Exercise 1

Define the complexity class $C$ to be the class of all languages that can be verified by a TM that has:

- Input tape: Read only, move in both directions.
- Witness tape: Read only, move in both directions.
- Work tape: Read-Write, move in both directions.

The machine itself is deterministic (the guesses are the value of the witness tape). The space complexity is the size of the work tape, and is bounded by $O(\log n)$. We say the machine accepts an input if and only if there exists a setting for the witness tape, with which the machine accepts.

Prove that $NP \subseteq C$. Conclude that if $P \neq NP$ then $NL \neq C$.

Exercise 2

Define the complexity class $C$ to be the class of all languages that can be verified by a TM that has:

- Input tape: Read only, move in both directions.
- Witness tape: Read only, move in both directions.
- Work tape: Read-Write, move in both directions.

The machine itself is deterministic (the guesses are the value of the witness tape). The space complexity is the size of the work tape, and is polynomial. We say the machine accepts an input if and only if there exists a setting for the witness tape, with which the machine accepts.

Prove: $C = PSPACE$.

Exercise 3

1. Let $\Sigma_2 SAT$ denote the following decision problem: given a quantified formula $\varphi$ of the form $\varphi = \exists x_1, \ldots, x_n \forall y_1, \ldots, y_m \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, where $\varphi$ is a CNF formula, decide whether $\varphi$ is true. Prove that if $P = NP$ then $\Sigma_2 SAT \in P$.
2. For $k_1, k_2 \in \mathbb{N}$ define the problem $(k_1, k_2) - ColoringExtension$ (in short, $(k_1, k_2) - CE$) as follows: given a graph $G$ and a sub-set of vertices $S$, decide whether any $k_1$-coloring of $S$ can be extended to a $k_2$-coloring of $G$. Show that $(2, 3) - CE \in \Pi_2$ and that $(2, 2) - CE \in coNP$.

Exercise 4

Alice keeps a string $x_1, \ldots, x_n \in \{0, 1\}^n$ and Bob keeps $y_1, \ldots, y_n \in \{0, 1\}^n$. They want to know wether it is true that $x = y$ or not. Also, Alice and Bob share an access to a random string $r = r_1, \ldots, r_n \in \{0, 1\}^n$.

Show a protocol satisfying the following:

1. Alice sends one bit of information to Bob.
2. If $x = y$ then Bob always returns $True$.
3. If $x \neq y$ then Bob returns $True$ probability at least $\frac{1}{2}$, i.e. $Pr_r[Bob \ returns \ True] \leq \frac{1}{2}$.

GOOD LUCK