subformulas can be extended to a full $\mathcal{M}$-legal assignment.

If $\mathcal{M}$ is finite this entails that $\vdash_{\mathcal{M}}$ is:

- Decidable
- Finitary (the compactness theorem obtains)

Every Coherent Calculus Has a Characteristic 2Nmatrix

Consider the canonical calculus G_0 over the language $\{\wedge, \rightsquigarrow\}$ with no canonical rules whatsoever:

ψ	φ	$\psi\wedge\varphi$
t	t	$\{\mathbf{t,f}\}$
t	f	$\{\mathbf{t,f}\}$
f	t	$\{\mathbf{t,f}\}$
f	f	$\{\mathbf{t,f}\}$

ψ	φ	$\psi\rightsquigarrow\varphi$
t	t	$\{\mathbf{t,f}\}$
t	f	$\{\mathbf{t,f}\}$
f	t	$\{\mathbf{t,f}\}$
f	f	$\{\mathbf{t,f}\}$

Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule $\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$

ψ	φ	$\psi \wedge \varphi$
t	t	{**t**,**f**}
t	f	{**f**}
f	t	{**f**}
f	f	{**f**}

ψ	φ	$\psi \rightsquigarrow \varphi$
t	t	{**t**,**f**}
t	f	{**t**,**f**}
f	t	{**t**,**f** }
f	f	{**t**,**f**}

Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule $\{\Rightarrow p_2\}/\Rightarrow p_1 \rightsquigarrow p_2$

ψ	φ	$\psi \wedge \varphi$
t	t	$\{\mathbf{t},\mathbf{f}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{f}\}$
f	f	$\{\mathbf{f}\}$

ψ	φ	$\psi \rightsquigarrow \varphi$
t	t	$\{\mathbf{t}\}$
t	f	$\{\mathbf{t},\mathbf{f}\}$
f	t	$\{\mathbf{t}\}$
f	f	$\{\mathbf{t},\mathbf{f}\}$

Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule $\{\Rightarrow p_1\ ;\ p_2 \Rightarrow\}/p_1 \rightsquigarrow p_2 \Rightarrow$

ψ	φ	$\psi \wedge \varphi$
t	t	{**t**,**f**}
t	f	{**f**}
f	t	{**f**}
f	f	{**f**}

ψ	φ	$\psi \rightsquigarrow \varphi$
t	t	{**t**}
t	f	{**f**}
f	t	{**t**}
f	f	{**t**,**f**}

Every 2Nmatrix Has a Corresponding Coherent Calculus

p_1	p_2	$p_1 \circ p_2$
t	t	$\{\mathbf{f}\}$
t	f	$\{\mathbf{f}\}$
f	t	$\{\mathbf{t},\mathbf{f}\}$
f	f	$\{\mathbf{t}\}$

$$\{\Rightarrow p_1 \; ; \; \Rightarrow p_2\} \; / \; p_1 \circ p_2 \Rightarrow$$

$$\{\Rightarrow p_1 \; ; \; p_2 \Rightarrow\} \; / \; p_1 \circ p_2 \Rightarrow$$

$$\{p_1 \Rightarrow \; ; \; p_2 \Rightarrow\} \; / \; \Rightarrow p_1 \circ p_2$$

This is not the most efficient form - “normal form” to be defined.

Exact Correspondence

Let G be a canonical calculus. The following statements concerning G are equivalent:

1. $\vdash_G$ is consistent (and so it is a logic).
2. G is coherent.
3. G has a characteristic 2Nmatrix.
4. G admits strong cut-elimination.
5. G admits cut-elimination.

1⇒2 - see below. 2⇒3 was demonstrated above. 3⇒4 - see below. 4⇒5 is trivial. 5⇒1 - see below.

$(1) \Rightarrow (2)$

$\vdash_G$ is consistent $\Rightarrow G$ is coherent

Suppose that there are two rules $\Theta_1 / \Rightarrow \diamond(p_1, \ldots, p_n)$ and $\Theta_2 / \diamond(p_1, \ldots, p_n) \Rightarrow$, such that $\Theta_1 \cup \Theta_2$ is classically consistent. Then there is a classical valuation v which satisfies $\Theta_1 \cup \Theta_2$. Let $\Pi' = \{p_i | \; 1 \leq i \leq n, v[p_i] = t\}$ and $\Sigma' = \{p_i | \; 1 \leq i \leq n, v[p_i] = f\}$. Let $\Theta'_j = \{\Pi, \Pi' \Rightarrow \Sigma, \Sigma' | \; \Pi \Rightarrow \Sigma \in \Theta_j\}$ for $j = 1, 2$. Then Θ'_1, Θ'_2 are sets of standard axioms. By applying the first rule on Θ'_1 we obtain $\Pi', \diamond(p_1, \ldots, p_n) \Rightarrow \Sigma'$. By applying the second rule on Θ'_2 we obtain $\Pi' \Rightarrow \Sigma', \diamond(p_1, \ldots, p_n)$. By cut we obtain a proof of $\Pi' \Rightarrow \Sigma'$. But then $p \Rightarrow q$ is provable for every $p \neq q$. Hence $\vdash_G$ is not consistent.

$(3) \Rightarrow (4)$

G has a characteristic 2Nmatrix $\Rightarrow$ G admits strong cut-elimination

Let $\mathcal{M}_G$ be a characteristic 2Nmatrix for G. Suppose that $\Gamma \Rightarrow \Delta$ has no proper proof from Θ in G (in which the only cuts are on formulas from Θ). We show that $\Theta \not\vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$. To define a refuting valuation, we first extend the sets Γ, Δ to sets Γ^*, Δ^*, such that: (i) $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$, (ii) $\Gamma^* \Rightarrow \Delta^*$ has no proper proof in G from Θ, (iii) For every rule $\Theta / \Rightarrow \diamond(p_1, \ldots, p_n)$ ($\Theta / \diamond(p_1, \ldots, p_n) \Rightarrow$): whenever $\diamond(\psi_1, \ldots, \psi_n) \in \Delta^*$ ($\diamond(\psi_1, \ldots, \psi_n) \in \Gamma^*$), there is some $\Sigma \Rightarrow \Pi \in \Theta$ and some $1 \leq i \leq n$, such that either $p_i \in \Sigma$ and $\psi_i \in \Gamma^*$, or $p_i \in \Pi$ and $\psi_i \in \Delta^*$, and (iv) for every ψ occurring in Θ, $\psi \in \Gamma^* \cup \Delta^*$. Next a refuting valuation v is defined, which satisfies the sequents in Θ, but does not satisfy $\Gamma \Rightarrow \Delta$. The $\mathcal{M}_G$-legality of v is guaranteed by the properties above.

$$\boxed{(5) \Rightarrow (1)}$$

G admits cut-elimination $\Rightarrow \vdash_G$ is consistent

Clauses which are not axioms can be proved only by cuts on atomic formulas. Thus if G admits cut-elimination, it must be consistent.

Resolution Normal Form of Canonical Rules

- A canonical rule S_1/C is at least as strong as the canonical rule S_2/C iff every clause in S_1 classically follows from S_2. *This is equivalent to saying that every clause in S_1 is subsumed by some clause that can be derived from the clauses of S_2 using resolutions.*

- Two canonical rules S_1/C and S_2/C are *equivalent* if S_1 and S_2 are classically equivalent (as sets of clauses).

- A canonical rule is in *Resolution Normal Form* (RNF) if its set of premises S does not include a standard axiom, and any resolvent of two elements of S is subsumed by some other element of S.

- Every canonical rule has an equivalent canonical rule in RNF.

Example

$$\{p_1 \Rightarrow p_2 \ ; \ p_2 \Rightarrow p_1 \ ; \ \Rightarrow p_1, p_2\} \ / \ \Rightarrow p_1 \wedge p_2$$

An equivalent rule in RNF:

$$\{ \Rightarrow p_1 \ ; \ \Rightarrow p_2\} \ / \ \Rightarrow p_1 \wedge p_2$$

Canonical Calculi in Normal Form

- A canonical calculus G is in *normal form* if all the rules of G are in RNF, and for each connective G has at most one left introduction rule and at most one right introduction rule.
- Any canonical calculus can be transformed into a cut-free equivalent normal form.

The Syntactic Method

Consider the following two rules for the binary connective X (representing XOR):

$$\{\Rightarrow p_1 \; ; \; p_2 \Rightarrow\} \, / \, \Rightarrow p_1 X p_2 \quad \{\Rightarrow p_2 \; ; \; p_1 \Rightarrow\} \, / \, \Rightarrow p_1 X p_2$$

$$\Downarrow$$

$$\{\Rightarrow p_1, p_2 \; ; \; p_1 \Rightarrow p_1 \; ; \; p_2 \Rightarrow p_2 \; ; \; p_1, p_2 \Rightarrow\} \, / \, \Rightarrow p_1 X p_2$$

$$\Downarrow$$

$$\{\Rightarrow p_1, p_2 \; ; \; p_1, p_2 \Rightarrow\} \, / \, \Rightarrow p_1 X p_2$$

The Semantic Method

If a characteristic 2Nmatrix $\mathcal{M}$ is given for a calculus, its equivalent normal form can be easily constructed as follows:

$(\diamond \Rightarrow)$:

$$\frac{\{\{p_i \mid x_i = t\} \Rightarrow \{p_i \mid x_i = f\} \mid t \in \widetilde{\diamond}[x_1, \ldots, x_n]\}}{\diamond(p_1, \ldots, p_n) \Rightarrow}$$

$(\Rightarrow \diamond)$:

$$\frac{\{\{p_i \mid x_i = t\} \Rightarrow \{p_i \mid x_i = f\} \mid f \in \widetilde{\diamond}[x_1, \ldots, x_n]\}}{\Rightarrow \diamond(p_1, \ldots, p_n)}$$

If $t \in \widetilde{\diamond}[x_1, \ldots, x_n]$ for all $x_1, \ldots, x_n$, then the first rule is redundant and can be discarded.

If $\widetilde{\diamond}[x_1, \ldots, x_n] = \{f\}$ for all $x_1, \ldots, x_n$ then the first rule does not have any premises (a nonstandard axiom).

Example: The XOR Connective

p_1	p_2	$p_1 X p_2$
t	t	$\{\mathbf{f}\}$
t	f	$\{\mathbf{t}\}$
f	t	$\{\mathbf{t}\}$
f	f	$\{\mathbf{f}\}$

$$\{\Rightarrow p_1, p_2\ ;\ p_1, p_2 \Rightarrow\} / \Rightarrow p_1 X p_2$$

$$\{p_1 \Rightarrow p_2\ ;\ p_2 \Rightarrow p_1\} / \ p_1 X p_2 \Rightarrow$$

When are Canonical Rules Invertible?

- A rule R is *invertible in a calculus G* if for every application of R it holds that whenever its conclusion is provable in G, also each of its premises is provable in G.
- Calculus G_1 (the first rule is invertible):

$$\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \,;\, \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

Application of the first rule:

$$\frac{\Gamma, \psi_1, \psi_2 \Rightarrow \Delta}{\Gamma \Rightarrow \psi_1 \wedge \psi_2, \Delta}$$

Invertibility of the first rule:

$$\frac{\Gamma, \psi_1 \wedge \psi_2 \Rightarrow \Delta \qquad \dfrac{\Gamma, \psi_1 \Rightarrow \Delta, \psi_1 \quad \Gamma, \psi_2 \Rightarrow \Delta, \psi_2}{\Gamma, \psi_1, \psi_2 \Rightarrow \Delta, \psi_1 \wedge \psi_2}}{\Gamma, \psi_1, \psi_2 \Rightarrow \Delta}$$

When are Canonical Rules Invertible?

- Calculus G_1:

$$\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \;;\; \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

- An (equivalent) calculus G_2:

$$\{p_1 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \;;\; \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$$

The first rules are NOT invertible: $\vdash_{G_2} \psi_1 \wedge \psi_2 \Rightarrow \psi_1$, but $\not\vdash_{G_2} \psi_1 \Rightarrow \psi_2$.

The explanation: G_2 is not in normal form.

Invertibility and Determinism

Let G be a coherent canonical calculus in normal form. The following statements are equivalent:

1. G has an invertible rule for $\diamond$.
2. $\tilde{\diamond}_{\mathcal{M}_G}$ (constructed as explained above) is deterministic.
3. G has a rule for $\diamond$, and all the rules for $\diamond$ are invertible.

Axiom Expansion

- Axiom expansion in a calculus allows for a reduction of logical axioms to the atomic case.
- An n-ary connective $\diamond$ *admits axiom expansion* in a calculus G if whenever $\diamond(p_1, ..., p_n) \Rightarrow \diamond(p_1, ..., p_n)$ is provable in G, it has a cut-free derivation in G from atomic axioms of the form $\{p_i \Rightarrow p_i\}_{1 \leq i \leq n}$.

Another Exact Correspondence

Let G be a coherent canonical calculus in normal form. The following statements are equivalent:

1. The rules of G are invertible,
2. G has a characteristic two-valued deterministic matrix.
3. Every connective of $\mathcal{L}$ admits axiom expansion in G.

Canonical Calculi: The Single-conclusion Case

- *A positive Horn clause:* a sequent of the form $\Pi \Rightarrow \{q\}$, where Π is a set of atomic formulas and q - an atomic formula.
- *A negative Horn clause:* a sequent of the form $\Pi \Rightarrow$, where Π is a set of atomic formulas.
- *A single-conclusioned sequent:* an expression $\Gamma \Rightarrow \psi$, where Γ is a set of $\mathcal{L}$-formulas and ψ - an $\mathcal{L}$-formula.

Canonical Single-conclusion Rules

- A canonical introduction rule:

$$\{\Pi_i \Rightarrow \Sigma_i\}_{1 \le i \le m} \ / \ \Rightarrow \diamond(p_1, ..., p_n)$$

where $m \ge 0$ *and for all* $1 \le i \le m$*:* $\Pi_i \Rightarrow \Sigma_i$ *is a positive Horn clause over* $\{p_1, ..., p_n\}$.

- A canonical elimination rule:

$$\{\Pi_i \Rightarrow \Sigma_i\}_{1 \le i \le m} \ / \ \diamond\,(p_1, ..., p_n) \Rightarrow$$

where $m \ge 0$ *and for all* $1 \le i \le m$*:* $\Pi_i \Rightarrow \Sigma_i$ *is a Horn clause (either positive or negative) over* $\{p_1, ..., p_n\}$.

Applications of Rules

Application of $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} \ / \ \diamond(p_1, ..., p_n) \Rightarrow$:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \varphi_i\}_{1 \leq i \leq m}}{\Gamma, \diamond(\psi_1, ..., \psi_n) \Rightarrow \theta}$$

where Π_i^* *is obtained from* Π_i *by substituting* ψ_j *for* p_j *for all* $1 \leq j \leq n$, $\varphi_i = \psi_j$ *in case* $\Sigma_i = \{p_j\}$, $\varphi_i = \theta$ *in case* Σ_i *is empty, and* Γ *is any finite set of formulas.*

Example 1

Conjunction rules:

$$\{p_1, p_2 \Rightarrow\} \;/\; p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1 \;;\; \Rightarrow p_2\} \;/\; \Rightarrow p_1 \wedge p_2$$

Their applications:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta}{\Gamma, \psi \wedge \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \psi \quad \Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \wedge \varphi}$$

Example 2

Implication rules:

$$\{p_1 \Rightarrow p_2\} \;/\; \Rightarrow p_1 \supset p_2 \qquad \{\Rightarrow p_1 \;;\; p_2 \Rightarrow\} \;/\; p_1 \supset p_2 \Rightarrow$$

Their applications:

$$\frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \psi \supset \varphi} \qquad \frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \supset \varphi \Rightarrow \theta}$$

Example 3

Semi-implication rules:

$$\{\Rightarrow p_1 \; ; \; p_2 \Rightarrow\} \; / \; p_1 \rightsquigarrow p_2 \Rightarrow \qquad \{\Rightarrow p_2\} \; / \; \Rightarrow p_1 \rightsquigarrow p_2$$

Their applications:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \rightsquigarrow \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \rightsquigarrow \varphi}$$

Yet Another Exact Correspondence

Let G be a single-conclusioned canonical calculus. The following statements concerning G are equivalent:

1. $\vdash_G$ is consistent (and so it is a logic).
2. G is coherent.
3. G admits cut-elimination.
4. G admits strong cut-elimination.

Signed Formulas: the Two-valued Case

A version of Gentzen-type calculi using signed formulas:

$$\frac{\psi, \varphi \Rightarrow}{\psi \wedge \varphi \Rightarrow} \qquad \frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \wedge \varphi}$$

$$\frac{\{\mathbf{f} : \psi, \mathbf{f} : \varphi\}}{\{\mathbf{f}\} : \psi \wedge \varphi} \qquad \frac{\{\mathbf{t} : \psi\} \quad \{\mathbf{t} : \varphi\}}{\{\mathbf{t}\} : \psi \wedge \varphi}$$

Signed Formulas: the General Case

Let $\mathcal{V}$ be a finite set of signs.

- *A signed formula*: an expression of the form $a : \psi$, where ψ is a formula and $a \in \mathcal{V}$.
- *A sequent*: a finite set of signed formulas.
- *A clause*: a sequent consisting of atomic signed formulas.
- A valuation v *satisfies* a signed formula $a : \psi$ if $v[\psi] = a$.
- v *satisfies* a set of signed formulas Ω, if it satisfies some signed formula in Ω. *Sequents are interpreted disjunctively.*
- v *satisfies* a set of sequents Θ, if it satisfies all sequents of Θ.

Signed Calculi

- A *logical axiom* for $\mathcal{V}$ is a sequent of the form $\{l : \psi \mid l \in \mathcal{V}\}$.
- The *cut and weakening* rules for $\mathcal{V}$ are defined as follows:

$$\frac{\Omega \cup \{l : \psi \mid l \in L_1\} \quad \Omega \cup \{l : \psi \mid l \in L_2\}}{\Omega \cup \{l : \psi \mid l \in L_1 \cap L_2\}}$$

where $L_1, L_2 \subseteq \mathcal{V}$

$$\frac{\Omega}{\Omega, l : \psi}$$

where $l \in \mathcal{V}$.

Canonical Signed Rules

- $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}$:

$$\{p_1 \Rightarrow p_2\} \;/\; \Rightarrow p_1 \supset p_2$$

$$\{\Rightarrow p_1 \;,\; p_2 \Rightarrow\} \;/\; p_1 \supset p_2 \Rightarrow$$

$$\{\; \{\mathbf{f} : p_1, \mathbf{t} : p_2\} \;\} \;/\; \{\mathbf{t}\} : p_1 \supset p_2$$

$$\{\{\mathbf{t} : p_1\} \;,\; \{\; \mathbf{f} : p_2\}\} \;/\; \{\mathbf{f}\} : p_1 \supset p_2$$

- $\mathcal{V} = \{a, b, c\}$:

$$\{\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}\} \;/\; \{a, c\} : \circ(p_1, p_2, p_3)$$

$$\{\; \{c : p_2\} \;,\; \{a : p_3, b : p_3\} \;,\; \{c : p_1\}\} \;/\; \{b, c\} : \circ(p_1, p_2, p_3)$$

Canonical Signed Rules

- A *signed canonical rule* for an n-ary connective $\diamond$:

$$\{\Sigma_1, ..., \Sigma_m\} \ / \ S : \diamond(p_1, \ldots, p_n)$$

where $S \subset \mathcal{V}$, $m \geq 0$ and for every $1 \leq j \leq m$: Σ_j is a clause consisting of atomic signed formulas of the form $a : p_k$ for $a \in \mathcal{V}$ and $1 \leq k \leq n$.

- An *application* of a rule $\{\Sigma_1, ..., \Sigma_m\} \ / \ S : \diamond(p_1, \ldots, p_n)$:

$$\frac{\Omega \cup \Sigma_1^* \quad \ldots \quad \Omega \cup \Sigma_m^*}{\Omega \cup S : \diamond(\psi_1, ..., \psi_n)}$$

where $\psi_1, ..., \psi_n$ are $\mathcal{L}$-formulas, Ω is a sequent, and for all $1 \leq i \leq m$: Σ_i^* is obtained from Σ_i by replacing p_j by ψ_j for every $1 \leq j \leq n$.

Example 1

Standard conjunction rules:

$$\{\{\mathbf{f}:p_1,\mathbf{f}:p_2\}\}\;/\;\{\mathbf{f}\}:p_1\wedge p_2 \qquad \{\{\mathbf{t}:p_1\},\{\mathbf{t}:p_2\}\}\;/\;\{\mathbf{t}\}:p_1\wedge p_2$$

Their applications:

$$\frac{\Omega\cup\{\mathbf{f}:\psi_1,\mathbf{f}:\psi_2\}}{\Omega\cup\{\mathbf{f}:\psi_1\wedge\psi_2\}} \qquad \frac{\Omega\cup\{\mathbf{t}:\psi_1\}\quad \Omega\cup\{\mathbf{t}:\psi_2\}}{\Omega\cup\{\mathbf{t}:\psi_1\wedge\psi_2\}}$$

Example 2

Rule for a trenary connective $\circ$:

$$\{\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}\} \;/\; \{a, c\} : \circ(p_1, p_2, p_3)$$

Its application:

$$\frac{\Omega \cup \{a : \psi_1, c : \psi_2\} \quad \Omega \cup \{a : \psi_3, b : \psi_2\}}{\Omega \cup \{a : \circ(\psi_1, \psi_2, \psi_3), c : \circ(\psi_1, \psi_2, \psi_3)\}}$$

Coherence

- Example for $\mathcal{V} = \{a, b, c\}$:

$$\{\{a : p_1\} \ , \ \{c : p_2\}\} \ / \ \{a, b\} : p_1 \diamond p_2$$

$$\{\{a : p_1\} \ , \ \{c : p_2\}\} \ / \ \{b, c\} : p_1 \diamond p_2$$

$$\{\{a : p_1\} \ , \ \{c : p_2\}\} \ / \ \{a, c\} : p_1 \diamond p_2$$

It is not enough to check only pairs of rules.

- A canonical signed calculus G is *coherent* if $\Theta_1 \cup ... \cup \Theta_m$ is inconsistent (i.e., the empty sequent can be derived by cuts) whenever $\Theta_1/S_1 : \diamond(p_1, \ldots, p_n), ..., \Theta_m/S_m : \diamond(p_1, \ldots, p_n)$ is a set of rules of G, such that $S_1 \cap ... \cap S_m = \emptyset$.

Examples

Coherent:

$$\{\{\mathbf{f}:p_1,\mathbf{f}:p_2\}\}\ /\ \{\mathbf{f}\}:p_1\wedge p_2 \qquad \{\{\mathbf{t}:p_1\},\{\mathbf{t}:p_2\}\}\ /\ \{\mathbf{t}\}:p_1\wedge p_2$$

$$\cfrac{\cfrac{\{\mathbf{t}:p_1\}\quad \{\mathbf{f}:p_1,\mathbf{f}:p_2\}}{\{\mathbf{f}:p_2\}}\ \mathbf{cut} \qquad \{\mathbf{t}:p_2\}}{\emptyset}\ \mathbf{cut}$$

Non-coherent:

$$\{\{a:p_1\},\{b:p_2\}\}\ /\ \{a,b\}:\circ(p_1,p_2,p_3)$$

$$\{\{a:p_2,c:p_3\}\}\ /\ \{c\}:\circ(p_1,p_2,p_3)$$

The set $\{\{a:p_1\},\{b:p_2\},\{a:p_2,c:p_3\}\}$ is satisfiable (and thus consistent).

Semantics for Canonical Signed Calculi

- We use Nmatrices of the form $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for canonical calculi over a set of signs $\mathcal{V}$.
- For a set of sets of signed formulas Θ and a set of signed formulas Ω: $\Theta \vdash_{\mathcal{M}} \Omega$ if whenever an $\mathcal{M}$-legal valuation satisfies Θ, it satisfies some signed formula of Ω.
- The connection: $\Gamma \vdash_{\mathcal{M}} \Delta$ iff $\vdash_{\mathcal{M}} \{\mathcal{D} : \psi \mid \psi \in \Delta\} \cup \{\mathcal{V} - \mathcal{D} : \psi \mid \psi \in \Gamma\}$

"Reading off" the Semantics from Canonical Rules

Let $\mathcal{V} = \{t, f, \top, \bot\}$. Let G be a calculus with no rules for $\circ$.

$\circ$	t	f	$\top$	$\bot$
t	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$
f	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$
$\top$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$
$\bot$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$	$\mathcal{V}$

"Reading off" the Semantics from Canonical Rules

Add the rule

$$\{\{f : p_1, f : p_2\}\} \;/\; \{\bot, f\} : p_1 \circ p_2$$

$\circ$	t	f	$\top$	$\bot$
t	$\mathcal{V}$	$\{\bot, \mathbf{f}\}$	$\mathcal{V}$	$\mathcal{V}$
f	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$
$\top$	$\mathcal{V}$	$\{\bot, \mathbf{f}\}$	$\mathcal{V}$	$\mathcal{V}$
$\bot$	$\mathcal{V}$	$\{\bot, \mathbf{f}\}$	$\mathcal{V}$	$\mathcal{V}$

“Reading off” the Semantics from Canonical Rules

Add the rule

$$\{\{t : p_1, \top : p_1\}\} \;/\; \{f\} : p_1 \circ p_2$$

$\circ$	t	f	$\top$	$\bot$
t	$\{\mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
f	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\bot, \mathbf{f}\}$
$\top$	$\{\mathbf{f}\}$	$\{\bot, \mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\bot$	$\mathcal{V}$	$\{\bot, \mathbf{f}\}$	$\mathcal{V}$	$\mathcal{V}$

Another Exact Correspondence

Let G be a canonical calculus. The following statements concerning G are equivalent.

1. G is coherent.
2. G has a strongly characteristic Nmatrix.
3. G admits strong analytic cut-elimination.
4. G admits analytic cut-elimination.

What about (Strong) Cut-elimination?

$$\{\{a : p_1\}\} \;/\; \{b, c\} : p_1 \circ p_2 \qquad \{\{a : p_1\}\} \;/\; \{a, b\} : p_1 \circ p_2$$

This calculus is coherent.

The sequent $\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}$ has an analytic proof:

$$\dfrac{\dfrac{\{a : p_1, b : p_1, c : p_1\}}{\{b : p_1, c : p_1, b : (p_1 \circ p_2), c : (p_1 \circ p_2)\}} \qquad \dfrac{\{a : p_1, b : p_1, c : p_1\}}{\{b : p_1, c : p_1, a : (p_1 \circ p_2), b : (p_1 \circ p_2)\}}}{\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}}$$

However, it has no cut-free proof.

Coherence does not imply strong cut-elimination. However, by adding the rule $\{\{a : p_1\}\} \;/\; \{b\} : p_1 \circ p_2$ *strong cut-elimination is guaranteed.*

Density

- A canonical signed calculus G is *dense* if for every two rules of G $\Theta_1/S_1 : \diamond(p_1, \ldots, p_n)$ and $\Theta_2/S_2 : \diamond(p_1, \ldots, p_n)$ in G there is some rule $\Theta/S : \diamond(p_1, \ldots, p_n)$, such that $\Theta \subseteq \Theta_1 \cup \Theta_2$ and $S \subseteq S_1 \cap S_2$.

- Density implies coherence.

- The following calculus is not dense:

$$\{\{a : p_1\}\} \;/\; \{a, b\} : p_1 \circ p_2 \qquad \{\{a : p_1\}\} \;/\; \{b, c\} : p_1 \circ p_2$$

Correction: add $\{\{a : p_1\}\} \;/\; \{b\} : p_1 \circ p_2$*, obtaining an equivalent calculus.*

Density Characterizes Strong Cut-elimination

Let G be a canonical calculus. The following statements concerning G are equivalent:

1. G is dense.
2. G admits cut-elimination.
3. G admits strong cut-elimination.

Quantifiers: the First-order Case and Beyond

- A unary quantifier:

$$\forall x \psi \qquad \exists x \psi$$

- An n-ary quantifier: $\mathcal{Q}x(\psi_1, \ldots, \psi_n)$

$$\overline{\forall} x(P(x), Q(x)) \equiv \forall x(P(x) \rightarrow Q(x))$$

$$\overline{\exists} x(P(x), Q(x)) \equiv \exists x(P(x) \wedge Q(x))$$

Canonical Rules with Quantifiers

Universal quantification rules:

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}$$

where z is free for w in A, z is not free in $\Gamma \cup \Delta \cup \{\forall w A\}$, and $\mathbf{t}$ is any term free for w in A.

$$\Downarrow$$

$$\frac{A\{\mathbf{t}/w\} \Rightarrow}{\forall w\, A \Rightarrow} \qquad \frac{\Rightarrow A\{z/w\}}{\Rightarrow \forall w\, A}$$

$$\Downarrow$$

$$\{p(c) \Rightarrow\}/\forall w\, p(w) \Rightarrow \qquad \{\Rightarrow p(y)\}/ \Rightarrow \forall w\, p(w)$$

Important: is an eigenvariable or a term used? Signify eigenvariable by a variable, and a term by a constant.

L^n - a Representation Language for Canonical Rules

- Represent $\mathcal{Q}x(\psi_1, ..., \psi_n)$ by $\mathcal{Q}x(p_1(x), ..., p_n(x))$.
- For an n-ary introduction rule, L^n has n unary predicate symbols $p_1, ..., p_n$ and a set of constants *(no logical connectives)*.
- Represent the case of a term-variable ($\mathbf{t}$) by a constant, and the case of an eigenvariable (y) by a variable.

Canonical n-ary Rules

A canonical rule of arity n is an expression of one of the forms:

$$\frac{\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}}{\Rightarrow \mathcal{Q}x(p_1(x),...,p_n(x))} \qquad \frac{\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}}{\mathcal{Q}x(p_1(x),...,p_n(x)) \Rightarrow}$$

where $\mathcal{Q}$ is an n-ary quantifier, $m \geq 0$ and for every $1 \leq i \leq m$:
$\Pi_i \Rightarrow \Sigma_i$ is a clause over L^n.

Instantiating Canonical Rules

Let $R = \Theta / \Rightarrow \mathcal{Q}x(p_1(x), \ldots, p_n(x))$ be an n-ary canonical rule, Γ - some context of L-formulas, and z a variable of L. A $\langle R, \Gamma, z\rangle$-mapping is any function $\mathcal{F}$ from the predicate symbols and terms of L^n to formulas and terms of L, satisfying:

- For every $1 \leq i \leq n$, $\mathcal{F}[p_i]$ is an L-formula, $\mathcal{F}[y]$ is a variable of L and $\mathcal{F}[c]$ is an L-term.
- $\mathcal{F}[x] \neq \mathcal{F}[y]$ for every two variables $x \neq y$ of L^n.
- For every $1 \leq i \leq n$ and every $p_i(\mathbf{t})$ occurring in Θ:
  $\mathcal{F}[\mathbf{t}]$ is a term free for z in $\mathcal{F}[p_i]$.
  - if $\mathbf{t}$ is a variable, then $\mathcal{F}[\mathbf{t}]$ does not occur free in
  $\Gamma \cup \{\mathcal{Q}z(\mathcal{F}[p_1], ..., \mathcal{F}[p_n])\}$.
- $\mathcal{F}[p_i(\mathbf{t})] = \mathcal{F}[p_i]\{\mathcal{F}[\mathbf{t}]/z\}$.

$\mathcal{F}$ is extended to sets of formulas: $\mathcal{F}(\Gamma) = \{\mathcal{F}(\psi) \mid \psi \in \Gamma\}$.

Applications of Canonical Rules

An application of $R = \{\Sigma_j \Rightarrow \Pi_j\}_{1 \leq j \leq m} \ / \ \Rightarrow \mathcal{Q}x(p_1(x), ..., p_n(x))$:

$$\frac{\{\Gamma, \mathcal{F}(\Sigma_j) \Rightarrow \Delta, \mathcal{F}(\Pi_j)\}_{1 \leq j \leq m}}{\Gamma \Rightarrow \Delta, \mathcal{Q}z(\mathcal{F}(p_1), ..., \mathcal{F}(p_n))}$$

where $\mathcal{F}$ is some $\langle R, \Gamma \cup \Delta, z \rangle$-mapping.

Example 1

The two standard introduction rules for the unary quantifier $\forall$ can be formulated as follows:

$$\{p(c) \Rightarrow\} \;/\; \forall x\, p(x) \Rightarrow \qquad \{\Rightarrow p(y)\} \;/\; \Rightarrow \forall x\, p(x)$$

Applications of these rules have the forms:

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}$$

where z is free for w in A, z is not free in $\Gamma \cup \Delta \cup \{\forall w A\}$, and $\mathbf{t}$ is any term free for w in A.

Example 2

Consider the bounded universal binary quantifier $\overline{\forall}$ (corresponding to $\forall x(p_1(x) \to p_2(x))$).

$$\{p_2(c) \Rightarrow\ ;\ \Rightarrow p_1(c)\}\ /\ \overline{\forall} x\ (p_1(x), p_2(x)) \Rightarrow$$

$$\{p_1(y) \Rightarrow p_2(y)\}\ /\ \Rightarrow \overline{\forall} x\ (p_1(x), p_2(x))$$

Applications of these rules are of the form:

$$\frac{\Gamma, \psi_2\{\mathbf{t}/z\} \Rightarrow \Delta \quad \Gamma \Rightarrow \psi_1\{\mathbf{t}/z\}, \Delta}{\Gamma, \overline{\forall} z\ (\psi_1, \psi_2) \Rightarrow \Delta} \qquad \frac{\Gamma, \psi_1\{w/z\} \Rightarrow \psi_2\{w/z\}, \Delta}{\Gamma \Rightarrow \overline{\forall} z\ (\psi_1, \psi_2), \Delta}$$

where $\mathbf{t}$ and w are free for z in ψ_1 and ψ_2, and w does not occur free in $\Gamma \cup \Delta \cup \{\overline{\forall} z(\psi_1, \psi_2)\}$.

Coherence

- A canonical calculus G is coherent if for every two canonical rules of G of the form $\Theta_1 / \Rightarrow A$ and $\Theta_2 / \; A \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent.
- *The coherence of a canonical calculus G is decidable.*
- Examples: Coherent:

$$\{p(c) \Rightarrow\} \; / \; \forall x \, p(x) \Rightarrow \qquad \{\Rightarrow p(y)\} \; / \; \Rightarrow \forall x \, p(x)$$

  Non-coherent:

$$\{\Rightarrow p(c)\} \; / \; \Rightarrow \mathcal{Q}xp(x) \qquad \{p(d) \Rightarrow\} \; / \; \mathcal{Q}xp(x) \Rightarrow$$

Canonical Calculi with Quantifiers and α-Equivalence

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}$$

$\forall x P(x) \Rightarrow \forall y P(y)$ is derivable:

$$\frac{\dfrac{P(y) \Rightarrow P(y)}{\forall x P(x) \Rightarrow P(y)}}{\forall x P(x) \Rightarrow \forall y P(y)}$$

α-Equivalence Axiom

$$\frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}$$

After discarding one rule for $\forall$, $\forall x P(x) \Rightarrow \forall y P(y)$ is no longer derivable.

Solution: add to canonical calculi an explicit axiom capturing the α-equivalence principle:

$$(\alpha) \;\; A \Rightarrow A' \;\; if \;\; A \equiv_\alpha A'$$

Canonical Calculi

- A substitution instance of $\Gamma \Rightarrow \Delta$ is any sequent of the form $\Gamma\{\mathbf{t}_1/x_1, \ldots, \mathbf{t}_n/x_n\} \Rightarrow \Delta\{\mathbf{t}_1/x_1, \ldots, \mathbf{t}_n/x_n\}$, where $\mathbf{t}_i$ is free for x_i in all the formulas of $\Gamma \cup \Delta$.

- The substitution rule:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'}\ Sub$$

  where $\Gamma' \Rightarrow \Delta'$ is a substitution instance of $\Gamma \Rightarrow \Delta$.

- A Gentzen-type calculus G with quantifiers is canonical if in addition to the α-axiom, the substitution rule and the standard structural rules, G has only canonical rules.

Unary Quantifiers in Deterministic Matrices

- A unary quantifier $\mathcal{Q}$ is usually interpreted as a function $\tilde{\mathcal{Q}} : P^{+}(\mathcal{V}) \rightarrow \mathcal{V}$.

- Examples:

H	$\tilde{\forall}$[H]
{t}	t
{t,f}	f
{f}	f

H	$\tilde{\exists}$[H]
{t}	t
{t,f}	t
{f}	f

- A natural generalization to Nmatrices: $\tilde{\mathcal{Q}} : P^{+}(\mathcal{V}) \rightarrow P^{+}(\mathcal{V})$.

Nmatrices with Unary Quantifiers

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a non-deterministic matrix (Nmatrix) for a language L with unary quantifiers if:

1. $\mathcal{V}$ is a nonempty set
2. $\emptyset \neq \mathcal{D} \subset \mathcal{V}$
3. for every n-ary connective $\diamond$ of L, $\mathcal{O}$ includes an operation $\widetilde{\diamond} : \mathcal{V}^n \to P^+(\mathcal{V})$
4. for every unary quantifier $\mathcal{Q}$ of L, $\mathcal{O}$ includes an operation $\widetilde{\mathcal{Q}} : P^+(\mathcal{V}) \to P^+(\mathcal{V})$.

Example

Consider the two-valued Nmatrix $\mathcal{M}_1 = \langle\{t, f\}, \{t\}, \mathcal{O}\rangle$ for a language L over $\{\mathcal{Q}, \forall, \neg\}$, where $\mathcal{O}$ contains the following operations:

H	$\tilde{\mathcal{Q}}$[H]
{t}	{t}
{t,f}	{t,f}
{f}	{f}

H	$\tilde{\forall}$[H]
{t}	{t}
{t,f}	{f}
{f}	{f}

a	$\neg a$
t	{t,f}
f	{t}

L-structures

An L-structure for an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a pair $S = \langle D, I \rangle$, where D is a non-empty domain and I satisfies the following properties:

- For every constant c of L: $I[c] \in D$.
- For an n-ary predicate symbol p of L: $I[p] : D^n \to \mathcal{V}$.
- For every n-ary function symbol f of L: $I[f] : D^n \to D$.

I is extended to interpret closed terms of L as follows:

$$I[f(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[f][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$$

For $S = \langle D, I \rangle$, the language extended by individual constants $\{\overline{a} \mid a \in D\}$ is denoted by $L(D)$. I is extended to interpret closed terms of $L(D)$: $I(\overline{a}) = a$.

Naive Definition of Valuations in Nmatrices

Let $S = \langle D, I \rangle$ be an L-structure. A *valuation* in an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for L is a function v from closed sentences of $L(D)$ to $\mathcal{V}$, satisfying:

- $v[p[\mathbf{t}_1, ..., \mathbf{t}_n]] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$.
- $v[\diamond[\psi_1, \ldots, \psi_n]] \in \widetilde{\diamond}[v[\psi_1], \ldots, v[\psi_n]]$.
- $v[\mathcal{Q}x\psi] \in \widetilde{\mathcal{Q}}[\{v[\psi\{\overline{a}/x\}] \mid a \in D\}]$.

Reflection of the Problem of α-Equivalence

Two α-equivalent sentences are not necessarily assigned the same truth-value:

H	$\tilde{\forall}$[H]
{t}	{t}
{t,f}	{f}
{f}	{f}

a	$\neg a$
t	{t,f}
f	{t}

Let $S = \langle \{a, b\}, I \rangle$, $I[p][a] = I[p][b] = t$.

Consider: $\neg \forall x p(x)$ and $\neg \forall y p(y)$

The Definition of a Non-deterministic Valuation - Correction

Let $S = \langle D, I \rangle$ be an L-structure. A *valuation* in an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for L is a function v from closed sentences of $L(D)$ to $\mathcal{V}$ satisfying:

- If $\psi_1 \equiv_\alpha \psi_2$, then $v[\psi_1] = v[\psi_2]$.
- $v[p[\mathbf{t}_1, ..., \mathbf{t}_n]] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$.
- $v[\diamond[\psi_1, \ldots, \psi_n]] \in \widetilde{\diamond}[v[\psi_1], \ldots, v[\psi_n]]$.
- $v[\mathcal{Q}x\psi] \in \widetilde{\mathcal{Q}}[\{v[\psi\{\overline{a}/x\}] \mid a \in D\}]$.

Other Problems

- Terms denoting the same objects cannot be used interchangeably.
- Void quantification for first-order quantifiers $\forall$ and $\exists$ *(e.g, first-order paraconsistent logics of da Costa)*.

  Consider: (i) $\neg p(c)$ and $\neg p(d)$, (ii) $\neg \forall x p(c)$ and $\neg p(c)$.

Solution: add appropriate congruence relations.

For instance, $A \sim_{void} QxA$ *if* $x \notin Fv(A)$.

Nmatrices with n-ary Quantifiers

- An n-ary quantifier $\mathcal{Q}$ in an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is interpreted by a function $\tilde{\mathcal{Q}} : P^+(\mathcal{V}^n) \to P^+(\mathcal{V})$.
- Example: for every $\mathcal{E} \in P^+(\{t, f\}^2)$:

$$\tilde{\tilde{\forall}}[\mathcal{E}] = \begin{cases} \{t\} & \textit{if } \langle t, f \rangle \notin \mathcal{E} \\ \{f\} & \textit{otherwise} \end{cases} \qquad \tilde{\tilde{\exists}}[\mathcal{E}] = \begin{cases} \{t\} & \textit{if } \langle t, t \rangle \in \mathcal{E} \\ \{f\} & \textit{otherwise} \end{cases}$$

The definition of an $\mathcal{M}$-valuation v is now modified as follows:

$$v[\mathcal{Q}x(\psi_1, ..., \psi_n)] \in \tilde{\mathcal{Q}}_{\mathcal{M}}[\{\langle v[\psi_1\{\overline{a}/x\}], ..., v(\psi_n\{\overline{a}/x\})\rangle \mid a \in D\}]$$

Example

$\mathbf{H}$	$\tilde{\tilde{\forall}}(\mathbf{H})$	$\tilde{\tilde{\exists}}(\mathbf{H})$	$\tilde{\mathcal{Q}}_2(\mathbf{H})$
$\{\langle \mathbf{t}, \mathbf{t}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{f}, \mathbf{f}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{f}, \mathbf{t}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t}\rangle, \langle \mathbf{t}, \mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t}\rangle, \langle \mathbf{f}, \mathbf{t}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t}\rangle, \langle \mathbf{f}, \mathbf{f}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{f}, \mathbf{t}\rangle, \langle \mathbf{t}, \mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{f}, \mathbf{t}\rangle, \langle \mathbf{f}, \mathbf{f}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{t}, \mathbf{f}\rangle, \langle \mathbf{f}, \mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{t}, \mathbf{t}\rangle, \langle \mathbf{t}, \mathbf{f}\rangle, \langle \mathbf{f}, \mathbf{t}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t}\rangle, \langle \mathbf{f}, \mathbf{f}\rangle, \langle \mathbf{f}, \mathbf{t}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{f}, \mathbf{t}\rangle, \langle \mathbf{t}, \mathbf{f}\rangle, \langle \mathbf{f}, \mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle \mathbf{f}, \mathbf{f}\rangle, \langle \mathbf{t}, \mathbf{f}\rangle, \langle \mathbf{f}, \mathbf{t}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\langle \mathbf{t}, \mathbf{t}\rangle, \langle \mathbf{t}, \mathbf{f}\rangle, \langle \mathbf{f}, \mathbf{t}\rangle, \langle \mathbf{f}, \mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

Construction of a Characteristic 2Nmatrix

$$\frac{\Rightarrow p(c_1) \quad p(c_2) \Rightarrow}{\Rightarrow \mathcal{Q}xp(x)} \qquad \frac{\Rightarrow p(y)}{\mathcal{Q}xp(x) \Rightarrow}$$

$\tilde{\mathcal{Q}}[\{t\}] = ?$

$\tilde{\mathcal{Q}}[\{t, f\}] = ?$

$\tilde{\mathcal{Q}}[\{f\}] = ?$

H	$\tilde{\mathcal{Q}}[H]$
$\{t\}$	$?$
$\{f, t\}$	$?$
$\{f\}$	$?$

Construction of a Characteristic 2Nmatrix

$$\frac{\Rightarrow p(c_1) \quad p(c_2) \Rightarrow}{\Rightarrow \mathcal{Q}xp(x)} \qquad \frac{\Rightarrow p(y)}{\mathcal{Q}xp(x) \Rightarrow}$$

$\tilde{\mathcal{Q}}[\{t\}] = \{f\}$ ($\{\Rightarrow p(y)\}$ is valid in a structure with distribution $\{t\}$)

$\tilde{\mathcal{Q}}[\{t, f\}] =?$

$\tilde{\mathcal{Q}}[\{f\}] =?$

H	$\tilde{\mathcal{Q}}[H]$
$\{t\}$	$\{f\}$
$\{f, t\}$	?
$\{f\}$	?

Construction of a Characteristic 2Nmatrix

$$\frac{\Rightarrow p(c_1) \quad p(c_2) \Rightarrow}{\Rightarrow \mathcal{Q}xp(x)} \qquad \frac{\Rightarrow p(y)}{\mathcal{Q}xp(x) \Rightarrow}$$

$\tilde{\mathcal{Q}}[\{t\}] = \{f\}$ ($\{\Rightarrow p(y)\}$ is valid in a structure with distribution $\{t\}$)

$\tilde{\mathcal{Q}}[\{t, f\}] = \{t\}$ ($\{\Rightarrow p(c_1), p(c_2) \Rightarrow\}$ is valid in a structure with distribution $\{f, t\}$)

$\tilde{\mathcal{Q}}[\{f\}] = ?$

H	$\tilde{\mathcal{Q}}[H]$
$\{t\}$	$\{f\}$
$\{f, t\}$	$\{t\}$
$\{f\}$	$?$

How to Construct a Characteristic 2Nmatrix?

$$\frac{\Rightarrow p(c_1) \quad p(c_2) \Rightarrow}{\Rightarrow \mathcal{Q}xp(x)} \qquad \frac{\Rightarrow p(y)}{\mathcal{Q}xp(x) \Rightarrow}$$

$\tilde{\mathcal{Q}}[\{t\}] = \{f\}$ ($\{\Rightarrow p(y)\}$ is valid in a structure with distribution $\{t\}$)

$\tilde{\mathcal{Q}}[\{t, f\}] = \{t\}$ ($\{\Rightarrow p(c_1), p(c_2) \Rightarrow\}$ is valid in a structure with distribution $\{f, t\}$)
$\tilde{\mathcal{Q}}[\{f\}] = \{f, t\}$ (There is no relevant rule)

H	$\tilde{\mathcal{Q}}[H]$
$\{t\}$	$\{f\}$
$\{f, t\}$	$\{t\}$
$\{f\}$	$\{t, f\}$

Cut-elimination

For every canonical calculus G:

- G is coherent iff it has a (strongly) characteristic 2Nmatrix.
- If G is coherent, then it admits cut-elimination.
- But cut-elimination does not imply coherence! *We can construct a calculus with binary quantifiers which is not coherent, but the only sequents provable in it are logical axioms.*

Exact Correspondence Still Holds for Strong Cut-elimination

The following statements concerning a canonical system G with n-ary quantifiers are equivalent:

1. G is coherent.
2. G has a strongly characteristic 2Nmatrix.
3. G admits strong cut elimination.