Expressive Completeness over Nat and Finite orders

- MLO=Automata=regular expressions (over finite orders).

Expressive Completeness over Nat and Finite orders

- MLO=Automata=regular expressions (over finite orders).
- MLO=ω-Automata=ω-regular expressions (over Nat).

Expressive Completeness over Nat and Finite orders

- MLO=Automata=regular expressions (over finite orders).
- MLO=ω-Automata=ω-regular expressions (over Nat).
- FOMLO=TL(U,S) (over Dedekind complete orders)

Expressive Completeness over Nat and Finite orders

- MLO=Automata=regular expressions (over finite orders).
- MLO=ω-Automata=ω-regular expressions (over Nat).
- FOMLO=TL(U,S) (over Dedekind complete orders)
- FOMLO= star free regular expressions (over finite orders)

Expressive Completeness over Nat and Finite orders

- MLO=Automata=regular expressions (over finite orders).
- MLO=ω-Automata=ω-regular expressions (over Nat).
- FOMLO=TL(U,S) (over Dedekind complete orders)
- FOMLO= star free regular expressions (over finite orders)
- FOMLO = Counter-free automata (over finite orders)

Counter-free automata

Def. A sequence of states $q_0,\ q_1, \ldots q_m$ (for $m > 0$) in an automaton A is a counter for a string u if $\delta(q_i, u) = q_{i+1}$ where by convention $q_0 = q_{m+1}$.

Counter-free automata

Def. A sequence of states $q_0,\ q_1, \ldots q_m$ (for $m > 0$) in an automaton A is a counter for a string u if $\delta(q_i, u) = q_{i+1}$ where by convention $q_0 = q_{m+1}$.

Def. An automaton is counter-free iff it does not have a counter.

Counter-free automata

Def. A sequence of states $q_0,\ q_1, \ldots q_m$ (for $m > 0$) in an automaton A is a counter for a string u if $\delta(q_i, u) = q_{i+1}$ where by convention $q_0 = q_{m+1}$.

Def. An automaton is counter-free iff it does not have a counter.

Theorem (MacNaughton) A language is definable by FOMLO formula iff it is accepted by a deterministic counter-free automaton iff it is definable by a star free regular expression.

The complexity of TL(U) over Nat

Theorem The satisfiability problem for TL(U) over Nat is in PSPACE.

The complexity of TL(U) over Nat

Theorem The satisfiability problem for TL(U) over Nat is in PSPACE.

Lemma(Small Model property) If ϕ is satisfiable then it is satisfiable on a *quasi-periodic* model uv^{ω} with u, v small ($O(2^{|\phi|} \times |\phi|)$

The complexity of TL(U) over Nat

Theorem The satisfiability problem for TL(U) over Nat is in PSPACE.

Lemma(Small Model property) If ϕ is satisfiable then it is satisfiable on a *quasi-periodic* model uv^{ω} with u, v small ($O(2^{|\phi|} \times |\phi|)$

Lemma The satisfiability of ϕ over small model can be checked in NPSPACE.

The complexity of TL(U) over Nat

Theorem The satisfiability problem for TL(U) over Nat is in PSPACE.

Lemma(Small Model property) If ϕ is satisfiable then it is satisfiable on a *quasi-periodic* model uv^ω with u, v small ($O(2^{|\phi|} \times |\phi|)$

Lemma The satisfiability of ϕ over small model can be checked in NPSPACE.

Homework: Prove PSPACE lower bound for the satifiability problem

The complexity of TL(U) over Nat

Theorem The satisfiability problem for TL(U) over Nat is in PSPACE.

Lemma(Small Model property) If ϕ is satisfiable then it is satisfiable on a *quasi-periodic* model uv^ω with u, v small ($O(2^{|\phi|} \times |\phi|)$

Lemma The satisfiability of ϕ over small model can be checked in NPSPACE.

Homework: Prove PSPACE lower bound for the satifiability problem

Hint: For every PSPACE TM M and a word u construct a formula $\phi_{M,u}$ which is satisfiable iff M accepts u.

Proof of a small model property

Notations: Sub(ϕ) - the set of subformulas of ϕ

Proof of a small model property

Notations: Sub(ϕ) - the set of subformulas of ϕ

Example $\phi = (XU(YU(Z \wedge \neg X)))$

The Subformulas of ϕ

$\{X.\neg X, Y, \neg Y, Z, \neg Z, YU(Z \wedge \neg X), \neg YU(Z \wedge \neg X), Z \wedge \neg X, \neg(Z \wedge \neg X)\} \cup \{\phi\}$

Proof of a small model property

Notations: Sub(ϕ) - the set of subformulas of ϕ
Example $\phi = (XU(YU(Z \wedge \neg X)))$
The Subformulas of ϕ
$\{X.\neg X, Y, \neg Y, Z, \neg Z, YU(Z \wedge \neg X), \neg YU(Z \wedge \neg X), Z \wedge \neg X, \neg(Z \wedge \neg X)\} \cup \{\phi\}$
Number of subformulas - $O(|\phi|)$

Proof of a small model property

Notations: Sub(ϕ) - the set of subformulas of ϕ
Example $\phi = (XU(YU(Z \wedge \neg X)))$
The Subformulas of ϕ
$\{X.\neg X, Y, \neg Y, Z, \neg Z, YU(Z \wedge \neg X), \neg YU(Z \wedge \neg X), Z \wedge \neg X, \neg(Z \wedge \neg X)\} \cup \{\phi\}$
Number of subformulas - $O(|\phi|)$

Def (Type) Let ϕ be a formula A be a linear order with monadic predicates and b an element of A.

$$type^{\phi}_{A}(b) = \{\psi \in Sub(\phi) \ : \ A, b \models \psi\}$$

Proof of a small model property

Assume $type^{\phi}_{A_1+A_2+A_3}(a) = type^{\phi}_{A_1+A_2+A_3}(b)$

Then

Proof of a small model property

Assume $type^{\phi}_{A_1+A_2+A_3}(a) = type^{\phi}_{A_1+A_2+A_3}(b)$

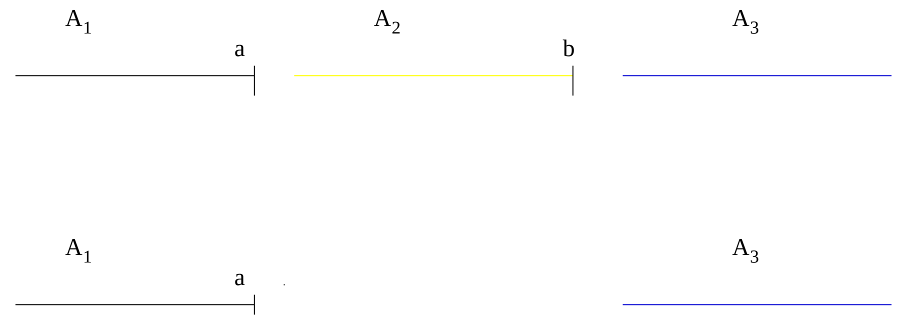
Then

1. For every $c \in A_3$

$$type^{\phi}_{A_1+A_2+A_3}(c) = type^{\phi}_{A_1+A_3}(c)$$

Proof of a small model property

Assume $type^{\phi}_{A_1+A_2+A_3}(a) = type^{\phi}_{A_1+A_2+A_3}(b)$



Then

1. For every $c \in A_3$

$$type^{\phi}_{A_1+A_2+A_3}(c) = type^{\phi}_{A_1+A_3}(c)$$

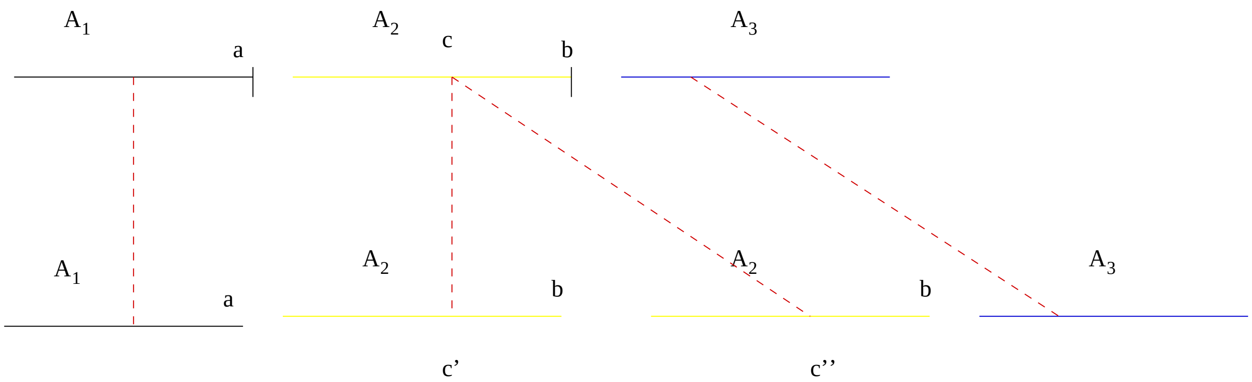
2. For every $c \in A_1$

$$type^{\phi}_{A_1+A_2+A_3}(c) = type^{\phi}_{A_1+A_3}(c)$$

Proof of a small model property

Additional transformations

Image of a point



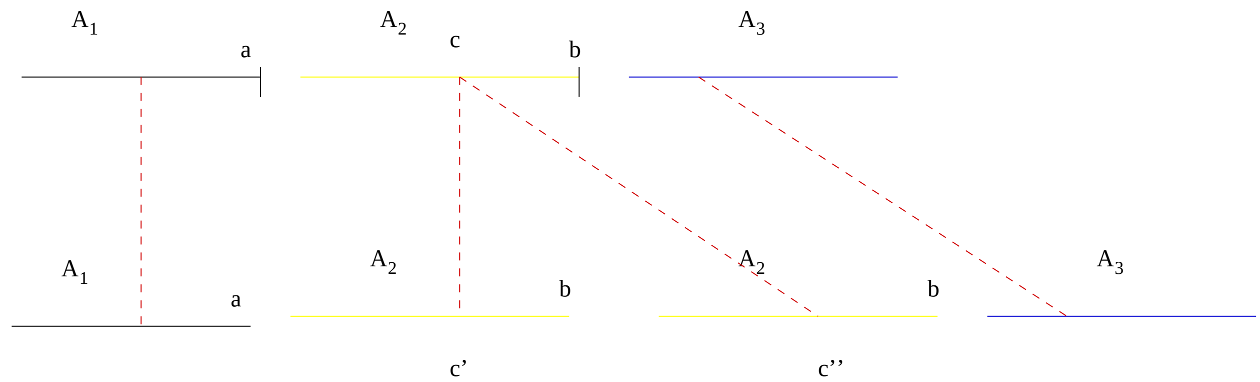
Assume $type^{\phi}_{A_1+A_2+A_3}(a) = type^{\phi}_{A_1+A_2+A_3}(b)$

Then

Proof of a small model property

Additional transformations

Image of a point



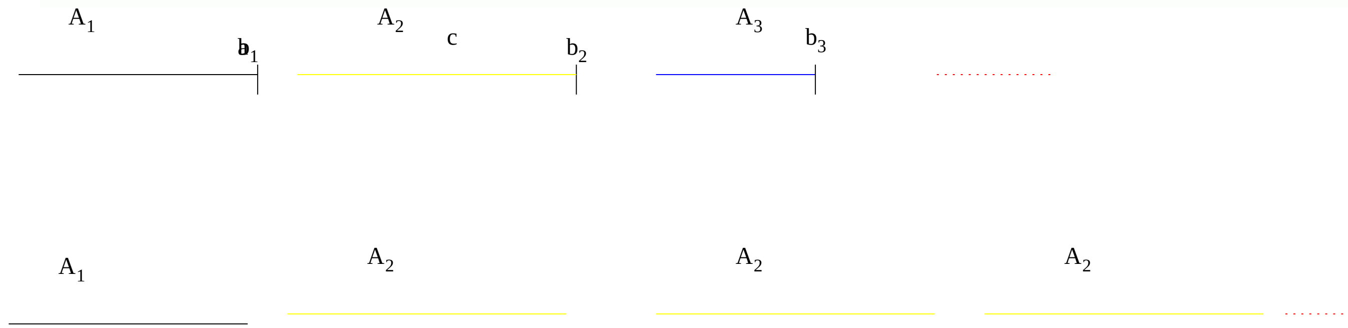
Assume $type^{\phi}_{A_1+A_2+A_3}(a) = type^{\phi}_{A_1+A_2+A_3}(b)$

Then

For every c and its image d

$$type^{\phi}_{A_1+A_2+A_3}(c) = type^{\phi}_{A_1+A_2+A_2+A_3}(d)$$

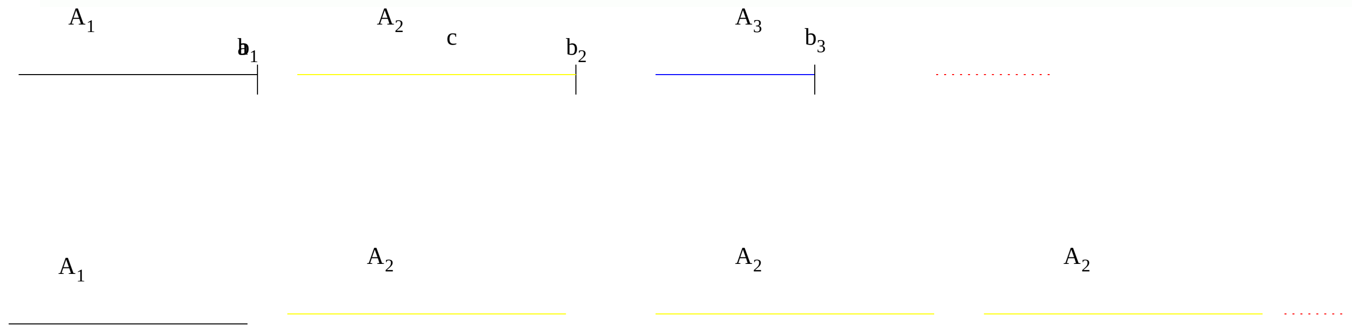
Proof of a small model property



Assume that b_i is an unbounded increasing sequence and

- $type^{\phi}_{A}(b_i) = type^{\phi}_{A}(b_j)$ for $i, j \in Nat$

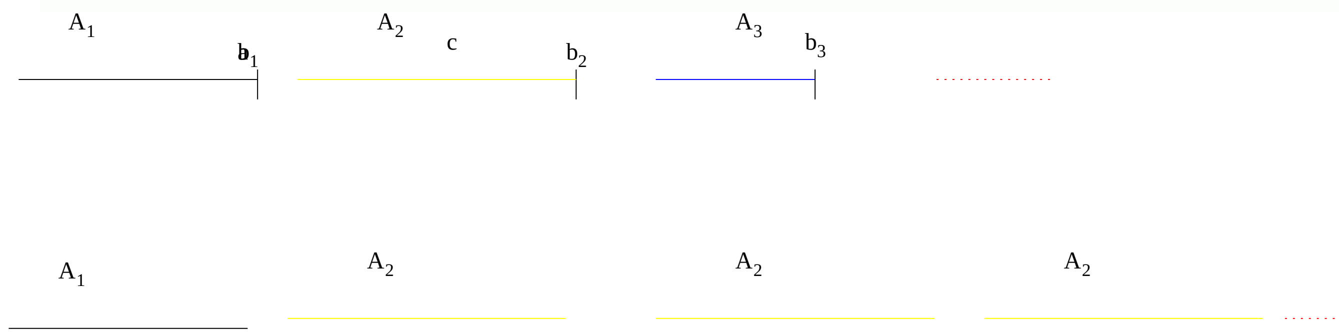
Proof of a small model property



Assume that b_i is an unbounded increasing sequence and

- $type^{\phi}_{A}(b_i) = type^{\phi}_{A}(b_j)$ for $i, j \in Nat$
- For $\phi_1 U \phi_2 \in type^{\phi}_{A}(b_1)$ there is $c \in A_2$ such that $A, c \models \phi_2$.

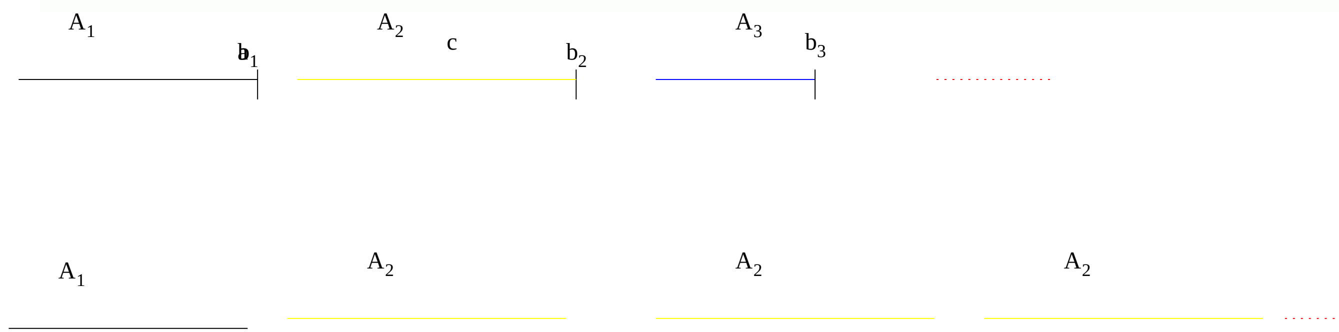
Proof of a small model property



Assume that b_i is an unbounded increasing sequence and

- $type_A^{\phi}(b_i) = type_A^{\phi}(b_j)$ for $i, j \in Nat$
- For $\phi_1 U \phi_2 \in type_A^{\phi}(b_1)$ there is $c \in A_2$ such that $A, c \models \phi_2$.

Proof of a small model property



Assume that b_i is an unbounded increasing sequence and

- $type^{\phi}_{A}(b_i) = type^{\phi}_{A}(b_j)$ for $i, j \in Nat$
- For $\phi_1 U \phi_2 \in type^{\phi}_{A}(b_1)$ there is $c \in A_2$ such that $A, c \models \phi_2$.

Then for every $d \in A_1 \cup A_2$ and $\psi \in Sub(\phi)$

$$A, d \models \psi \text{ iff } A_1 + \omega \times A_2, d \models \psi$$

Proof of a small model property

Hence

- if ϕ is satisfiable over a linear structure without a maximal element then it is satisfiable over a structure $A_1 + \omega A_2$.

Proof of a small model property

Hence

- if ϕ is satisfiable over a linear structure without a maximal element then it is satisfiable over a structure $A_1 + \omega A_2$.
- if ϕ is satisfiable over the discrete time then it is satisfiable over a quasiperiodic structure uv^ω.

Proof of a small model property

Hence

- if ϕ is satisfiable over a linear structure without a maximal element then it is satisfiable over a structure $A_1 + \omega A_2$.
- if ϕ is satisfiable over the discrete time then it is satisfiable over a quasiperiodic structure uv^{ω}.

Proof of a small model property

Hence

- if ϕ is satisfiable over a linear structure without a maximal element then it is satisfiable over a structure $A_1 + \omega A_2$.
- if ϕ is satisfiable over the discrete time then it is satisfiable over a quasiperiodic structure uv^ω.

What is the length of u?

Proof of a small model property

Hence

- if ϕ is satisfiable over a linear structure without a maximal element then it is satisfiable over a structure $A_1 + \omega A_2$.
- if ϕ is satisfiable over the discrete time then it is satisfiable over a quasiperiodic structure uv^ω.

What is the length of u?

$|u| \leq$ the number of types of $\phi \leq 2^{|\phi|}$.

Proof of a small model property

Hence

- if ϕ is satisfiable over a linear structure without a maximal element then it is satisfiable over a structure $A_1 + \omega A_2$.
- if ϕ is satisfiable over the discrete time then it is satisfiable over a quasiperiodic structure uv^ω.

What is the length of u?

$|u| \leq$ the number of types of $\phi \leq 2^{|\phi|}$.

What is the length of v?

Proof of a small model property

Hence

- if ϕ is satisfiable over a linear structure without a maximal element then it is satisfiable over a structure $A_1 + \omega A_2$.
- if ϕ is satisfiable over the discrete time then it is satisfiable over a quasiperiodic structure uv^ω.

What is the length of u?

$|u| \leq$ the number of types of $\phi \leq 2^{|\phi|}$.

What is the length of v?

$|v| \leq 2^{|\phi|} \times |\phi|$.

The complexity of satisfiability for TL(U)

The small model property Lemma implies

The complexity of satisfiability for TL(U)

The small model property Lemma implies

Theorem The satisfiability problem for TL(U) over Nat is in NEXPTIME.

The complexity of satisfiability for TL(U)

The small model property Lemma implies

Theorem The satisfiability problem for TL(U) over Nat is in NEXPTIME.

Algorithm If ϕ is satisfiable then there are exponentially small u and v such that $uv^\omega, 0 \models \phi$. An Algorithm guesses u and v and checks that the guesses are correct.

The complexity of satisfiability for TL(U)

The small model property Lemma implies
Theorem The satisfiability problem for TL(U) over Nat is in NEXPTIME.

Algorithm If ϕ is satisfiable then there are exponentially small u and v such that $uv^{\omega}, 0 \models \phi$. An Algorithm guesses u and v and checks that the guesses are correct.

The algorithm can be implemented on the fly (without explicit construction of u and v) in PSPACE.
Hence

The complexity of satisfiability for TL(U)

The small model property Lemma implies

Theorem The satisfiability problem for TL(U) over Nat is in NEXPTIME.

Algorithm If ϕ is satisfiable then there are exponentially small u and v such that $uv^{\omega}, 0 \models \phi$. An Algorithm guesses u and v and checks that the guesses are correct.

The algorithm can be implemented on the fly (without explicit construction of u and v) in PSPACE.
Hence

Theorem The satisfiability problem for TL(U) over Nat is in PSPACE.

From TL(U) to Automata

Theorem. For every $\phi \in TL(U)$ there is a Street automata of size $2^{\phi|}$ that is equivalent to ϕ.

From TL(U) to Automata

Theorem. For every $\phi \in TL(U)$ there is a Street automata of size $2^{\phi|}$ that is equivalent to ϕ.

Def. a set S of formulas is boolean consistent iff

1. $\phi_1 \wedge \phi_2 \in S$ iff $\phi_1 \in S$ and $\phi_2 \in S$.
2. $\neg\psi \in S$ iff $\psi \notin S$

From TL(U) to Automata

Theorem. For every $\phi \in TL(U)$ there is a Street automata of size $2^{\phi|}$ that is equivalent to ϕ.

Def. a set S of formulas is boolean consistent iff

1. $\phi_1 \wedge \phi_2 \in S$ iff $\phi_1 \in S$ and $\phi_2 \in S$.
2. $\neg\psi \in S$ iff $\psi \notin S$

Observation $type_A^\phi$ is a maximal boolean consistent subset of the subformulas of ϕ.

From TL(U) to Automata

States. The maximal Consistent subsets of $Sub(\phi)$.

From TL(U) to Automata

States. The maximal Consistent subsets of $Sub(\phi)$.

Alphabet Let $X_1, \ldots X_n$ be the atomic propositions in ϕ. The alphabet is the subsets of $\{1, \ldots, n\}$.

From TL(U) to Automata

States. The maximal Consistent subsets of $Sub(\phi)$.

Alphabet Let $X_1, \ldots X_n$ be the atomic propositions in ϕ. The alphabet is the subsets of $\{1, \ldots, n\}$.

Transitions Let a be the set of atomic propositions which are true at a state s. From s only a transitions are enabled.

$s \rightarrow_a s'$ iff for every $\phi_1 U \phi_2 \in S$
either $\phi_2 \in s'$ or $\phi_1 \in s'$ and $\phi_1 U \phi_2 \in s'$

From TL(U) to Automata

Notations. G_ψ the set of states that contain formula ψ.

From TL(U) to Automata

Notations. G_ψ the set of states that contain formula ψ.

The Initial States: G_ϕ

From TL(U) to Automata

Notations. G_ψ the set of states that contain formula ψ.

The Initial States: G_ϕ

The Street Acceptance conditions: For every $\phi_1 U \phi_2 \in Sub(\phi)$ we have the pair $\langle G_{\phi_1 U \phi_2},\ G_{\phi_2} \rangle$ (i.e. if $\phi_1 U \phi_2$ holds infinitely often then ϕ_2 holds infinitely often,)

From TL(U) to Automata

Notations. G_ψ the set of states that contain formula ψ.

The Initial States: G_ϕ

The Street Acceptance conditions: For every $\phi_1 U \phi_2 \in Sub(\phi)$ we have the pair $\langle G_{\phi_1 U \phi_2},\ G_{\phi_2} \rangle$ (i.e. if $\phi_1 U \phi_2$ holds infinitely often then ϕ_2 holds infinitely often,)

Theorem Let $\sigma = s_0 a_0 s_1, a_1 \ldots$ be a run of the automaton and let $u = a_0 a_1 \ldots$ be the corresponding ω string. Then

σ is an accepting run if and only if

$u, 0 \models \phi$ and $s_i = type_u^\phi(i)$.

From TL(U) to Automata

Notations. G_ψ the set of states that contain formula ψ.

The Initial States: G_ϕ

The Street Acceptance conditions: For every $\phi_1 U \phi_2 \in Sub(\phi)$ we have the pair $\langle G_{\phi_1 U \phi_2},\ G_{\phi_2} \rangle$ (i.e. if $\phi_1 U \phi_2$ holds infinitely often then ϕ_2 holds infinitely often,)

Theorem Let $\sigma = s_0 a_0 s_1, a_1 \ldots$ be a run of the automaton and let $u = a_0 a_1 \ldots$ be the corresponding ω string. Then

σ is an accepting run if and only if

$u, 0 \models \phi$ and $s_i = type_u^\phi(i)$.

Proof. The if direction is easy. The only if direction: by structural induction on formula for all i simultaneously show: if σ is an accepting run then $\psi \in s_i$ iff $u, i \models \psi$.