Cut-free Sequent Calculi for C-systems with Generalized Finite-valued Semantics

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Abstract

In [5], a general method was developed for generating cut-free ordinary sequent calculi for logics that can be characterized by finite-valued semantics based on non-deterministic matrices (Nmatrices). In this paper, a substantial step towards automation of paraconsistent reasoning is made by applying that method to a certain crucial family of thousands of paraconsistent logics, all belonging to the class of *C-systems*. For that family, the method produces in a modular way uniform Gentzen-type rules corresponding to a variety of axioms considered in the literature.

1 Introduction

One of the main drawbacks of classical logic (CL) is that it fails to accommodate the fact that knowledge bases containing contradictory data may still produce useful answers to queries. This is because in CL a single inconsistency leads to trivialization of the whole knowledge base. Accordingly, over the last decades there has been a growing interest in computer science applications of paraconsistent logics — logics which allow non-trivial inconsistent theories. [1] Integration of information from multiple sources in large knowledge bases, negotiations among agents with conflicting goals, and complex software specifications in which different stake-holders have incompatible requirements are just a few cases in point. Recently, suggestions have even been made (see, e.g., [27]) to adopt paraconsistent logic as a foundational concept for future information systems engineering.

One of the oldest and best known approaches to paraconsistency is da Costa's approach ([24, 25, 28]), which seeks to allow the use of classical logic whenever it is safe to do so, but behaves completely differently when contradictions are involved. This approach has led to the introduction of the family of *Logics of Formal (In)consistency (LFIs)* ([17, 18]).[2] The LFI family is based on two key ideas. The first is that propositions should be divided into two sorts: the "normal" (or consistent), and the "abnormal" (or inconsistent) ones.

[1] (See, e.g., [19, 26, 33] for some concrete examples of applications of paraconsistent logics in computer science.

[2] The reason we write "(In)consistency" rather than "Inconsistency" is that the primitive operator $\circ$ used in this paper (and in most other works on LFIs) actually denotes *consistency* rather than inconsistency. In fact, the inconsistency of φ can usually be expressed by $\varphi \wedge \neg\varphi$ (or $\neg\varphi \wedge \varphi$). In contrast, formally expressing consistency is a much more subtle problem.

While classical logic can be applied freely to normal propositions, its application to the abnormal ones is restricted. The second idea is to reflect this classification within the language used. In the most important class of LFIs called *C-systems* ([18]), this is done by employing a special (either primitive or defined) connective $\circ$, where the intuitive meaning of $\circ\varphi$ is "φ is consistent" (or "φ is normal").

For a long time, the class of C-systems had two major shortcomings, which in our opinion prevented it from becoming a widely-used logical formalism for reasoning with inconsistent data and theories. The first was that originally those systems lacked corresponding intuitive and useful semantics which would provide real insight into them. Later, bivaluations semantics and possible translations semantics were introduced for those systems ([20, 30, 17, 18]). However, both those types of semantics are problematic from the crucial viewpoint of *analyticity*. Roughly, a semantics is *analytic* if to determine whether φ follows from T one need not consider the set of complete "orthodox" models, but only the *parts* of those models which involve the subformulas of $T \cup \{\varphi\}$.[3] This property usually guarantees decidability if T is finite. Unfortunately, neither bivaluations semantics nor possible translations semantics are satisfactory in this respect, as their analyticity is not guaranteed a priori. Accordingly, the corresponding proposition should be proved from scratch (if it is true at all) for any potentially useful instance of these types of semantics. This unfortunate state of affairs was finally remedied in [4, 1, 3], where simple, modular and analytic semantics for practically all the propositional C-systems considered in the literature were introduced. Those semantics were based on the use of *non-deterministic matrices (Nmatrices)*, which provide a natural (and still analytic) generalization of the class of many-valued matrices.[4] In this generalization, the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options. The analyticity of this kind of semantics guarantees that a logic which has a finite characteristic Nmatrix is necessarily decidable.

The second shortcoming of C-systems was that their formulation was originally given in terms of Hilbert-type calculi, and for many years no analytic[5] calculi were available for most of them. At first, most of the efforts towards finding such calculi concentrated on da Costa's historical system C_1. After an aborted attempt by Raggio in the sixties ([34]), Beziau proposed in [14] somewhat peculiar sequent rules for C_1, using an intuitive translation of certain semantical conditions. Later he proved a general completeness theorem which explains why this intuitive translation works. Proving cut-elimination using his "monstrous" rules (as he himself described them in [15]) was another non-trivial task. At about the same time, Carnielli et al. introduced a tableau system for C_1 ([21, 17, 22]). Recently some analytic calculi have been intro-

[3] An exact definition of analyticity for the case when "models" are certain functions defined for all formulas of the language is implicit in Proposition 18 below.

[4] In turn, bivaluations semantics, and especially possible translations semantics, can be viewed as a generalization of the semantics of Nmatrices (see [16]) — but one in which the property of analyticity is *lost*.

[5] Note that in this context we have in mind the usual syntactic analyticity of a calculus, as opposed to the semantic analyticity described above.

duced also for a few other C-systems ([32, 33, 29]). However, since each of those calculi was tailored to some specific system, their rules were introduced in a sort of an ad-hoc manner, and so they do not have a uniform structure. Therefore, even a slight modification in any of those systems would practically mean starting the search for a corresponding analytic calculus all over again.

In this paper we show that for a very large class of C-systems a remedy for the above deficiency consists in applying the algorithm given in [5] for constructing an analytic Gentzen-type system for a logic with a characteristic finite-valued Nmatrix whose language is sufficiently expressive for that Nmatrix. The resulting sequent calculus automatically enjoys cut-admissibility, and its rules have a uniform form, closely related to that used in calculi for classical logic and other well known calculi. Based on that algorithm, we provide a *uniform* and *modular* method for a *systematic* generation of cut-free sequent calculi for a very large family of thousands of C-systems[6]. We believe that these results can open the door to construction and implementation of efficient theorem provers based on this type of paraconsistent logics, which in turn will lead to their useful new applications for reasoning under uncertainty and inconsistency.

2 Preliminaries

In what follows, $\mathcal{L}$ is a propositional language, and $Frm_{\mathcal{L}}$ is its set of wffs. The metavariables φ, ψ range over $\mathcal{L}$-formulas, p, q range over atomic formulas, T, S range over sets of $\mathcal{L}$-formulas, and Γ, Δ range over finite sets of $\mathcal{L}$-formulas.

Definition 1 A (Tarskian) *consequence relation* (tcr) for a language $\mathcal{L}$ is a binary relation $\vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, satisfying the following three conditions:

Reflexivity: if $\psi \in T$ then $T \vdash \psi$.

Monotonicity: if $T \vdash \psi$ and $T \subseteq T'$ then $T' \vdash \psi$.

Transitivity: if $T \vdash \psi$ and $T, \psi \vdash \varphi$ then $T \vdash \varphi$.

Definition 2 A *propositional logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a tcr for $\mathcal{L}$ which satisfies the following:

Structurality: if $T \vdash \varphi$ then $\sigma(T) \vdash \sigma(\varphi)$, where σ is a substitution in $\mathcal{L}$.

Non-triviality: $p \not\vdash q$, where p and q are distinct propositional variables.

The notion of paraconsistency (with respect to $\neg$) is usually defined as follows (see, e.g., [17]):

Definition 3 Let $\mathcal{L}$ be a language which includes a unary connective $\neg$. A propositional logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is *paraconsistent* (with respect to $\neg$) if there are formulas $\psi, \varphi \in Frm_{\mathcal{L}}$ such that $\psi, \neg\psi \not\vdash \varphi$.[7]

[6] A preliminary description of the method, together with an illustrative example, was given in [8].

[7] As $\vdash$ is structural, it is enough to require that there are propositional variables p, q such that $p, \neg p \not\vdash q$.

2.1 A Taxonomy of LFIs

Notation 4 Let $\mathcal{L}_{cl}$ be the classical propositional language with the set of connectives $\{\wedge, \vee, \supset, \neg\}$, and let $\mathcal{L}_{cl}^{+}$ be its positive fragment (i.e., $\mathcal{L}_{cl}^{+}$ is the propositional language whose set of connectives is $\{\wedge, \vee, \supset\}$).

Logics of Formal (In)consistency (LFIs) form a large family of paraconsistent logics, in which the notion of consistency is expressed in the language of the logic itself. Namely, a paraconsistent logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is an LFI if there is a propositional variable p and a set $X(p)$ of $\mathcal{L}$-formulas containing only the variable p such that $\psi, \neg\psi, X\{\psi/p\} \vdash \varphi$[8] for every $\psi, \varphi \in Frm_{\mathcal{L}}$. A particularly useful subclass of LFIs is that of *C-systems*, in which $X(p)$ is a singleton:

Definition 5 Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with $\mathcal{L}$ containing $\mathcal{L}_{cl}$. We say that $\mathbf{L}$ is a *C-system* if the following holds:

1. $\mathbf{L}$ contains the $\mathcal{L}_{cl}^{+}$-fragment of classical logic,
2. $\mathbf{L}$ is paraconsistent,
3. $\mathbf{L}$ has a (primitive or defined) unary connective $\circ$, for which the following axioms are valid in $\mathbf{L}$:

$$(\mathbf{t})\ \neg\varphi \vee \varphi \qquad (\mathbf{b})\ \circ\varphi \supset (\varphi \wedge \neg\varphi \supset \psi) \qquad (\mathbf{k})\ \circ\varphi \vee (\varphi \wedge \neg\varphi)$$

Notation 6 In what follows, we take $\circ$ to be a primitive connective of the language. We shall denote by $\mathcal{L}_C$ the propositional language whose set of connectives is $\{\wedge, \vee, \supset, \neg, \circ\}$.

Definition 7 Let $\mathbf{HCL}^{+}$ be a standard Hilbert-style system which has Modus Ponens as the only inference rule, and is sound and strongly complete for the positive fragment (i.e., the $\mathcal{L}_{cl}^{+}$-fragment) of classical propositional logic.

1. The system $\mathbf{B}$ for $\mathcal{L}_C$ is obtained by adding to $\mathbf{HCL}^{+}$ the axioms ($\mathbf{t}$) and ($\mathbf{b}$).
2. The system $\mathbf{BK}$ (for $\mathcal{L}_C$) is obtained by adding to $\mathbf{B}$ the axiom ($\mathbf{k}$).

Remark 8 According to our definition of a C-system, the system **BK** introduced above is the minimal (and most basic) C-system. It should be noted, though, that our notion of a "C-system" is somewhat narrower than that used in [17, 18], and corresponds to what would be called there "a C-system based on classical logic in which axioms ($\mathbf{t}$) and ($\mathbf{k}$) are valid". However, *all* the "C-systems" which are studied in [17, 18] are based on classical logic, and in all of them ($\mathbf{t}$) is valid. Therefore the actual difference is that in [17, 18] the validity of ($\mathbf{k}$) is not required. Accordingly, the system **B** (called **mbC** there) is the one which is considered there to be the most basic C-system. Nevertheless, we find it much more appropriate to choose **BK** for this role, for the following reasons:

[8]The substitution $X\{\psi/p\}$ is understood in the standard way.

1. Given the intended meaning of $\circ\varphi$ as "φ is consistent", the meaning of axiom **(b)** is that no formula is both consistent and contradictory. Axiom **(k)** complements this by saying that every formula is either consistent or contradictory. This last principle seems to be as essential for the intended meaning of $\circ\varphi$ as that expressed by axiom **(b)**.

2. Another strong indication that **BK** is the most natural basic C-system is that in the Gentzen-type system for this logic which is presented in Subsection 3.2 the right and left introduction rules for all the connectives other than $\neg$ are dual: including one of them guarantees the invertibility (Definition 43) of the other in **BK** (see Proposition 44 below). *This also applies to the rules for $\circ$ corresponding to axioms* (**b**) *and* (**k**).

3. (**k**) is anyway a theorem of almost every important C-system ever studied. This is due to the fact that it is derivable in **B** from each of the three most important axioms concerning $\circ$ which have been studied in the literature: the axiom denoted below by (**i**), as well as axioms (**l**) $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$, and (**d**) $\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$, which are not handled in this paper[9]. Those dependencies are easily established (see Example 42 for the case of (**l**)).

Next, we provide a list of axioms which are frequently used for defining individual C-systems:

Definition 9 Let **A** be the following set of axioms for $\sharp \in \{\wedge, \supset, \vee\}$:

$$
\begin{array}{llll}
(\mathbf{c}) & \neg\neg\varphi \supset \varphi & (\mathbf{e}) & \varphi \supset \neg\neg\varphi \\
(\mathbf{i_1}) & \neg\circ\varphi \supset \varphi & (\mathbf{i_2}) & \neg\circ\varphi \supset \neg\varphi \\
(\mathbf{o}^1_\sharp) & \circ\varphi \supset \circ(\varphi\sharp\psi) & (\mathbf{o}^2_\sharp) & \circ\psi \supset \circ(\varphi\sharp\psi) \\
(\mathbf{a}_\sharp) & (\circ\varphi \wedge \circ\psi) \supset \circ(\varphi\sharp\psi) & (\mathbf{a}_\neg) & \circ\varphi \supset \circ\neg\varphi
\end{array}
$$

Remark 10 The literature on LFIs usually mentions an axiom denoted by **(i)** which is the conjunction of our ($\mathbf{i_1}$) and ($\mathbf{i_2}$). Similarly, the axiom ($\mathbf{o}_\sharp$) frequently mentioned in the literature is the conjunction of our ($\mathbf{o}^1_\sharp$) and ($\mathbf{o}^2_\sharp$). Note also that the extensions of **B** with **(c)**, and with both **(c)** and **(i)**, are denoted in [29] by **bC** and **Ci**, respectively.

Definition 11 For any $A \subseteq \mathbf{A}$, $\mathbf{BK}[A]$ is the system obtained out of **BK** by extending it with the axioms from A.

Notation 12 In the sequel we shall usually omit the various brackets, and write, e.g., $\mathbf{BKco}^1_\wedge$ instead of $\mathbf{BK}[\{(\mathbf{c}), (\mathbf{o}^1_\wedge)\}]$. Moreover, we shall write, e.g., **BKi** instead of $\mathbf{BK}[\{(\mathbf{i_1}), (\mathbf{i_2})\}]$ and **BKa** instead of $\mathbf{BK}[\{(\mathbf{a}_\wedge), (\mathbf{a}_\vee), (\mathbf{a}_\supset)\}]$. We will also use similar abbreviations for the (**o**)-axioms.

[9] Unfortunately, the problem of handling axioms (**l**) and (**d**) is outside the scope of this paper, because (as shown in [4]) C-systems which include one of them cannot be given a finite semantic characterization in terms of Nmatrices. As result, the method used in this paper does not apply to such systems. However, recently we have found (see [9]) that also for such systems it is possible to develop in a modular way cut-free sequent calculi (albeit of a more complex nature), but this requires an essential change in the method.

Remark 13 Not all of the systems of the form $\mathbf{BK}[A]$ for $A \subseteq \mathbf{A}$ are different from each other. Thus, e.g., $(\mathbf{a}_{\neg})$ is equivalent to $(\mathbf{c})$ in $\mathbf{BK}$. For this reason, we do not mention the former in the sequel. All other dependencies can be checked mechanically using the semantics provided below. See Corollary 32 and Remark 33 below for details on the dependencies between the axioms in $\mathbf{A}$. Another dependency not mentioned there (due to the fact that the $(\mathbf{k})$-axiom is not included in $\mathbf{A}$) is that $(\mathbf{i})$ implies $(\mathbf{k})$ in $\mathbf{BK}$.

2.2 Non-deterministic Matrices

Our main semantic tool in what follows will be the following generalization of the concept of a many-valued matrix introduced in [10, 11] (for a comprehensive survey on non-deterministic matrices, see also [13]):

Definition 14

1. A non-deterministic matrix (Nmatrix) for a language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where: $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ (the set of designated truth values) is a non-empty proper subset of $\mathcal{V}$, and $\mathcal{O}$ includes an interpretation function $\tilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to P^+(\mathcal{V})$ for every n-ary connective $\diamond$ (where $P^+(\mathcal{V})$ is the set of nonempty subsets of $\mathcal{V}$). We say that $\mathcal{M}$ is *finite* if so is $\mathcal{V}$.

2. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix. Let F be some set of $\mathcal{L}$-formulas closed under subformulas. An $\mathcal{M}$-valuation on F is a function $v : F \to \mathcal{V}$ which satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in F$ such that $\diamond(\psi_1, \ldots, \psi_n) \in F$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}_{\mathcal{M}}(v(\psi_1), \ldots, v(\psi_n))$$

   A *full $\mathcal{M}$-valuation* is an $\mathcal{M}$-valuation on $Frm_{\mathcal{L}}$.

3. Let F be as above, and let $\psi \in F$. An $\mathcal{M}$-valuation v on F *satisfies* ψ, denoted by $v \models_{\mathcal{M}} \psi$, if $v(\psi) \in \mathcal{D}$. v satisfies a set $T \subseteq F$ of formulas, denoted by $v \models_{\mathcal{M}} T$, if it satisfies every formula of T.

4. Let F be as above, and let v be an $\mathcal{M}$-valuation on F. A sequent $\Gamma \Rightarrow \Delta$ such that $\Gamma \cup \Delta \subseteq F$ is *satisfied by* v if $v \models_{\mathcal{M}} \psi$ for some $\psi \in \Delta$, or $v \not\models_{\mathcal{M}} \psi$ for some $\psi \in \Gamma$. A sequent is *valid in* $\mathcal{M}$ if it is satisfied by every full $\mathcal{M}$-valuation.

5. $\vdash_{\mathcal{M}}$, the consequence relation induced by $\mathcal{M}$, is defined by: $T \vdash_{\mathcal{M}} \psi$ if $v \models_{\mathcal{M}} \psi$ for every full $\mathcal{M}$-valuation v such that $v \models_{\mathcal{M}} T$.

Notation 15 Below we shall frequently write just $\diamond$ instead of $\tilde{\diamond}_{\mathcal{M}}$, relying on the context to indicate whether we mean the connective itself or its interpretation in some Nmatrix $\mathcal{M}$.

Nmatrices enjoy many of the attractive properties of usual (deterministic) finite-valued matrices. These properties include the following (see ([11]) for the full proofs):

Proposition 16 *For every Nmatrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.*

Proposition 17 (Compactness) *If $\mathcal{M}$ is finite, then $T \vdash_{\mathcal{M}} \psi$ implies that there is a finite $\Gamma \subseteq T$ such that $\Gamma \vdash_{\mathcal{M}} \psi$.*

Proposition 18 (Semantic Analyticity) *Let F be a set of $\mathcal{L}$-formulas closed under subformulas, and let $\mathcal{M}$ be an Nmatrix for $\mathcal{L}$. Any $\mathcal{M}$-valuation on F can be extended to a full $\mathcal{M}$-valuation.*

Corollary 19 *If $T \cup \{\psi\} \subseteq F$, then $T \vdash_{\mathcal{M}} \psi$ iff ψ is satisfied by every $\mathcal{M}$-valuation on F which satisfies T.*

Corollary 20 (Decidability) *Given a finite Nmatrix $\mathcal{M}$, a finite theory Γ, and a formula ψ, it is decidable whether $\Gamma \vdash_{\mathcal{M}} \psi$ or not.*

The following notion will be useful in the sequel:

Definition 21 Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$. $\mathcal{M}_2$ is a *simple refinement* of $\mathcal{M}_1$ if $\mathcal{V}_1 = \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2$ and for every n-ary connective $\diamond$ and every $a_1, \ldots, a_n \in \mathcal{V}_1$, $\tilde{\diamond}_{\mathcal{M}_2}(a_1, \ldots, a_n) \subseteq \tilde{\diamond}_{\mathcal{M}_1}(a_1, \ldots, a_n)$.

Proposition 22 ([4]) *If $\mathcal{M}_2$ is a simple refinement of $\mathcal{M}_1$, then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$.*

3 A Systematic Generation of Analytic Calculi

Our method for a systematic construction of Gentzen-type calculi for all the C-systems presented in this paper is based on the following two facts:

Fact 1: All the systems presented above have semantic characterizations in terms of finite-valued (in fact, three-valued) Nmatrices (this was first shown in [1], and is proved again in Subsection 3.1 below). Those characterizations can be obtained in a modular way within the finite-valued non-deterministic semantic framework developed in [1, 3, 4].

Fact 2: [5] provides an algorithm for constructing cut-free Gentzen-type systems for logics which have a characteristic finite-valued Nmatrix $\mathcal{M}$, and the language of which is *sufficiently expressive* with respect to $\mathcal{M}$ (see Definition 35 below for the meaning of this notion).

We will shortly see that the language of the C-systems studied in this paper is sufficiently expressive with respect to all the three-valued Nmatrices mentioned in Fact 1 above. Hence we can indeed exploit the algorithm mentioned in Fact 2 in order to construct cut-free Gentzen-type systems for all the paraconsistent logics presented in the foregoing.

We start by recalling some basic relevant definitions:

Definition 23 We say that an Nmatrix $\mathcal{M}$ is *characteristic* for a Gentzen-type system $\mathbf{G}$ if, for every Γ and Δ, $\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$ holds iff $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}$.

Remark 24 If $\mathcal{M}$ is characteristic for $\mathbf{G}$, then $\vdash_{\mathbf{G}} \Gamma \Rightarrow \psi$ iff $\Gamma \vdash_{\mathcal{M}} \psi$. By the compactness theorem (Proposition 17), if $\mathcal{M}$ is finite, then the foregoing implies that $\vdash_{\mathcal{M}}=\vdash_{\mathbf{G}}$.

Below we define non-deterministic three-valued semantics for $\mathbf{BK}[A]$ for all $A \subseteq \mathbf{A}$, and then introduce the corresponding Gentzen-type systems for those logics.

3.1 Non-deterministic Semantics

Our non-deterministic semantics is based on the following four truth values, the intuition being that a formula φ is assigned a truth value of the form $\langle x, y\rangle$, where $x = 1$ iff φ is "true", and $y = 1$ iff $\neg\varphi$ is "true":

$$t = \langle 1, 0\rangle, f = \langle 0, 1\rangle, \top = \langle 1, 1\rangle, \bot = \langle 0, 0\rangle$$

First we note that the axiom $(\mathbf{t})$ $\varphi \vee \neg\varphi$, included already in $\mathbf{B}$, rules out the fourth truth value $\bot$ (as $(\mathbf{t})$ intuitively means that φ and $\neg\varphi$ cannot be both "false"), and so we are left with three truth values: t, f and $\top$. Semantics for systems without the axiom **(t)** (which are obtained from the positive fragment of classical logic by adding some axioms from $\mathbf{A}$) can be provided in a similar way using the above four truth values.

We start by defining the Nmatrix $\mathcal{M}^3$ for $\mathbf{BK}$:

Definition 25 The Nmatrix $\mathcal{M}^3 = (\{t, f, \top\}, \{t, \top\}, \mathcal{O})$ for $\mathcal{L}_C$ is defined as follows:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$
f	$\{t, \top\}$	$\{t, \top\}$

$\wedge$	t	$\top$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

$\vee$	t	$\top$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$

$\supset$	t	$\top$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

The next theorem was first proved in [1]. In order to make this paper self-contained, we provide here a direct proof (which is very similar to the proof of Theorem 2 in [4]).

Theorem 26 *$T \vdash_{\mathcal{M}^3} \psi$ iff $T \vdash_{\mathbf{BK}} \psi$.*

Proof: Proving soundness is easy. For completeness, assume that $\mathbf{T}$ is a theory and φ_0 a formula such that $\mathbf{T} \not\vdash_{\mathbf{BK}} \varphi_0$. We construct a model of $\mathbf{T}$ in $\mathcal{M}^3$ which is not a model of φ_0. For this, we extend $\mathbf{T}$ to a maximal theory $\mathbf{T}^*$ such that $\mathbf{T}^* \not\vdash_{\mathbf{BK}} \varphi_0$. Then $\mathbf{T}^*$ has the following properties:

1. $\psi \notin \mathbf{T}^*$ iff $\psi \supset \varphi_0 \in \mathbf{T}^*$.

2. If $\psi \notin \mathbf{T}^*$ then $\psi \supset \varphi \in \mathbf{T}^*$ for every formula φ of $\mathcal{L}_C$.

3. $\varphi \vee \psi \in \mathbf{T}^*$ iff either $\varphi \in \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.

4. $\varphi \wedge \psi \in \mathbf{T}^*$ iff both $\varphi \in \mathbf{T}^*$ and $\psi \in \mathbf{T}^*$.

5. $\varphi \supset \psi \in \mathbf{T}^*$ iff either $\varphi \notin \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.

6. For every formula φ of $\mathcal{L}_C$, either $\varphi \in \mathbf{T}^*$ or $\neg\varphi \in \mathbf{T}^*$.

7. $\circ\varphi \in \mathbf{T}^*$ iff either $\varphi \notin \mathbf{T}^*$ or $\neg\varphi \notin \mathbf{T}^*$.

The proofs of Properties 1–5 are exactly as in the case of $\mathbf{HCL}^+$ (see definition 7): Property 1 follows from the deduction theorem (which is obviously valid for **BK**) and the maximality of $\mathbf{T}^*$. Property 2 is proved first for $\psi = \varphi_0$ as follows: by 1, if $\varphi_0 \supset \varphi \notin \mathbf{T}^*$ then $(\varphi_0 \supset \varphi) \supset \varphi_0 \in \mathbf{T}^*$. Hence $\varphi_0 \in \mathbf{T}^*$ by the positive tautology $((\varphi_0 \supset \varphi) \supset \varphi_0) \supset \varphi_0$ — a contradiction. Property 2 then follows for all $\psi \notin \mathbf{T}^*$ by 1 and the transitivity of implication. Properties 3–5 are easy corollaries of 1, 2 and the closure of $\mathbf{T}^*$ under positive classical inferences (for example: suppose $\varphi \vee \psi \in \mathbf{T}^*$, but neither $\varphi \in \mathbf{T}^*$, nor $\psi \in \mathbf{T}^*$. By Property 1, $\varphi \supset \varphi_0 \in \mathbf{T}^*$ and $\psi \supset \varphi_0 \in \mathbf{T}^*$. Since φ_0 follows in positive classical logic from $\varphi \vee \psi$, $\varphi \supset \varphi_0$, and $\psi \supset \varphi_0$, we get $\varphi_0 \in \mathbf{T}^*$ — a contradiction). Finally, Property 6 is immediate from Property 3 and Axiom **(t)**, and Property 7 easily follows from Axioms (**b**) and (**k**) and Properties 3,4.

Now define a valuation v in $\mathcal{M}^3$ as follows:

$$v(\psi) = \begin{cases} f & \text{if} \quad \psi \notin \mathbf{T}^* \\ t & \text{if} \quad \neg\psi \notin \mathbf{T}^* \\ \top & \text{if} \quad \psi \in \mathbf{T}^*, \neg\psi \in \mathbf{T}^* \end{cases}$$

Then, by Property 6, v is well-defined and $v(\psi) \in \mathcal{D} = \{\top, t\}$ iff $\psi \in \mathbf{T}^*$. We use this to prove that v is a legal valuation, i.e., it respects the interpretations of the connectives in $\mathcal{M}^3$. That this is the case for the positive connectives easily follows from Properties 3–5 of $\mathbf{T}^*$. We prove next the cases of $\neg$ and $\circ$:

- Assume $v(\psi) = f$. Then $\psi \notin \mathbf{T}^*$. Hence $\neg\psi \in \mathbf{T}^*$ by Property 6 of $\mathbf{T}^*$, and $\circ\psi \in \mathbf{T}^*$ by Property 7. Thus $v(\neg\psi) \in \{\top, t\}$ and $v(\circ\psi) \in \{\top, t\}$.

- Assume $v(\psi) = t$. By definition, this implies $\neg\psi \notin \mathbf{T}^*$, and so $\circ\psi \in \mathbf{T}^*$ by Property 7. By the definition of v it follows that $v(\neg\psi) = f$, and $v(\circ\psi) \in \{\top, t\}$.

- Assume $v(\psi) = \top$. By definition, this implies $\psi \in \mathbf{T}^*$ and $\neg\psi \in \mathbf{T}^*$. The latter implies $v(\neg\psi) \in \{\top, t\}$. Together with the former, it also implies that $\circ\psi \notin \mathbf{T}^*$, by Property 7 of $\mathbf{T}^*$. Hence $v(\circ\psi) = f$.

Since $v(\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}^*$, $v(\psi) \in \mathcal{D}$ for every $\psi \in \mathbf{T}$, while $v(\varphi_0) \notin \mathcal{D}$. Hence v is a model of $\mathbf{T}$ which is not a model of φ_0. ∎

Next we turn to providing non-deterministic semantics for the extensions of **BK** with axioms from **A**. Our semantics is *modular* in the following sense: each axiom $ax \in \mathbf{A}$ corresponds to some finite set $\mathsf{C}(ax)$ of semantic conditions.

These conditions lead to simple refinements of the basic Nmatrix $\mathcal{M}^3$ (which amount to reducing the level of non-determinism in $\mathcal{M}^3$). The semantics of $\mathbf{BK}[A]$ is then obtained by straightforwardly combining the semantic effects of all the axioms from A.

Tables 1 and 2 below include the various semantic conditions that correspond to the axioms in $\mathbf{A}$. Most of them are either taken from [4], or are easily derivable using the method employed in that paper.

Example 27 By way of example, we will explain the derivation of $\mathsf{C}(\mathbf{o}^{\mathbf{1}}_{\vee})$ from Table 2. For this purpose, assume that v is a valuation in $\mathcal{M}^3$. If $v(\varphi) = \top$, then v certainly satisfies $\circ\varphi \supset \circ(\varphi \vee \psi)$. Otherwise it satisfies this formula iff it satisfies $\circ(\varphi \vee \psi)$, which is the case iff $v(\varphi \vee \psi) \neq \top$. This again necessarily holds if $v(\varphi) = v(\psi) = f$. In the remaining five cases, all we know is that $v(\varphi \vee \psi) \in \{t, \top\}$. Hence to ensure that indeed $v(\varphi \vee \psi) \neq \top$ we have to force it to be t in those cases. This requires five basic semantic conditions, which can be conveniently grouped as follows: (i) $t \vee t = t \vee \top = t \vee f = \{t\}$ (i.e., $t \vee x = \{t\}$ for $x \in \{t, \top, f\}$), and (ii) $f \vee t = f \vee \top = \{t\}$. These are exactly the elements of $\mathsf{C}(\mathbf{o}^{\mathbf{1}}_{\vee})$ given in Table 2.

Definition 28 For any $A \subseteq \mathbf{A}$, the Nmatrix $\mathcal{M}^3[A]$ is the weakest simple refinement of $\mathcal{M}^3$ in which $\mathsf{C}(ax)$ (from Tables 1 and 2) holds for every $ax \in A$.

Remark 29 It is easy to check that none of the combinations of the conditions in Tables 1 and 2 is contradictory. Hence $\mathcal{M}^3[A]$ is well-defined for every $A \subseteq \mathbf{A}$.

Remark 30 The semantic conditions for **(c)**, **(e)**, $(\mathbf{i_1})$, $(\mathbf{i_2})$ correspond to their respective axioms already in the framework of the system $\mathbf{B}$. For other conditions, this holds only for $\mathbf{BK}$.

Theorem 31 *For any* $A \subseteq \mathbf{A}$*,* $T \vdash_{\mathcal{M}^3[A]} \psi$ *iff* $T \vdash_{\mathbf{BK}[A]} \psi$*.*

Proof: The proof is similar to that of Theorem 26, with $\mathbf{BK}[A]$ used instead of $\mathbf{BK}$. We only have to show that the extra conditions on a valuation imposed by the axioms from A are respected by the valuation v defined in that proof. This is easy, and by way of example we show it for the case where $(\mathbf{o}^{\mathbf{1}}_{\vee}) \in A$. So assume that $v(\varphi)$ is in $\{t, f\}$. Like in the proof of Theorem 26, this implies that $\circ(\varphi) \in \mathbf{T}^*$, and so $\circ(\varphi \vee \psi) \in \mathbf{T}^*$ by $\mathbf{o}^{\mathbf{1}}_{\vee}$. If in addition $v(\varphi) \in \{t, \top\}$ or $v(\psi) \in \{t, \top\}$ then either $\varphi \in \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$, and so $\varphi \vee \psi \in \mathbf{T}^*$ (by positive classical logic). But if both $\varphi \vee \psi$ and $\circ(\varphi \vee \psi)$ are in $\mathbf{T}^*$ then $\neg((\varphi \vee \psi) \notin \mathbf{T}^*$ (by Property 7 of $\mathbf{T}^*$ — see the proof of Theorem 26). Hence $v(\varphi \vee \psi) = t$ under the two assumptions. ■

Corollary 32 *For* $\sharp \in \{\wedge, \vee, \supset\}$*, each of* $(\mathbf{o}^{\mathbf{1}}_{\sharp})$ *and* $(\mathbf{o}^{\mathbf{2}}_{\sharp})$ *implies* $(\mathbf{a}_{\sharp})$ *in* $\mathbf{BK}$*.*

Remark 33 A mechanical check using the semantic conditions from Tables 1,2 shows that there are no other dependencies among the axioms in $\mathbf{A}$.

	ax	$\mathsf{C}(ax)$	$\mathsf{R}(ax)$
$(\mathbf{c})$	$\neg\neg\varphi \supset \varphi$	$\neg f = \{t\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$
$(\mathbf{e})$	$\varphi \supset \neg\neg\varphi$	$\neg\top = \{\top\}$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$
$(\mathbf{i_1})$	$\neg\circ\varphi \supset \varphi$	$\circ f = \{t\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg \circ \varphi \Rightarrow \Delta}$
$(\mathbf{i_2})$	$\neg\circ\varphi \supset \neg\varphi$	$\circ t = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg \circ \varphi \Rightarrow \Delta}$
$(\mathbf{a}_\wedge)$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$	$t \wedge t = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\mathbf{a}_\vee)$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi)$	$t \vee t = t \vee f = \{t\}$ $t \vee t = f \vee t = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{a}_\supset)$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \supset \psi)$	$f \supset t = f \supset f = \{t\}$ $f \supset t = t \supset t = \{t\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \neg\psi, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \neg\varphi, \varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$

Table 1: Axioms, semantic conditions and Gentzen-type rules

	ax	$\mathsf{C}(ax)$	$\mathsf{R}(ax)$
$(\mathbf{o}^{\mathbf{1}}_{\wedge})$	$\circ\varphi \supset \circ(\varphi \wedge \psi)$	$t \wedge t = t \wedge \top = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{2}}_{\wedge})$	$\circ\psi \supset \circ(\varphi \wedge \psi)$	$t \wedge t = \top \wedge t = \{t\}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{1}}_{\vee})$	$\circ\varphi \supset \circ(\varphi \vee \psi)$	$t \vee x = \{t\}$ $f \vee t = f \vee \top = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{2}}_{\vee})$	$\circ\psi \supset \circ(\varphi \vee \psi)$	$x \vee t = \{t\}$ $t \vee f = \top \vee f = \{t\}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{1}}_{\supset})$	$\circ\varphi \supset \circ(\varphi \supset \psi)$	$t \supset t = t \supset \top = \{t\}$ $f \supset x = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$
$(\mathbf{o}^{\mathbf{2}}_{\supset})$	$\circ\psi \supset \circ(\varphi \supset \psi)$	$x \supset t = \{t\}$ $f \supset f = \{t\}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$

Table 2: Axioms, semantic conditions and rules (for $x \in \{t, \top, f\}$) — continued

Example 34 • The truth tables for $\circ$ and $\neg$ in the Nmatrix $\mathcal{M}^3[\{(\mathbf{c}),(\mathbf{e})\}]$ (which is characteristic for the system **BKce**) are defined as follows:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t,\top\}$
$\top$	$\{\top\}$	$\{f\}$
f	$\{t\}$	$\{t,\top\}$

- The system denoted by **Cie** in [18, 17] is equivalent to our **BKcie** (or just **Bcie**, by Remark 13). In the corresponding characteristic Nmatrix $\mathcal{M}^3[\{(\mathbf{c}),(\mathbf{e}),(\mathbf{i_1}),(\mathbf{i_2})\}]$, the truth tables for $\circ$ and $\neg$ are as follows[10]:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t\}$
$\top$	$\{\top\}$	$\{f\}$
f	$\{t\}$	$\{t\}$

- The Nmatrix $\mathcal{M}^3[\{(\mathbf{a}_\vee),(\mathbf{a}_\wedge),(\mathbf{a}_\supset)\}]$, which is characteristic for **BKa**, is defined as follows:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{f\}$
f	$\{t,\top\}$	$\{t,\top\}$

$\wedge$	t	$\top$	f
t	$\{t\}$	$\{t,\top\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

$\vee$	t	$\top$	f
t	$\{t\}$	$\{t,\top\}$	$\{t\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
f	$\{t\}$	$\{t,\top\}$	$\{f\}$

$\supset$	t	$\top$	f
t	$\{t\}$	$\{t,\top\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
f	$\{t\}$	$\{t,\top\}$	$\{t\}$

3.2 Corresponding Gentzen-type Systems

Before constructing cut-free Gentzen-type systems for all the logics discussed above, we need some preparatory technicalities.

Definition 35 The language $\mathcal{L}$ is *sufficiently expressive* for an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ if for any $x \in \mathcal{V}$ there exist natural numbers $l_x, m_x \geq 0$ and formulas A^x_j, B^x_k $(1 \leq j \leq l_x, 1 \leq k \leq m_x)$ such that the following conditions are satisfied:

(i) $Fv(A^x_j) = Fv(B^x_k) = \{p_1\}$ for every $1 \leq j \leq l_x$ and $1 \leq k \leq m_x$.

(ii) For any valuation v in $\mathcal{M}$ and any formula φ of $\mathcal{L}$, $v(\varphi) = x$ if and only if $A^x_j\{\varphi/p_1\} \Rightarrow$ and $\Rightarrow B^x_k\{\varphi/p_1\}$ are satisfied by v for every $1 \leq j \leq l_x$ and $1 \leq k \leq m_x$.[11]

[10] An alternative possible-translations semantics for this logic was presented in [31].

[11] See Item 4 of Definition 14 for the semantic meaning of a sequent.

As we have said above, [5] provides a method for constructing a cut-free, sound and complete Gentzen-type system for a given finite Nmatrix $\mathcal{M}$ whose language is sufficiently expressive for $\mathcal{M}$. The next proposition implies that this method is applicable to the logics investigated here:

Proposition 36 *$\mathcal{L}_C$ is sufficiently expressive for every simple refinement of $\mathcal{M}^3$.*

Proof: It is easy to verify that the following conditions hold in $\mathcal{M}^3$ and any simple refinement of that Nmatrix:

- $v(\psi) = t$ iff v does not satisfy $\neg\psi$ (i.e., $v(\neg\psi) \notin \mathcal{D}$).
- $v(\psi) = f$ iff v does not satisfy ψ (i.e., $v(\psi) \notin \mathcal{D}$).
- $v(\psi) = \top$ iff v satisfies both ψ and $\neg\psi$ (i.e., $v(\psi) \in \mathcal{D}$ and $v(\neg\psi) \in \mathcal{D}$).

Hence, taking $l_t = l_f = m_\top = 0$, $m_t = m_f = 1$, $l_\top = 2$, $B_1^t = \neg p_1$, $B_1^f = p_1$, $A_1^\top = p_1$, and $A_2^\top = \neg p_1$, we can conclude that $\mathcal{L}_C$ is sufficiently expressive for every simple refinement of $\mathcal{M}^3$. ■

Now the way the algorithm of [5] works (see below for an explanation and demonstration), together with the proof of Proposition 36, imply that each application of one of the Gentzen-type rules we obtain in the present case has the following uniform form:

1. It introduces exactly one formula in its conclusion, on exactly one of its two sides;
2. The formula being introduced is either of the form $\diamond(\psi_1, \ldots, \psi_n)$ or of the form $\neg\diamond(\psi_1, \ldots, \psi_n)$, where $\diamond$ is a primitive connective of the language;
3. Let $\diamond(\psi_1, \ldots, \psi_n)$ be the formula mentioned in Item 2. Then all the principal formulas in the premises belong to the set $\{\psi_1, \ldots, \psi_n, \neg\psi_1, \ldots, \neg\psi_n\}$;
4. There are no restrictions on the side formulas (i.e., every context is legitimate).

Our next definition formalizes this observation in precise terms, and introduces notation that will be useful for the process of deriving rules.

Definition 37

1. A *quasi-canonical rule* of arity n is an expression of the form

$$\frac{\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}}{C}$$

   where $m \geq 0$, $\Pi_i, \Sigma_i \subseteq \{p_1, \neg p_1, p_2, \neg p_2, \ldots, p_n, \neg p_n\}$ for $1 \leq i \leq m$, and C has one of the following forms:

$$\diamond(p_1, p_2, \ldots, p_n) \Rightarrow \qquad \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$$

$$\neg\diamond(p_1, p_2, \ldots, p_n) \Rightarrow \qquad \Rightarrow \neg\diamond(p_1, p_2, \ldots, p_n)$$

2. An *application* of a quasi-canonical rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} \,/\, \diamond(p_1,\ldots,p_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \diamond(\psi_1,\ldots,\psi_n) \Rightarrow \Delta}$$

where Π_i^* and Σ_i^* are obtained from Π_i and Σ_i (respectively) by substituting ψ_j for p_j (for all $1 \leq j \leq n$), and Γ, Δ are any sets of formulas. An application of a quasi-canonical rule of one of the other three forms is defined similarly.

3. A Gentzen-type system in which all rules are either structural or quasi-canonical is called *quasi-canonical*.

Remark 38 All the Gentzen-type systems developed in this paper are quasi-canonical. Now quasi-canonical systems provide a natural generalization of the class of canonical systems ([10, 11, 13]) — the type of the standard Gentzen-type systems for classical logic. The difference is that canonical systems allow only rules which introduce formulas of the form $\diamond(\psi_1,\ldots,\psi_n)$ in their conclusion, while the principal formulas in their premises are taken just from the set $\{\psi_1,\ldots,\psi_n\}$. It should be noted that quasi-canonical Gentzen-type systems have already been used extensively in the proof theory of non-classical logics (see e.g. [2]). As far as we know, this cannot be said about any Gentzen-type formulation of a C-system that has been suggested before.

Now we are ready to apply the method described in [5]. This method has two stages.

1. At the first (and more important) stage, every entry of every truth table of $\mathcal{M}$ is translated into a set of rules. In the case of $\mathcal{M}^3$ and its simple refinements the general process can be significantly simplified, since each nontrivial subset of their set of truth values can easily be characterized using one or two very simple sequents:

   - $v(\psi) = t$ iff $\neg\psi \Rightarrow$ is satisfied by v.
   - $v(\psi) = f$ iff $\psi \Rightarrow$ is satisfied by v.
   - $v(\psi) = \top$ iff $\Rightarrow \psi$ and $\Rightarrow \neg\psi$ are both satisfied by v.
   - $v(\psi) \in \{f, \top\}$ iff $\Rightarrow \neg\psi$ is satisfied by v.
   - $v(\psi) \in \{t, \top\}$ iff $\Rightarrow \psi$ is satisfied by v.
   - $v(\psi) \in \{t, f\}$ iff $\psi, \neg\psi \Rightarrow$ is satisfied by v.

   Using these six facts[12], we can directly and easily translate every entry of every truth table of $\mathcal{M}^3$ (or any of its simple refinements) into either a *single rule*, or a *pair of rules* in case of an entry $o(x_1, ..., x_n) = \{\top\}$. Examples of how this is done are provided below.

[12] Note that since for every valuation v and every formula ψ exactly one of the sequents $\Rightarrow \psi$ and $\psi \Rightarrow$ is satisfied by v, the fourth fact is actually equivalent to the first, the fifth to the second, and the sixth to the third, so we actually use here just three semantic facts.

2. The first stage usually results in a large set of often complicated, unwieldy rules. Accordingly, at the second stage the obtained rules are combined and simplified in order to get an optimal set of rules. This is done by using streamlining principles whose applications to a system in which the cut rule is admissible preserve that property. The set of principles that we use here for this purpose is a modified variant of the streamlining principles which were introduced in [7] (and used in [5]). If we denote the system under consideration by $\mathcal{R}$, then the said streamlining principles consist in: deleting from the premises of a rule a sequent derivable from the remainder of the premises (Princ. 1), replacing a sequent in the premises of a rule by one of its subsequents[13] if the latter is derivable from the premises (Princ. 2), and combining two rules with the same conclusion (Princ. 3):

   Principle 1 If $\rho = \dfrac{S}{\Sigma}$ (where S is a finite set of premises) is a rule in $\mathcal{R}$, $\Pi \in S$ and Π is derivable from $S \setminus \{\Pi\}$, then ρ can be replaced with $\dfrac{S \setminus \{\Pi\}}{\Sigma}$.

   Principle 2 If $\rho = \dfrac{S}{\Sigma}$ is a rule in $\mathcal{R}$, $\Pi \in S, \Pi^* \subseteq \Pi$, and Π^* is derivable from S, then ρ can be replaced with $\dfrac{S^*}{\Sigma}$, where S^* is obtained from S by replacing Π with Π^*.

   Principle 3 Rules $\dfrac{\Sigma_1 \quad \dots \quad \Sigma_k}{\Sigma}$ and $\dfrac{\Sigma'_1 \quad \dots \quad \Sigma'_l}{\Sigma}$ can be replaced with the rule $\dfrac{\{\Sigma_i \cup \Sigma'_j\}_{1 \leq i \leq k, 1 \leq j \leq l}}{\Sigma}$

   By saying in Principles 1,2 that a sequent $\Pi = \Gamma \Rightarrow \Delta$ is derivable from a set of sequents S' we mean in the present case that either Γ and Δ have some literal in common, or Π can be derived using cuts and weakening from $S' \cup \{\Rightarrow p, \neg p \mid p \in At(\Pi)\}$, where $At(\Pi)$ is the set of atomic formulas occurring in Π. Note however that the cut rule is *not* needed for deriving the original set of rules from that obtained using the above three principles. In consequence, by using these principles for simplifying a system in which the cut rule is admissible we do not lose this property. Hence the method of [5] guarantees the admissibility of the cut rule in all the systems obtained below.

To see how the whole process of rule generation and streamlining works, take for example the truth table for $\vee$ in $\mathcal{M}^3$. We will show how the quasi-canonical rules for $\vee$ are derived from that table, using for better readability φ and ψ instead of p_1 and p_2.

[13] Here we assume that inclusion and union of sequents are defined componentwise, i.e.: if $\Sigma_1 = \Gamma_1 \Rightarrow \Delta_1, \Sigma_2 = \Gamma_2 \Rightarrow \Delta_2$ then $\Sigma_1 \subseteq \Sigma_2$ iff $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$; and $\Sigma_1 \cup \Sigma_2$ is the sequent $\Gamma_1 \cup \Gamma_2 \Rightarrow \Delta_1 \cup \Delta_2$.

First, the entry $f \vee f = \{f\}$ is translated into the condition: if $\varphi \Rightarrow$ is satisfied by a valuation v, and $\psi \Rightarrow$ is satisfied by v, then $\varphi \vee \psi \Rightarrow$ is satisfied by v. From this, we obtain the rule:

$$\frac{\varphi \Rightarrow \qquad \psi \Rightarrow}{\varphi \vee \psi \Rightarrow}$$

Or with the context explicitly mentioned:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$$

Next, the entry $f \vee \top = \{t, \top\}$ is translated into the condition: if $\varphi \Rightarrow$, $\Rightarrow \psi$ and $\Rightarrow \neg\psi$ are all satisfied by v, then so is $\Rightarrow \varphi \vee \psi$. This gives rise to the rule:

$$(i) \quad \frac{\varphi \Rightarrow \quad \Rightarrow \psi \quad \Rightarrow \neg\psi}{\Rightarrow \varphi \vee \psi}$$

Similarly, the entry $f \vee t = \{t, \top\}$ is translated into the rule

$$(ii) \quad \frac{\varphi \Rightarrow \qquad \neg\psi \Rightarrow}{\Rightarrow \varphi \vee \psi}$$

Since rules (i), (ii) have the same conclusion, we can combine them using Principle 3. What we get is the rule which has the conclusion $\Rightarrow \varphi \vee \psi$ and the following 6 premises:

(1) $\varphi \Rightarrow$; (2) $\varphi \Rightarrow \psi$; (3) $\varphi \Rightarrow \neg\psi$; (4) $\neg\psi, \varphi \Rightarrow$; (5) $\neg\psi \Rightarrow \psi$; (6) $\neg\psi \Rightarrow \neg\psi$
In this rule, Premise (6) is an axiom, and so can be deleted by Principle 1. Further, Premises (2), (3), (4) can all be derived from Premise (1) using weakening, and so can also be deleted by Principle 1. We are left with two premises: (1) $\varphi \Rightarrow$ and (5) $\neg\psi \Rightarrow \psi$. However, by Principle 2, premise (5) can be replaced by $\Rightarrow \psi$, because $\Rightarrow \psi \subseteq \neg\psi \Rightarrow \psi$ and $\Rightarrow \psi$ can be derived from $\neg\psi \Rightarrow \psi$ by a cut on $\neg\psi$ with the derivable sequent $\Rightarrow \psi, \neg\psi$. As a result, rules (i), (ii) finally combine into the rule:

$$\frac{\varphi \Rightarrow \qquad \Rightarrow \psi}{\Rightarrow \varphi \vee \psi}$$

From the semantic viewpoint, this rule says that if φ is assigned f, while ψ is assigned a value in $\{t, \top\}$, then $\varphi \vee \psi$ should get a valued in $\{t, \top\}$. This is precisely what is said by the two entries of the truth table for $\vee$ which have given rise to this rule.

Continuing in this way, we can show that the 8 entries of the above table different from $f \vee f = \{f\}$ are finally subsumed by the rule

$$\frac{\Rightarrow \varphi, \psi}{\Rightarrow \varphi \vee \psi}$$

Or with the context explicitly mentioned:

$$\frac{\Gamma, \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Remark 39 Instead of first deriving eight different rules, and then combining them, we can actually derive the last rule directly by observing that the relevant eight entries taken together mean that if either $v(\varphi) \in \{t, \top\}$ or $v(\psi) \in \{t, \top\}$ then $v(\varphi \vee \psi) \in \{t, \top\}$. This directly translates into two basic rules: from $\Rightarrow \varphi$ infer $\Rightarrow \varphi \vee \psi$, and from $\Rightarrow \psi$ infer $\Rightarrow \varphi \vee \psi$. By combining these rules using Principle 3, we obtain the same final rule reached above.

The procedure we have just described can be justified as followed: it is obvious that it produces sound rules. It is also easy to see that any rule which is obtained from one of the relevant entries can be derived using only weakenings from one of the basic rules which are produced by the direct method. Hence the completeness and cut-admissibilty of the set of rules obtained by the original method imply those properties for the set of rules obtained by the direct one.

The method described above provides the following system $\mathbf{G_K}$ for $\vdash_{\mathcal{M}^3}$:

Axioms of $\mathbf{G_K}$: $\varphi \Rightarrow \varphi$

Rules of $\mathbf{G_K}$: Cut, Weakening, and the following logical rules:

$$(\wedge\Rightarrow)\quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad\qquad (\Rightarrow\wedge)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee\Rightarrow)\quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad\qquad (\Rightarrow\vee)\quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\supset\Rightarrow)\quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \supset \psi \Rightarrow \Delta} \qquad\qquad (\Rightarrow\supset)\quad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta}$$

$$(\Rightarrow \neg)\quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\circ \Rightarrow)\quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \circ\varphi \Rightarrow \Delta} \qquad\qquad (\Rightarrow \circ)\quad \frac{\Gamma, \varphi, \neg\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \circ\varphi, \Delta}$$

Remark 40 An alternative, equivalent formulation of $\mathbf{G_K}$ is obtained by deleting the weakening rule, and instead taking as axioms all sequents of the form $\Gamma, \varphi \Rightarrow \Delta, \varphi$. As this formulation shortens the proofs, it is used in some of the examples below.

Now from the results of [5] we directly obtain the following two facts (both of which can easily be proved directly):

Proposition 41

1. **BK** *is equivalent to* $\mathbf{G_K}$.
2. $\mathbf{G_K}$ *enjoys cut-admissibility.*

Example 42 Below we provide a proof that (**l**) $\Rightarrow$ (**k$_2$**) in the system which is obtained from $\mathbf{G_K}$ by deleting the rule ($\Rightarrow \circ$), where **(l)** is the axiom which is implicit in da Costa's historical system C_1 mentioned in Item 3 of Remark 8. The proof of (**l**) $\Rightarrow$ (**k$_1$**) is similar.

$$\dfrac{\dfrac{\dfrac{\varphi, \neg\varphi \Rightarrow \circ\varphi, \neg\varphi}{\varphi \wedge \neg\varphi \Rightarrow \circ\varphi, \neg\varphi}\ (\wedge \Rightarrow)}{\Rightarrow \neg(\varphi \wedge \neg\varphi), \circ\varphi, \neg\varphi}\ (\Rightarrow \neg) \qquad \circ\varphi \Rightarrow \circ\varphi, \neg\varphi}{\dfrac{\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi \Rightarrow \circ\varphi, \neg\varphi}{\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi \Rightarrow \circ\varphi \vee \neg\varphi}\ (\Rightarrow \vee)}\ (\supset\Rightarrow)$$

It is important to note that the rules in $\mathbf{G_K}$, except of the rule for negation, are particularly well-behaved in the following sense:

Definition 43 An introduction rule ρ is *invertible in a Gentzen-type system* $\mathbf{G}$ if each of the premises of ρ has a derivation in $\mathbf{G}$ from the conclusion of ρ.

Proposition 44 *All the rules for the positive connectives ($\wedge, \vee, \supset$ and $\circ$) are invertible in* $\mathbf{G_K}$.

Proof: For the connectives of positive classical logic, the proofs are like in the classical case. We show the proof for the rules for $\circ$. The following is a derivation of $\Gamma, \varphi, \neg\varphi \Rightarrow \Delta$ from $\Gamma \Rightarrow \circ\varphi, \Delta$ (which proves the derivability of the converse of ($\Rightarrow \circ$)) in $\mathbf{G_K}$:

$$\dfrac{\Gamma \Rightarrow \circ\varphi, \Delta \qquad \dfrac{\Gamma, \varphi \Rightarrow \Delta, \varphi \quad \Gamma, \neg\varphi \Rightarrow \Delta, \neg\varphi}{\Gamma, \circ\varphi, \varphi, \neg\varphi \Rightarrow \Delta}\ (\circ \Rightarrow)}{\Gamma, \varphi, \neg\varphi \Rightarrow \Delta}\ cut$$

The derivation of $\Gamma \Rightarrow \varphi, \Delta$ from $\Gamma, \circ\varphi \Rightarrow \Delta$ (which proves the derivability of one converse of ($\circ \Rightarrow$)) in $\mathbf{G_K}$ is as follows:

$$\dfrac{\Gamma, \circ\varphi \Rightarrow \Delta \qquad \dfrac{\Gamma, \varphi, \neg\varphi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \varphi, \circ\varphi, \Delta}\ (\Rightarrow \circ)}{\Gamma \Rightarrow \varphi, \Delta}\ cut$$

The derivation of $\Gamma \Rightarrow \neg\varphi, \Delta$ from $\Gamma, \circ\varphi \Rightarrow$ is similar. ■

The method we have just used for $\mathcal{M}^3$ can be applied to each of its simple refinements separately. In this way, we can obtain a cut-free Gentzen-type formulation for each of the C-systems we have considered above. However, as described in Remark 39, this can be done much easier in a modular way, by translating the semantic effect of each extra axiom into rules (and using the streamlining principles to simplify the outcome). This shorter procedure can be justified using the same kind of reasoning that was used in Remark 39.

The results of this process are again given in Tables 1 and 2.

Example 45 To see how the Gentzen-type rules from Tables 1,2 are derived, we revisit the axiom $(\mathbf{o}^{\mathbf{1}}_{\vee})$. As explained in Example 27, the validity of this axiom is equivalent to the combination of the following two conditions: (i) $t \vee x = t$ and (ii) $f \vee t = f \vee \top = \{t\}$. Now (i) can be reformulated as follows: if $\neg\varphi \Rightarrow$ is true, then $\neg(\varphi \vee \psi) \Rightarrow$ is true. By adding context, we obtain:

$$\frac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$$

In turn, (ii) can be reformulated as follows: if $\varphi \Rightarrow$ and $\Rightarrow \psi$ are true, then so is $\neg(\varphi \vee \psi) \Rightarrow$. Again, by adding context we get the following rule:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$$

Taken together, these two Gentzen-type rules correspond to the axiom $(\mathbf{o}^{\mathbf{1}}_{\vee})$.

Definition 46 For each $ax \in \mathbf{A}$, the set of Gentzen-type rules $\mathsf{R}(ax)$, corresponding to ax, is defined as in Tables 1,2. For $A \subseteq \mathbf{A}$, $\mathbf{G_K}[A]$ is the Gentzen-type system obtained by adding to $\mathbf{G_K}$ the set of rules $\mathsf{R}(ax)$ for every $ax \in A$.

Example 47 Figure 1 at the end of this paper provides three examples of cut-free proofs in three of our systems: $\mathbf{BKa}_{\wedge}$, $\mathbf{BKo}^{\mathbf{1}}_{\wedge}$, and the basic system $\mathbf{BK}$. Note that the first example in Figure 1 shows that axiom $(\mathbf{a}_{\wedge})$ indeed has a cut-free proof in $\mathbf{BKa}_{\wedge}$, even though the Gentzen-type rule $(\mathbf{a}_{\wedge})$ corresponding to this axiom does not even mention the connective $\circ$. The second example does the same for axiom $(\mathbf{o}^{\mathbf{1}}_{\wedge})$ and the system $\mathbf{BK}(\mathbf{o}^{\mathbf{1}}_{\wedge})$. Finally, the third example shows that the axiom which corresponds to the semantic condition $t \vee t = \{t\}$ follows from the axiom $(\mathbf{a}_{\vee})$ in the basic system $\mathbf{BK}$.

Theorem 48 *Let $A \subseteq \mathbf{A}$.*

1. $\mathbf{BK}[A]$ *is equivalent to* $\mathbf{G_K}[A]$.
2. $\mathbf{G_K}[A]$ *enjoys cut-admissibility.*

Proof: It is not difficult to see that $\mathbf{G_K}[A]$ is the calculus obtained for $\mathcal{M}^3[A]$ using the algorithm from [5]. Thus the theorem follows from Theorem 31 and the results of [5]. ■

Remark 49 The process by which we have derived a Gentzen-type system G from a Hilbert-style system H ensures the equivalence of the two systems in the sense that $T \vdash_H \psi$ iff $T \vdash_G \psi$ (where the consequence relation $\vdash_{\mathbf{G}}$ is defined, as usual, by: $T \vdash_{\mathbf{G}} \psi$ if there is a finite $\Gamma \subseteq T$ such that $\vdash_{\mathbf{G}} \Gamma \Rightarrow \psi$). In particular, ψ is a theorem of H iff $\vdash_{\mathbf{G}} \Rightarrow \psi$. It is also possible of course (and easy) to establish this equivalence directly in the usual way (with the help of the cut rule).

Remark 50 The modularity of our method can be easily further increased as follows. From Remark 30 it follows that our method can also handle any extension of **B** with a subset of the set $\{(\mathbf{c}), (\mathbf{e}), (\mathbf{i_1}), (\mathbf{i_2}), (\mathbf{k_1}), (\mathbf{k_2})\}$ (but not systems like $\mathbf{Bk_1a!}$). For this purpose, we first derive the semantic conditions for $(\mathbf{k_1})$ and $(\mathbf{k_2})$:

$$\mathsf{C}(\mathbf{k_1}): \quad \circ t = \{t, \top\}$$
$$\mathsf{C}(\mathbf{k_2}): \quad \circ f = \{t, \top\}$$

We then translate these conditions $\mathsf{C}(\mathbf{k_1})$ and $\mathsf{C}(\mathbf{k_2})$ into the following rules:

$$(\mathsf{R}(\mathbf{k_1})) \quad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \circ\psi} \qquad\qquad (\mathsf{R}(\mathbf{k_2})) \quad \frac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \circ\psi}$$

Now we can develop cut-free Gentzen type systems for this family of logics by first deleting $(\Rightarrow \circ)$ from $G_{\mathbf{K}}$ (and obtaining in this way a cut-free system for **B**), then using in a modular way these two rules together with the rules associated in Table 1 with the other four axioms. Since $(\mathbf{i_1})$ entails $(\mathbf{k_1})$ already in **B**, and $(\mathbf{i_2})$ entails $(\mathbf{k_2})$, there are exactly twenty logics in this family which are not extensions of **BK**: namely, the extensions of either **B**, $\mathbf{Bk_1}$, $\mathbf{Bk_2}$, $\mathbf{Bi_1}$ or $\mathbf{Bi_2}$ with some subset of $\{(\mathbf{c}), (\mathbf{e})\}$.[14] It should be noted that in the cut-free systems for the extensions of $\mathbf{Bi_1}$ ($\mathbf{Bi_2}$) with some subset of $\{(\mathbf{c}), (\mathbf{e})\}$ one should include both $\mathsf{R}(\mathbf{k_1})$ and $\mathsf{R}(\mathbf{i_1})$ (both $\mathsf{R}(\mathbf{k_2})$ and $\mathsf{R}(\mathbf{i_2})$).

Remark 51 [3] treats many other classical tautologies involving $\neg$ (but not $\circ$) in exactly the same way the axioms in **A** have been treated here: it modularly associates with each of them a semantic condition on $\mathcal{M}^3$ and its translation to a quasi-canonical Gentzen-type rule. Table 3 includes the list of axioms treated there which are not in **A**, together with the corresponding semantic conditions and Gentzen-type rules.

Using the rules from Table 3, one can provide in a modular way a cut-free quasi-canonical Gentzen-type system for almost every logic **L** which is obtained by adding some subset of the axioms listed in Table 3 to any of the systems treated here so far (including those mentioned in Remark 50). The only condition is that **L** should have a simple refinement of $\mathcal{M}^3$ as a characteristic Nmatrix. This, in turn, is the case iff the various semantic conditions imposed by the axioms of **L** do not contradict each other (this claim can be shown rather easily by applying the same proof method as used for showing Theorem 31). Now it is straightforward to find out all such contradictions. Here is their full list:

1. If **L** includes both $(\mathbf{nr}_\supset)$ and $(\mathbf{o}^{\mathbf{1}}_\supset)$ then $t \supset \top$ causes a conflict.
2. If **L** includes both $(\mathbf{nr}_\vee)$ and $(\mathbf{o}^{\mathbf{1}}_\vee)$ then $f \vee \top$ causes a conflict.
3. If **L** includes both $(\mathbf{nr}_\vee)$ and $(\mathbf{o}^{\mathbf{2}}_\vee)$ then $\top \vee f$ causes a conflict.

[14] One of those systems is the system **Bc**, for which a cut-free Gentzen-type system was provided in [29] (where it was called **bC**).

	ax	$\mathsf{C}(ax)$	$\mathsf{R}(ax)$
$(\mathbf{nr}^{\mathbf{1}}_{\wedge})$	$\neg\varphi \supset \neg(\varphi \wedge \psi)$	$\top \wedge t = \top \wedge \top = \{\top\}$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$
$(\mathbf{nr}^{\mathbf{2}}_{\wedge})$	$\neg\psi \supset \neg(\varphi \wedge \psi)$	$t \wedge \top = \top \wedge \top = \{\top\}$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$
$(\mathbf{nl}_{\wedge})$	$\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$	$t \wedge t = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\mathbf{nr}_{\vee})$	$(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$	$f \vee \top = \top \vee f = \top \vee \top = \{\top\}$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$
$(\mathbf{nl}^{\mathbf{1}}_{\vee})$	$\neg(\varphi \vee \psi) \supset \neg\varphi$	$t \vee x = \{t\}$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{nl}^{\mathbf{2}}_{\vee})$	$\neg(\varphi \vee \psi) \supset \neg\psi$	$x \vee t = \{t\}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\mathbf{nr}_{\supset})$	$(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$	$t \supset \top = \top \supset \top = \{\top\}$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \supset \psi)}$
$(\mathbf{nl}^{\mathbf{1}}_{\supset})$	$\neg(\varphi \supset \psi) \supset \varphi$	$f \supset x = \{t\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$
$(\mathbf{nl}^{\mathbf{2}}_{\supset})$	$\neg(\varphi \supset \psi) \supset \neg\psi$	$x \supset t = \{t\}$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$

Table 3: More axioms, conditions, and Gentzen-type rules (for $x \in \{t, \top, f\}$)

4. If $\mathbf{L}$ includes both $(\mathbf{nr^1_\wedge})$ and $(\mathbf{o^2_\wedge})$ then $\top \wedge t$ causes a conflict.

5. If $\mathbf{L}$ includes both $(\mathbf{nr^2_\wedge})$ and $(\mathbf{o^1_\wedge})$ then $t \wedge \top$ causes a conflict.

It is not difficult to show that in the first three cases the resulting $\mathbf{L}$ is classical logic, with $\lambda x \in \{t, f\}.t$ taken as the interpretation of $\circ$. Things are more complicated in cases 4 and 5. Here we leave it as an exercise for the reader to show that in both cases φ follows from a theory T in the resulting logic $\mathbf{L}$ iff it follows from it in both of the logics which are induced by the following two (deterministic) *matrices*: the one which is obtained from $\mathcal{M}^3$ by deleting $\top$, and the one which is obtained from $\mathcal{M}^3$ by deleting t. Since q does not follow from $\{p, \neg p\}$ in the second matrix, $\mathbf{L}$ is still paraconsistent in this case. However, $\neg\varphi \supset (\varphi \supset \neg\psi)$ is valid in it, and $\mathbf{L}$ can actually be axiomatized by adding this schema to $\mathbf{BK}$ as a new axiom. Classical logic itself is obtained from this logic by adding to it any axiom from those investigated in this paper which forces the availability of t (e.g. $(\mathbf{c})$).

Tables 1,2,3 can also be used to find out easily all the dependencies (over $\mathbf{BK}$) that exist among the axioms dealt with in these tables. In fact, a full list is obtained by adding the following dependencies to those mentioned in Corollary 32: $(\mathbf{nl_\wedge})$ is equivalent to $(\mathbf{a_\wedge})$ (and so it follows from $(\mathbf{o^1_\wedge})$, as well as from $(\mathbf{o^2_\wedge})$); $(\mathbf{nl^1_\vee})$ follows from $(\mathbf{o^1_\vee})$; $(\mathbf{nl^2_\vee})$ follows from $(\mathbf{o^2_\vee})$; $(\mathbf{a_\vee})$ follows from $\{(\mathbf{nl^1_\vee}), (\mathbf{nl^2_\vee})\}$; $(\mathbf{nl^1_\supset})$ follows from $(\mathbf{o^1_\supset})$; $(\mathbf{nl^2_\supset})$ follows from $(\mathbf{o^2_\supset})$; $(\mathbf{a_\supset})$ follows from $\{(\mathbf{nl^1_\supset}), (\mathbf{nl^2_\supset})\}$.

4 Conclusions and Further Research

In this paper we provide a *uniform* way to *systematically* construct analytic calculi for a large family of thousands of C-systems, each having a semantic characterization in terms of a three-valued Nmatrix. We believe that these results will help produce efficient tools for automated reasoning with inconsistency, eventually making LFIs a more appealing formalism for reasoning under uncertainty. A first step in this direction has been recently taken in [23]: an algorithm for a fully automatic generation of non-deterministic semantics and cut-free sequent calculi for practically all the C-systems studied in this paper (and many more) has been implemented there in Prolog.

The most immediate directions for further research include:

- Extending the method to LFIs like **Ba**, which have finite-valued non-deterministic semantics with more than three truth values. The easiest natural step to achieve that would be to try to adapt the method given here to the use of the five-valued semantics for extensions of $\mathbf{B}$ with axioms from $\mathbf{A} \cup \{(\mathbf{k_1}), (\mathbf{k_2})\}$, given in [4].

- It is clear that in order to build theorem provers based on C-systems for real-life applications the results of this paper need to be extended to the first-order case. To the best of our knowledge, currently there are no known analytic systems available on the first-order level. However, [12] provided finite non-deterministic semantics for first-order C-systems, which could be exploited along the lines of the approach presented here.

- As we have noted in Remark 38, all the sequent systems presented in this paper are what we call "quasi-canonical". Now there exists a quite well-developed theory of *canonical systems* ([11, 13]). Thus it is known that such systems have semantic characterizations in terms of two-valued Nmatrices, and that there is a strong connection between their semantics, their non-triviality, and the admissibility of the cut rule in them. There is also a strong connection between the determinism of their semantics, and their possession of the invertibility and axiom-expansion properties ([6]). A similar theory should be developed for quasi-canonical systems.

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A proof of $\Rightarrow (\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$ in $\mathbf{BKa}_{\wedge}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi,\psi,\neg(\varphi\wedge\psi)\Rightarrow\varphi,\psi}{\varphi\wedge\psi,\neg(\varphi\wedge\psi)\Rightarrow\varphi,\psi}\;(\wedge\Rightarrow)}{\Rightarrow\circ(\varphi\wedge\psi),\varphi,\psi}\;(\Rightarrow\circ)\qquad\dfrac{\dfrac{\neg(\varphi\wedge\psi),\varphi,\psi\Rightarrow\varphi,\neg\psi}{\neg(\varphi\wedge\psi),\varphi\wedge\psi\Rightarrow\varphi,\neg\psi}\;(\wedge\Rightarrow)}{\Rightarrow\circ(\varphi\wedge\psi),\varphi,\neg\psi}\;(\Rightarrow\circ)}{\circ\psi\Rightarrow\circ(\varphi\wedge\psi),\varphi}\;(\circ\Rightarrow)\qquad\dfrac{\dfrac{\dfrac{\neg(\varphi\wedge\psi),\varphi,\psi\Rightarrow\neg\varphi,\psi}{\neg(\varphi\wedge\psi),\varphi\wedge\psi\Rightarrow\neg\varphi,\psi}\;(\wedge\Rightarrow)}{\Rightarrow\circ(\varphi\wedge\psi),\neg\varphi,\psi}\;(\Rightarrow\circ)\qquad\dfrac{\dfrac{\dfrac{\neg\varphi,\varphi,\psi\Rightarrow\neg\varphi,\neg\psi\qquad\neg\psi,\varphi,\psi\Rightarrow\neg\varphi,\neg\psi}{\neg(\varphi\wedge\psi),\varphi,\psi\Rightarrow\neg\varphi,\neg\psi}\;(\mathbf{a}_{\wedge})}{\neg(\varphi\wedge\psi),\varphi\wedge\psi\Rightarrow\neg\varphi,\neg\psi}\;(\wedge\Rightarrow)}{\Rightarrow\circ(\varphi\wedge\psi),\neg\varphi,\neg\psi}\;(\Rightarrow\circ)}{\circ\psi\Rightarrow\circ(\varphi\wedge\psi),\neg\varphi}\;(\circ\Rightarrow)}{\dfrac{\circ\varphi,\circ\psi\Rightarrow\circ(\varphi\wedge\psi)}{\circ\varphi\wedge\circ\psi\Rightarrow\circ(\varphi\wedge\psi)}\;(\wedge\Rightarrow)}\;(\circ\Rightarrow)}{\Rightarrow(\circ\varphi\wedge\circ\psi)\supset\circ(\varphi\wedge\psi)}\;(\Rightarrow\supset)
$$

A proof of $\Rightarrow \circ\varphi \supset \circ(\varphi \wedge \psi)$ in $\mathbf{BKo}^{\mathbf{1}}_{\wedge}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg(\varphi\wedge\psi),\varphi,\psi\Rightarrow\varphi}{\neg(\varphi\wedge\psi),\varphi\wedge\psi\Rightarrow\varphi}\;(\wedge\Rightarrow)}{\Rightarrow\circ(\varphi\wedge\psi),\varphi}\;(\circ\Rightarrow)\qquad\dfrac{\dfrac{\dfrac{\varphi,\psi,\neg\varphi\Rightarrow\neg\varphi\qquad\varphi,\psi\Rightarrow\neg\varphi,\psi}{\neg(\varphi\wedge\psi),\varphi,\psi\Rightarrow\neg\varphi}\;(\mathbf{o}^{\mathbf{1}}_{\wedge})}{\neg(\varphi\wedge\psi),\varphi\wedge\psi\Rightarrow\neg\varphi}\;(\wedge\Rightarrow)}{\Rightarrow\circ(\varphi\wedge\psi),\neg\varphi}\;(\circ\Rightarrow)}{\circ\varphi\Rightarrow\circ(\varphi\wedge\psi)}\;(\circ\Rightarrow)}{\Rightarrow\circ\varphi\supset\circ(\varphi\wedge\psi)}\;(\Rightarrow\supset)
$$

A proof of $(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi) \Rightarrow \neg(\varphi \vee \psi) \supset (\neg\varphi \vee \neg\psi)$ in $\mathbf{BK}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\varphi\vee\psi),\neg\varphi,\varphi\Rightarrow\neg\varphi,\neg\psi}{\neg(\varphi\vee\psi)\Rightarrow\circ\varphi,\neg\varphi,\neg\psi}\;(\Rightarrow\circ)\qquad\dfrac{\neg(\varphi\vee\psi),\neg\psi,\psi\Rightarrow\neg\varphi,\neg\psi}{\neg(\varphi\vee\psi)\Rightarrow\circ\psi,\neg\varphi,\neg\psi}\;(\Rightarrow\circ)}{\neg(\varphi\vee\psi)\Rightarrow(\circ\varphi\wedge\circ\psi),\neg\varphi,\neg\psi}\;(\Rightarrow\wedge)\qquad\dfrac{\dfrac{\dfrac{\neg(\varphi\vee\psi),\psi\Rightarrow\varphi,\psi,\neg\varphi}{\neg(\varphi\vee\psi)\Rightarrow\varphi,\psi,\neg\varphi,\neg\psi}\;(\Rightarrow\neg)}{\neg(\varphi\vee\psi)\Rightarrow\varphi\vee\psi,\neg\varphi,\neg\psi}\;(\Rightarrow\vee)\qquad\neg(\varphi\vee\psi)\Rightarrow\neg(\varphi\vee\psi),\neg\varphi,\neg\psi}{\neg(\varphi\vee\psi),\circ(\varphi\vee\psi)\Rightarrow\neg\varphi,\neg\psi}\;(\circ\Rightarrow)}{\dfrac{(\circ\varphi\wedge\circ\psi)\supset\circ(\varphi\vee\psi),\neg(\varphi\vee\psi)\Rightarrow\neg\varphi,\neg\psi}{(\circ\varphi\wedge\circ\psi)\supset\circ(\varphi\vee\psi),\neg(\varphi\vee\psi)\Rightarrow\neg\varphi\vee\neg\psi}\;(\Rightarrow\vee)}\;(\supset\Rightarrow)}{(\circ\varphi\wedge\circ\psi)\supset\circ(\varphi\vee\psi)\Rightarrow\neg(\varphi\vee\psi)\supset(\neg\varphi\vee\neg\psi)}\;(\Rightarrow\supset)
$$

Figure 1: Selected examples of proofs