Constructing worst-case instances for semidefinite programming based approximation algorithms

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MAX CUT

Find a large cut in a graph.

[GW'95] gives an approximation algorithm with a performance guarantee of $\alpha \simeq 0.87856$.

MAX CUT of size at most $2m/3$

Find a large cut in a graph in which all cuts are of size at most $2m/3$.

[Z'99] gives an approximation algorithm with a performance guarantee of $\alpha' \simeq 0.89178$.

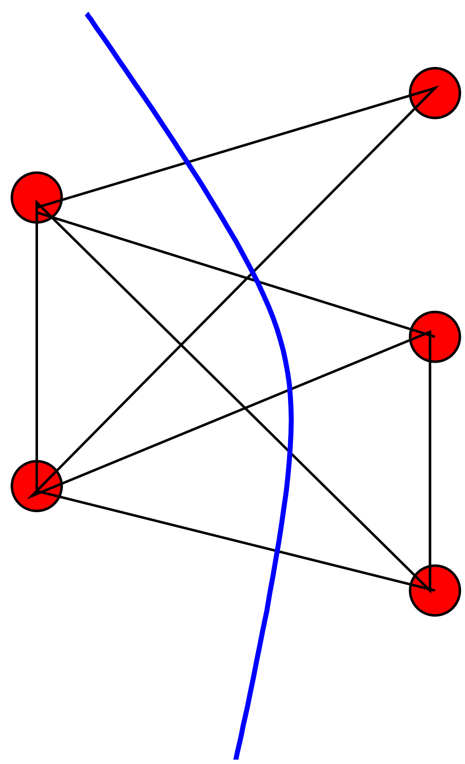
MAX NAE-{3}-SAT

Find an assignment that satisfies many clauses of a NAE-$\{3\}$-SAT instance.

Interesting special case: Find a large cut in a **3-uniform hypergraph**.

[Z'99] gives an approximation algorithm with a performance guarantee of $\beta \simeq 0.90871$.

MAX CUT



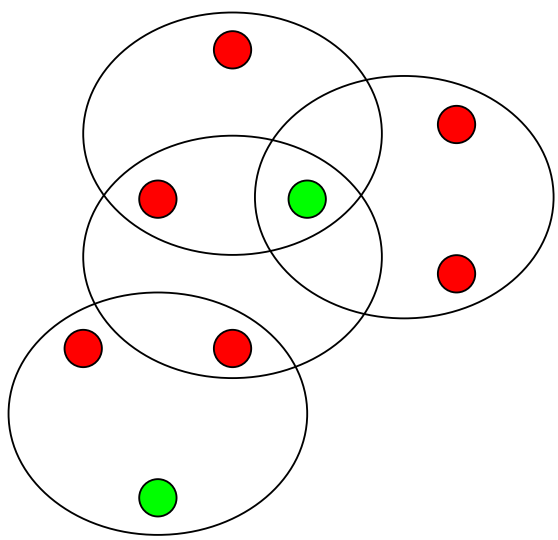
MAX NAE-{3}-SAT

$$\langle x_1, x_3, \bar{x}_5 \rangle \quad \langle \bar{x}_2, x_4, \bar{x}_7 \rangle \quad \langle x_1, x_2, \bar{x}_8 \rangle \quad \langle x_5, x_6, \bar{x}_7 \rangle$$

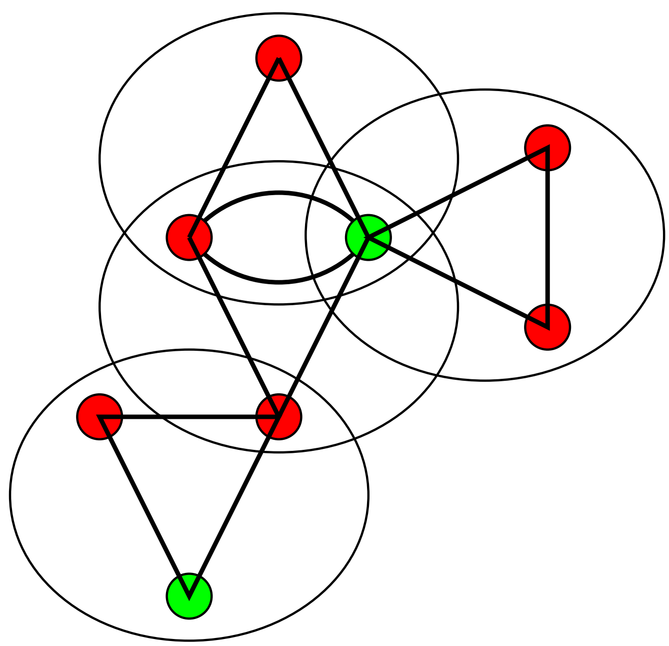
$$\langle x_3, x_8, \bar{x}_9 \rangle \quad \langle \bar{x}_3, \bar{x}_7, \bar{x}_9 \rangle \quad \langle \bar{x}_2, x_8, \bar{x}_9 \rangle \quad \langle x_4, \bar{x}_6, \bar{x}_9 \rangle$$

$$\ldots$$

MAX CUT in 3-uniform hypergraphs



MAX CUT in 3-uniform hypergraphs



The MAX CUT algorithm of Goemans and Williamson - extended version [GW'95],[Z'99]

1. Find an (almost) optimal solution $v_1, v_2, \ldots, v_n$ of the SDP program

$$\begin{array}{ll} \min & \sum_{i<j} w_{ij} \frac{1 - v_i \cdot v_j}{2} \\ \text{s.t.} & \|v_i\| = 1 \quad , \quad 1 \le i \le n \end{array}$$

2. Find unit vectors $u_1, u_2, \ldots, u_n$ such that

$$u_i \cdot u_j = \rho\,(v_i \cdot v_j) \quad , \quad \textbf{for } i \neq j$$

(Outward rotation).

3. Choose a random vector r and let

$$S = \{ i \mid u_i \cdot r \ge 0 \} \quad .$$

The MAX NAE-{3}-SAT approximation algorithm [Z'99]

1. Find an (almost) optimal solution $v_1, v_2, \ldots, v_n$ of the SDP program

$$\begin{array}{ll} \min & \sum_{i,j,k} w_{ijk} \frac{3 - v_i \cdot v_j - v_i \cdot v_k - v_j \cdot v_k}{4} \\ \text{s.t.} & v_i \cdot v_j + v_i \cdot v_k + v_j \cdot v_k \geq -1 \\ & v_i \cdot v_j - v_i \cdot v_k - v_j \cdot v_k \geq -1 \\ & -v_i \cdot v_j + v_i \cdot v_k - v_j \cdot v_k \geq -1 \\ & -v_i \cdot v_j - v_i \cdot v_k + v_j \cdot v_k \geq -1 \\ & ||v_i|| = 1 \quad , \quad 1 \leq i \leq n \end{array}$$

2. Find unit vectors $u_1, u_2, \ldots, u_n$ such that

$$u_i \cdot u_j = \rho\,(v_i \cdot v_j) \quad , \quad \textbf{for } i \neq j$$

(Outward rotation).

3. Choose a random vector r and let

$$S = \{i \mid u_i \cdot r \geq 0\} \quad .$$

Analysis of the MAX CUT algorithm [GW'95]

Goemans and Williamson prove, using a simple **local analysis**, that the performace ratio of their algorithm is **at least**

$$\alpha = \min_{0<\theta\leq\pi} \frac{2}{\pi}\frac{\theta}{1-\cos\theta} \simeq 0.87856 \quad .$$

The minimum is attained at

$$\theta_0 \simeq 2.331122 \quad .$$

To show that the analysis is tight,
find a graph $G = (V, E)$ for which:

1. Optimal value of SDP = size of max cut.
2. The SDP has an optimal solution in which $v_i \cdot v_j = \cos\theta_0$, for every $(i, j) \in E$.

Worst-case instances for MAX CUT
(Karloff '96) (Alon-Sudakov '00)

The Hamming graph $H(m, b)$

$$\mathbf{x} \in V \text{ iff } \mathbf{x} = (x_1, x_2, \ldots x_m) \in \{-1, +1\}^m.$$

$$(\mathbf{x}, \mathbf{y}) \in E \text{ iff } d_H(\mathbf{x}, \mathbf{y}) = b.$$

1. Number of vertices is 2^m.
2. Number of edges is $2^{m-1}\binom{m}{b}$.
3. opt(SDP) $=$ max cut $= 2^{m-1}\binom{m-1}{b-1}$.
4. $v_{\mathbf{x}} = \frac{\mathbf{x}}{\sqrt{m}}$, for $\mathbf{x} \in V$,
   is an optimal solution of SDP.
5. If $(\mathbf{x}, \mathbf{y}) \in E$, then $v_{\mathbf{x}} \cdot v_{\mathbf{y}} = \frac{m-2b}{m}$.

Choose b such that $\frac{m-2b}{m} \simeq \cos\theta_0$.

Basic Lemma

Lemma 1. *Let $G = (V, E)$ be a multigraph, where $V = \{1, 2, \ldots, n\}$, with adjacency matrix $A = (a_{ij})$. Let λ_n be the* ***smallest eigenvalue*** *of A. Let $v_1, \ldots, v_n$ be unit vectors in $R^k, k > 0$. Then*

$$\sum_{i<j} a_{ij} \frac{1 - v_i \cdot v_j}{2} \leq \frac{|E|}{2} - \frac{\lambda_n |V|}{4}.$$

In addition, if $B = (b_{ij})$ is the $n \times k$ matrix whose rows are the vectors $v_1^T, \ldots, v_n^T$, then equality holds if and only if each column of B is an eigenvector of A with eigenvalue λ_n.

Eigenvalues and eigenvectors of Hamming graphs

The Hamming graph $H(m, b)$ is a **Cayley** graph of the multiplicative group Z_2^m.

The eigenvectors are the **multiplicative characters** $\chi_I(x) = \prod_{i \in I} x_i$, where $I \subseteq \{1, \ldots, m\}$.

The eigenvalues are the values of the **Krawtchouk polynomial** at $0 \leq k \leq m$,

$$P_b^m(k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{m-k}{b-j} .$$

[Alon-Sudakov'00] show that for m large enough, the smallest eigenvalue is

$$\lambda = P_b^m(1) = \frac{m-2b}{m} \binom{m}{b} .$$

Worst-case instances for MAX CUT of size at most $2m/3$

Find graphs $G = (V, E)$ for which:

1. Opt(SDP) $=$ size of max cut $= 2m/3$.
2. The SDP has an optimal solution such that for every edge $(i, j) \in E$,

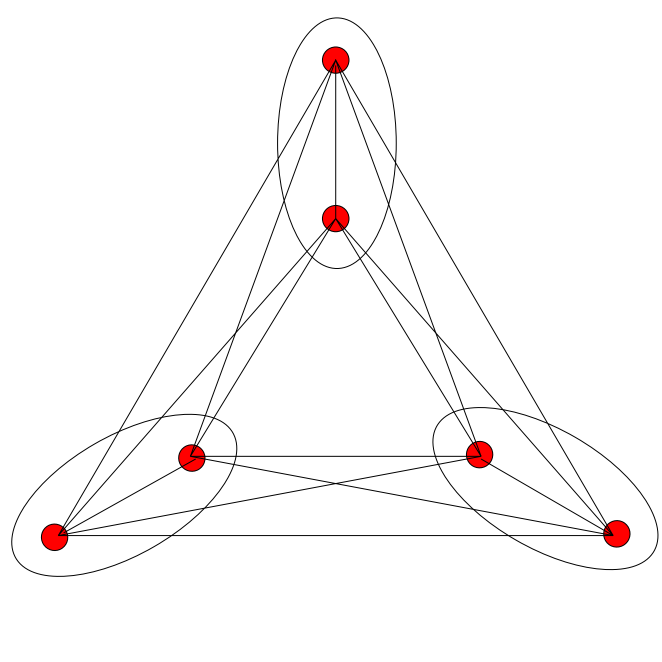
   $$v_i \cdot v_j = \cos\theta_1 \quad \textbf{or} \quad v_i \cdot v_j = 1 \ ,$$

   where $\theta_1 \simeq 2.20217$.

The required graphs are **compositions** / **tensor products** of **Hammimg graphs** and **expanders.**

Composition/tensor product of graphs

Given a graph A on n vertices and a bipartite graph B on $2k$ vertices, k on each side, replace each vertex of A by a clique on k vertices, and each edge of A by a copy of B:



Tensor products

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}$$

Worst-case instances for MAX NAE-{3}-SAT: requirements

Find 3-uniform hypergraphs $H = (V, E)$ s.t.:

1. Optimal value of SDP = size of max cut.
2. The SDP has an optimal solution in which for every $(i, j, k) \in E$,

$$\{v_i \cdot v_j \ , \ v_i \cdot v_k \ , \ v_j \cdot v_k\} = \{\eta, \eta, 1\} \quad \textbf{or}$$

$$\{v_i \cdot v_j \ , \ v_i \cdot v_k \ , \ v_j \cdot v_k\} = \{-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\} ,$$

where $\eta \simeq -0.74336$.

Worst-case instances for MAX NAE-{3}-SAT: construction

Let M_n be a perfect matching on $2n$ vertices.

Let $K_{n,n}$ be a complete bipartite graph with n vertices on each side.

Construct the following two graphs,
on the same set of vertices:

$$H(m, \frac{2m}{3}) \otimes M_n \quad , \quad H(m, \frac{1-\eta}{2}m) \otimes K_{n,n}$$

The edges of the 3-uniform hypergraph are the **triangles** of this combined graph.

Open problems

1. Is there a better approximation algorithm for **MAX CUT**?
2. Is there a better approximation algorithm for **MAX NAE-{3}-SAT**?
3. Is the local analysis of the **Feige** and **Goemans** **MAX 2-SAT** and **MAX DI-CUT** approximation algorithms tight?