# Null cone membership for the left right action on tuples of matrices

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# Outline

1. **Background and problem statement**
   - Problem statement
   - Invariant theory
2. **Using Gurvits algorithm**
3. **Progress via Blow-ups**
   - Regularity
   - Algorithmic and degree bounds
   - Degree bounds
   - Polynomial bound - degree of generation
   - Main lemma and blow ups using division algebras
   - Proof of the main lemma
   - Matrix of maximum rank
   - Division algebras

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## Problem statement

- $\mathsf{M}at(n, \mathbb{F})$ - $n \times n$ matrices with entries in $\mathbb{F}$.
- $B_1, B_2, \ldots, B_m \in \mathsf{M}at(n, \mathbb{F})$.

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**Shrunk subspaces**

A subspace $U \leqslant \mathbb{F}^n$ is $c$-shrunk by $\mathcal{B}$ if there is a subspace $W \subseteq \mathbb{F}^n$ such that dim W $\leqslant \dim U - c$, and for all matrices $B$ in $\mathcal{B}$, $\langle BU \rangle \subseteq W$.

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## Non commutative rank

$n - \max(c \in \{0, 1, \ldots, n\} \mid \exists subspace$ $c$-shrunk by $\mathcal{B})$ [FR04].

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Problem NCrk: What is the noncommutative rank?

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**Theorem - Gurvits**

Over $\mathbb{Q}$, given a matrix space $\langle \mathcal{B} \rangle$ there is a deterministic polynomial time algorithm which will output Rk=$n$, or NCrk $< n$, and its output is guaranteed to be correct when either $NCrk(\mathcal{B}) < n$ or $\mathrm{Rk}(\mathcal{B}) = n$.

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The algorithm may give a wrong answer in the case when $n = NCrk > Rk$.

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Question How long do we go on?

# Implications of degree bound $\sigma$

**Theorem**

[IQS15a] Over $\mathbb{Q}$, if the nullcone is defined by elements of degree $\leqslant \sigma = \sigma(n, m)$,there exists a deterministic poly$(n, m, \sigma)$ algorithm deciding if $(B_1, B_2, \ldots, B_m)$ is in the null cone.

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- Output NCrk$(\mathcal{B}) < n$.

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However...finding a nonsingular matrix in the span will be difficult.

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- Using $M_{i-1}$, update and get $M_i$, achieving some measurable progress.

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# Regularity of Blow-ups

**Main Lemma**

For $\mathcal{B} \leqslant Mat(n, \mathbb{F})$ and $\mathcal{A} = \mathcal{B}^{\{d,d\}}$, assume that $|\mathbb{F}| > 2rd$. Given a matrix $A \in \mathcal{A}$ with $\mathrm{rk} A > (r-1)d$, there exists a deterministic algorithm that returns $\widetilde{A} \in \mathcal{A}$ and an $r \times r$ window $W$ in $\widetilde{A}$ s.t. $W$ is nonsingular (of rank $rd$). This algorithm uses $\mathrm{poly}(nd)$ operations and, over $\mathbb{Q}$, the algorithm runs in polynomial time.

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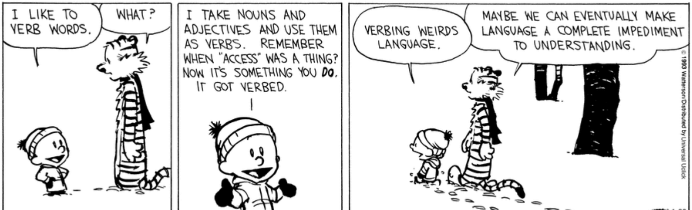

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Central division algebras almost do our job.

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5. Use regularity of blow-ups to get a matrix of rank $(r+1)*(r+1)$ in the blow up.
6. Loop back to step 2 with $\mathcal{B} = \mathcal{A}$ and $r = r+1$.

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- Identifying the shrunk subspace, if any.
- Knowing when to stop.

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- [IQS15b] Show $\sigma \leqslant O(n^2)$ over all large fields. Two proofs - a constructive version of [DM15] and a simple proof based on regularity under blow-up. Get the above results.

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- So the largest ranked matrix in a $n+1 \times n+1$ window is $(n+1)*(n-1) = n^2-1$.
- But we add to such a matrix at most $2n$ linearly independent rows and columns.
- So rank is upper bounded by $n^2-1+2n$, cannot be $(n+2)*n$. Regularity says rank is at most $(n+2)*(n-1) = n^2+n-2$. QED

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## Claim

Let $\mathbb{F}'$ be an extension field of $\mathbb{F}$, and Let $D$ be a central division algebra over $\mathbb{F}'$ of dimension $d^2$ over $\mathbb{F}'$, and let $\mathbb{K}$ be a maximal field in $D$ with extension degree $d$ over $\mathbb{F}'$. Let $\rho : D \to Mat(d, \mathbb{K})$ be a representation of $D$ over $\mathbb{K}$. Then every matrix in $Mat(n, \mathbb{F}) \otimes_{\mathbb{F}} \rho(D)$ has rank divisible by $d$ over $\mathbb{K}$.

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- Since $D \otimes D^{op} \cong \mathrm{Mat}(d, \mathbb{F}') \subset \mathrm{Mat}(\mathbb{K})$, the centralizer of the action of $\mathrm{Mat}(n, \mathbb{F}) \otimes_{\mathbb{F}} \rho(D)$ is $\mathrm{id} \otimes D^{op} \cong D^{op}$.

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- $D \otimes \mathbb{K} \simeq Mat(\mathbb{K})$. Explicit matrices describing the $\mathbb{F}'$-algebra $D \simeq D \otimes 1$ can be written down easily.
- Regard $\mathbb{K}^{dn} \simeq \mathbb{F}'^{d^2 n}$ as a module over $Mat(n, \mathbb{F}) \otimes_{\mathbb{F}} \rho(D)$.
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- For all $A$ in $Mat(n, \mathbb{F}) \otimes_{\mathbb{F}} \rho(D)$, $A\mathbb{F}'^{d^2 n}$ is a $D^{op}$-submodule, and so its dimension over $\mathbb{F}'$ is divisible by $d^2$, so dimension over $\mathbb{K}$ is divisible by $d$. But this is the rank of $A'$.

# Recap

**Main Lemma**

For $\mathcal{B} \leqslant Mat(n, \mathbb{F})$ and $\mathcal{A} = \mathcal{B}^{\{d,d\}}$, assume that $|\mathbb{F}| > 2rd$. Given a matrix $A \in \mathcal{A}$ with $\mathrm{rk} A > (r-1)d$, there exists a deterministic algorithm that returns $\widetilde{A} \in \mathcal{A}$ and an $r \times r$ window $W$ in $\widetilde{A}$ s.t. $W$ is nonsingular (of rank $rd$). This algorithm uses $\mathrm{poly}(nd)$ operations and, over $\mathbb{Q}$, the algorithm runs in polynomial time.

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- Wlog we have matrix of rank at least $(n - 1)d + 1$ with the principal $n - 1$ window having a nonsingular matrix.

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- Because $\mathbb{F} \geqslant 2nd$, we can find a matrix in $\mathcal{A}$ of rank $nd$ using ideas from [dGIR96].
- We need to construct division algebras, and be able to compute with them, at each stage

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- Let $B_1, \ldots, B_l$ be a $\mathbb{F}$ basis of $\mathcal{B}$. Then $A = a_1' B_1 + a_2' B_2 + \ldots + a_l' B_l$, and there is a $r \times r$ window in $A$ with nonzero determinant, say the principal $r$ window.

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- Complete the proof by recursion, substituting values for $a'_2, a'_3, \ldots, a'_l$.

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**Definition**

Given $(A, \mathcal{B})$, $A \in \mathrm{Mat}(n, \mathbb{F})$ and $\mathcal{B} \leqslant \mathrm{Mat}(n, \mathbb{F})$, the second Wong sequence of $(A, \mathcal{B})$ is the following sequence of subspaces in $\mathbb{F}^n$: $W_0 = 0$, $W_1 = \mathcal{B}(A^{-1}(W_0))$, ..., $W_i = \mathcal{B}(A^{-1}(W_{i-1}))$, ....

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Given $(A, \mathcal{B})$, $A \in \mathrm{Mat}(n, \mathbb{F})$ and $\mathcal{B} \leqslant \mathrm{Mat}(n, \mathbb{F})$, the second Wong sequence of $(A, \mathcal{B})$ is the following sequence of subspaces in $\mathbb{F}^n$: $W_0 = 0$, $W_1 = \mathcal{B}(A^{-1}(W_0)), \ldots,$ $W_i = \mathcal{B}(A^{-1}(W_{i-1})), \ldots.$

- $W_0 < W_1 < W_2 < \cdots < W_\ell = W_{\ell+1} = \ldots$ for some $\ell \in \{0, 1, \ldots, n\}$. $W_\ell$ is then called the limit of this sequence, denoted as $W^*$.

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- When $A \in \mathcal{B}$, $W^* \leqslant \mathrm{im}(A)$ if and only if there exists a corank$(A)$-shrunk subspace
- $A$ is of maximum rank and $A^{-1}(W^*)$ is a corank($A$)-shrunk subspace.

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## Incrementing rank

Let $\mathcal{B} \leqslant \mathrm{Mat}(n, \mathbb{F})$ and let $\mathcal{A} = \mathcal{B}^{\{d,d\}}$. Assume that we are given a matrix $A \in \mathcal{A}$ with $\mathrm{rk}(A) = rd$, and $|\mathbb{F}|$ is $\Omega(ndd')$, where $d' > r$ is any positive integer. There exists a deterministic algorithm that returns either an $(n-r)d$-shrunk subspace for $\mathcal{A}$ (equivalently, an $(n-r)$-shrunk subspace for $\mathcal{B}$), or a matrix $B \in \mathcal{A} \otimes \mathrm{Mat}(d', \mathbb{F})$ of rank at least $(r+1)dd'$.

# Cyclic algebras and the construction of Dickson

- Let $\mathbb{K}/\mathbb{F}$ be a Galois extension with cyclic Galois group. Let $\sigma$ be a generator of the Galois group and $s = dim_{\mathbb{F}}(\mathbb{K})$.
- Take $f \in \mathbb{F}$ and a symbol $x$, and consider $D = \mathbb{K} \bigoplus \mathbb{K} \cdot x \bigoplus \mathbb{K} \cdot x^2 + \ldots \mathbb{K} \cdot x^{s-1}$.
- Multiply elements in $D$ using the distributive law and using $x^s = f$ and $x \cdot b = \sigma(b)x$ for all $b \in K$.
- $\mathbb{F}$ i in the center of $D$ and so $D$ is an $\mathbb{F}$-algebra. Dimension over $\mathbb{F}$ is $s^2$.
- Wedderburn - if $f, f^2, \ldots, f^{s-1}$ are not in Norm$(\mathbb{K})$, then $D$ is a division algebra, and in this case $D \otimes_{\mathbb{F}} \mathbb{K} \cong Mat(\mathbb{K})$.

# Blowing-down a shrunk subspace

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## Shrinking by a factor of $d$

If $\mathcal{A} = \mathcal{B}^{\{d,d\}}$ has an $s$-shrunk subspace, then $\mathcal{A}$ has an $s'$-shrunk subspace with $s' \geqslant s$ and s.t. $d$ divides $s'$. $\mathcal{B}$ has an $s'/d$-shrunk subspace.

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Idea Maximal shrunk subspaces are of the form $U_o \otimes \mathbb{F}^d$ and their image under $\mathcal{A}$ is of the form $W_o \otimes \mathbb{F}^d$.

# Blowing-down

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### Reducing the size of blow-ups

Let $\mathcal{B} \leqslant \mathrm{Mat}(n, \mathbb{F})$, and $d > n+1$. Assume we are given a matrix $A \in \mathcal{B}^{\{d,d\}}$ of rank $dn$. Then there exists a deterministic polynomial-time procedure that constructs $A' \in \mathcal{B}^{\{d-1,d-1\}}$ of rank $(d-1)n$.

## Construction of division algebras

Let $L$ be a cyclic extension of degree $d$ of a field $K'$. Let $\sigma$ be a generator of the Galois group. Consider the transcendental extension $L(Z)$ of $L$. Then $\sigma$ extends to an automorphism (denoted again by $\sigma$) of $L(Z)$ such that the fixed field of $\sigma$ is $K'(Z)$. Thus $L(Z)$ is a cyclic extension of $K'(Z)$. Consider the $K'(Z)$-algebra $D$ generated by (a basis for) $L$ and by an element $U$ with relations $U^d = Z$ and $Ua = a^\sigma U$ ($\forall a \in L(Z)$, or, equivalently $\forall a \in$ the basis for $L$). Then $D$ is a central division algebra of index $d$ over $K'(Z)$.

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- Get a combinatorial algorithm in characteristic zero.
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- Orbit closure problem for the left right action .. NNL for this invariant ring.

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