Decidable Verification under Localized Release-Acquire Concurrency (Extended Version)

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Abstract. State reachability for finite state concurrent programs running under Release-Acquire (RA) semantics is known to be undecidable, while under a weaker variant, called Weak-Release-Acquire (WRA), the problem is decidable. However, WRA allows many counterintuitive behaviors not allowed under RA, in which threads locally oscillate between observed values. We propose a strengthening of WRA in the form of a new memory model, which we call Localized Release-Acquire (LRA), that prunes these oscillatory behaviors. We provide semantics for LRA and show that verification under LRA is decidable by extending the potential-based technique used to prove decidability under WRA. The LRA model is still weaker than RA, and thus our results can be used to soundly verify programs under RA.

Keywords: Relaxed Memory Concurrency · State Reachability · Release-Acquire Semantics

1 Introduction

The *Release-Acquire* memory model (RA), a prominent fragment of the C/C++ shared-memory concurrency specifications from 2011 [13, 16, 17, 27], has recently gained a lot of attention (see, e.g., [2, 7, 18, 23–25, 30]). For programmers, RA combines the essential guarantees of coherence [11] (a.k.a. "sequential consistency per-location") and causal consistency [10, 20], which enable the implementation of various concurrent algorithms and synchronization mechanisms with very few barriers. For implementors, RA is weaker than the Total Store Order model (TSO) [29, 32], which enables efficient mapping of memory accesses to Intel's x86 processors. Moreover, unlike TSO, RA is "monotone" [33], which, roughly speaking, means that replacing parallel composition with sequential composition can never introduce additional behaviors [26].

Unfortunately, the fundamental problem of state reachability in finite-state concurrent programs running under RA was recently shown to be undecidable [2]. This is in contrast with state reachability assuming the well-known model of sequential consistency (SC) [28], which amounts to standard reachability in a finite state system, as well as with state reachability assuming TSO, which was shown to be decidable [4,5,12] using the framework of well-structured transition systems (WSTS) [1, 15]. More recently, decidability of state reachability was established for two variants of RA [21,22], called Strong Release-Acquire (SRA) and Weak Release-Acquire (WRA), which bound RA from above (every behavior allowed by SRA is allowed by RA) and below (every behavior allowed by RA is allowed by WRA). In particular, verification under WRA can be used to obtain sound (but incomplete) verification under RA, since any buggy program under RA is also buggy under WRA. The gap, however, between WRA and RA includes some dubious behaviors:

Example 1. The annotated behaviors in the three litmus tests below are allowed by WRA but disallowed by RA:

(Oscillation 1)		(Oscillation 2)			(Oscillation 3)		
x := 2	x := 1 b := x // 2 c := x // 1	x := 2	a := x // 1 b := x // 2 c := x // 1	x := 1	x := 2	a := y // 1 b := x // 2 c := x // 1	x := 1 y := 1

Intuitively speaking, a thread in WRA can "change its mind" about the order of concurrent writes. In RA, every shared variable is governed by a "modification order" which dictates the (globally agreed upon) order of concurrent writes, and reads have to respect that order.

In this paper, we aim to narrow the gap between models with decidable reachability problem and RA by providing a model that lies between WRA and RA and still allows for decidable verification. More concretely, we propose to strengthen WRA in a way that eliminates the above oscillatory behaviors, while still (1) being weaker than RA and (2) inducing a decidable state reachability problem. The proposed model, which we call Localized Release-Acquire (LRA), is obtained by adding one constraint (a.k.a. axiom) to WRA's declarative consistency predicate. In turn, decidability is established similarly to [22], by carefully designing an operational "lossy" semantics based on maintaining *thread potentials*, so that it fits well in the framework of WSTS, and it is equivalent to LRA. Our proof establishes the equivalence of the lossy potential-based system with LRA using forward simulation in one direction and backward simulation in the converse.

2 Preliminaries

In this section we present the formal preliminaries for our results, including the representation of concurrent programs, memory systems, and declarative execution graphs. We employ the following *finite* domains (and metavariables ranging over them):

$$\textit{thread identifiers } \tau, \pi \in \mathsf{Tid} = \{\mathtt{T}_1, \mathtt{T}_2, ...\}$$
$$\textit{variables } x, y \in \mathsf{Loc} \triangleq \{\mathtt{x}, \mathtt{y}, ...\}$$
$$\textit{values } v \in \mathsf{Val} \triangleq \{0, 1, 2, ...\}$$

We represent concurrent programs as labeled transition systems. A *labeled transition system* (LTS, for short) A over an alphabet Σ is a triple $\langle Q, Q_0, T \rangle$, where

Q is a set of *states*, $Q_0 \subseteq Q$ is the set of *initial states*, and $T \subseteq Q \times \Sigma \times Q$ is a set of *transitions*. We denote by $A.\mathtt{Q}$, $A.\mathtt{Q_0}$, and $A.\mathtt{T}$ the three components of an LTS A; we write $\xrightarrow{\sigma}_A$ for the relation $\{\langle q, q'\rangle \mid \langle q, \sigma, q'\rangle \in A.\mathtt{T}\}$ and $\rightarrow_A$ for $\bigcup_{\sigma\in\Sigma} \xrightarrow{\sigma}_A$. A state $q \in A.\mathtt{Q}$ is *reachable* in A if $q_0 \rightarrow_A^* q$ for some $q_0 \in A.\mathtt{Q_0}$. A sequence $\sigma_1, \dots, \sigma_n$ is a *trace* of A if $q_0 \xrightarrow{\sigma_1}_A q_1 \xrightarrow{\sigma_2}_A \cdots q_{n-1} \xrightarrow{\sigma_n}_A q_n$ for some $q_0 \in A.\mathtt{Q_0}$ and $q_1, \dots, q_n \in A.\mathtt{Q}$.

For brevity, we elide the definition of how concurrent programs in a programming language are interpreted as LTSs (see [22] for such definition), but only note that these LTSs are *finite-state* and they employ labels (a.k.a. "program transition labels") from the set $\mathsf{ProgLab} \triangleq \mathsf{Tid} \times (\mathsf{Lab} \cup \{\epsilon\})$, where Lab denotes the set of *action labels*, representing interactions that a program may have with the memory, and ϵ denotes a thread-internal transition. Action labels $l \in \mathsf{Lab}$ take one of the following forms: a read $\mathtt{R}(x, v_\mathtt{R})$, a write $\mathtt{W}(x, v_\mathtt{W})$, or a read-modify-write $\mathtt{RMW}(x, v_\mathtt{R}, v_\mathtt{W})$, where $x \in \mathsf{Loc}$ and $v_\mathtt{R}, v_\mathtt{W}, \in \mathsf{Val}$. The functions `typ`, `loc`, $\mathtt{val_R}$, and $\mathtt{val_W}$ respectively retrieve (when applicable) the type (R/W/RMW), variable (x), read value ($v_\mathtt{R}$), and written value ($v_\mathtt{W}$) of an action label. Furthermore, for a program transition label $\alpha \in \mathsf{ProgLab}$, the functions `tid` and `lab` respectively retrieve the thread identifier (τ) and the action label (or ϵ) of α, and the functions on action labels (`typ`, `loc`, ...) are lifted to program transition labels in the obvious way.

To represent concurrent programs running under a particular memory model, we synchronize the transitions of a program Pr with a memory system. A memory system is another LTS $\mathcal{M}$ (but, possibly infinite-state) whose set of transition labels consists of non-silent program transition labels (elements of $\mathsf{Tid} \times \mathsf{Lab}$) as well as a (disjoint) set $\mathcal{M}.\mathbf{\Theta}$ of memory-internal actions. Then, the composition of a program Pr and a memory system $\mathcal{M}$, denoted by $Pr \bowtie \mathcal{M}$, is the LTS whose transition labels are the elements of $\mathsf{ProgLab} \cup \mathcal{M}.\mathbf{\Theta}$; states are pairs $\langle \overline{p}, M\rangle \in Pr.\mathtt{Q} \times \mathcal{M}.\mathtt{Q}$; initial state is $\langle \overline{p}_{\mathsf{Init}}, \mathcal{M}.\mathtt{Q_0}\rangle$; and transitions are given by:

$$\frac{\alpha \in \mathsf{Tid} \times \mathsf{Lab} \quad \overline{p} \xrightarrow{\alpha}_{Pr} \overline{p}' \quad M \xrightarrow{\alpha}_{\mathcal{M}} M'}{\langle \overline{p}, M\rangle \xrightarrow{\alpha}_{Pr\bowtie\mathcal{M}} \langle \overline{p}', M'\rangle} \qquad \frac{\alpha \in \mathsf{Tid} \times \{\epsilon\} \quad \overline{p} \xrightarrow{\alpha}_{Pr} \overline{p}'}{\langle \overline{p}, M\rangle \xrightarrow{\alpha}_{Pr\bowtie\mathcal{M}} \langle \overline{p}', M\rangle} \qquad \frac{\alpha \in \mathcal{M}.\mathbf{\Theta} \quad M \xrightarrow{\alpha}_{\mathcal{M}} M'}{\langle \overline{p}, M\rangle \xrightarrow{\alpha}_{Pr\bowtie\mathcal{M}} \langle \overline{p}, M'\rangle}$$

The state reachability problem for a memory system $\mathcal{M}$ receives as input a program Pr and a state $\overline{p} \in Pr.\mathtt{Q}$ and asks whether $\langle \overline{p}, M\rangle$ is reachable in $Pr \bowtie \mathcal{M}$ for some $M \in \mathcal{M}.\mathtt{Q}$.

Finally, we also need the notion of a *declarative* memory model, which accepts/rejects program behaviors based on constraints on the generated *execution graphs*.

Definition 1. An *execution graph* G is a pair $\langle E, rf\rangle$, where:

- E is a finite set of *events*. An *event* e is a tuple $\langle \tau, s, l\rangle$, where $\tau \in \mathsf{Tid}$, called the event's *thread identifier*; $s \in \mathbb{N}$, called the event's *serial identifier*, and $l \in \mathsf{Lab}$, called the event's *label*. The functions `tid`, `sn`, and `lab` return the thread identifier (τ), identifier (s), and action label (l) of an event. All functions on action labels (typ, loc, ...) are lifted to events in the obvious way. We denote by E the set of all events, and define the following subsets:

$$\mathsf{R} \triangleq \{e \in \mathsf{E} \mid \mathtt{typ}(e) \in \{\mathtt{R}, \mathtt{RMW}\}\} \qquad \mathsf{W} \triangleq \{e \in \mathsf{E} \mid \mathtt{typ}(e) \in \{\mathtt{W}, \mathtt{RMW}\}\}$$
$$\mathsf{RMW} \triangleq \mathsf{R} \cap \mathsf{W} \qquad \mathsf{E}^\tau = \{e \in \mathsf{E} \mid \mathtt{tid}(e) = \tau\}$$

– rf is a *reads-from relation for* E, that is a relation on E satisfying:
  - If $\langle w, r\rangle \in rf$, then $w \in \mathsf{W}$ and $r \in \mathsf{R}$.
  - If $\langle w, r\rangle \in rf$, then $\mathtt{loc}(w) = \mathtt{loc}(r)$ and $\mathtt{val}_\mathtt{W}(w) = \mathtt{val}_\mathtt{R}(r)$.
  - $w_1 = w_2$ whenever $\langle w_1, r\rangle, \langle w_2, r\rangle \in rf$ (each read reads from at most one write).
  - For every $r \in E \cap \mathsf{R}$, there exists some $w \in E$ such that $\langle w, r\rangle \in rf$ (each read reads from some write).

We denote the components of G by $G.\mathtt{E}$ and $G.\mathtt{rf}$. For any set $E' \subseteq \mathsf{E}$, we write $G.E'$ for $G.\mathtt{E} \cap E'$ (e.g., $G.\mathsf{W} = G.\mathtt{E} \cap \mathsf{W}$). The *program order* induced by an execution graph G, denoted by $G.\mathrm{po}$, is defined as $G.\mathrm{po} \triangleq \{\langle e_1, e_2\rangle \in E \times E \mid \mathtt{sn}(e_1) < \mathtt{sn}(e_2) \land \mathtt{tid}(e_1) = \mathtt{tid}(e_2)\}$.

Given a set E of events, $\tau \in \mathsf{Tid}$, and $l \in \mathsf{Lab}$, $\mathsf{NextEvent}(E, \tau, l)$ denotes the event with thread identifier τ, label l, and a minimal fresh serial identifier w.r.t. E, i.e., $\mathsf{NextEvent}(E, \tau, l) \triangleq \langle \tau, s, l\rangle$, where $s = \min\{n \in \mathbb{N} \mid \langle \tau, n, l\rangle \notin E\}$.

Definition 2. An execution graph G is *generated by* a program Pr with final state $\overline{p} \in Pr.\mathtt{Q}$ if $\langle \overline{p}_0, G_0\rangle \rightarrow^* \langle \overline{p}, G\rangle$ for some $\overline{p}_0 \in Pr.\mathtt{Q}_0$, where G_0 denotes the empty execution graph (given by $G_0 \triangleq \langle \emptyset, \emptyset\rangle$) and $\rightarrow$ is defined by:

$$\frac{\overline{p} \xrightarrow{\tau,l}_{Pr} \overline{p}' \qquad E' = E \cup \{\mathsf{NextEvent}(E, \tau, l)\} \qquad rf \subseteq rf' \qquad \langle E', rf'\rangle \text{ is an execution graph}}{\langle \overline{p}, \langle E, rf\rangle\rangle \rightarrow \langle \overline{p}', \langle E', rf'\rangle\rangle} \qquad \frac{\overline{p} \xrightarrow{\tau,\varepsilon}_{Pr} \overline{p}'}{\langle \overline{p}, G\rangle \rightarrow \langle \overline{p}', G\rangle}$$

Using the above definitions, a declarative memory model can be identified with a set of so-called *consistent* execution graphs, and a program state $\overline{p}$ is 'emphreachable under a declarative memory model if some consistent execution graph G is generated by Pr with final state $\overline{p}$.

3 The Localized Release-Acquire Model

In this section we introduce the Localized Release-Acquire (LRA) model, starting with its declarative presentation. LRA is obtained by adding a single constraint, called "local-read-coherence", to WRA. We first briefly repeat the three constraints of WRA (see [20] for more details). Figure 1 summarizes the four constraints of LRA.

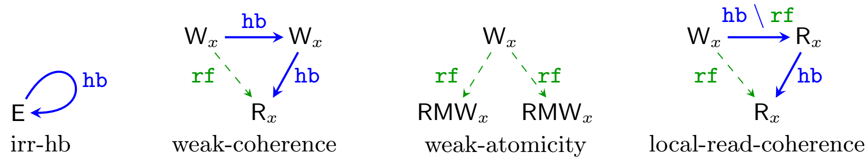


Fig. 1. Illustration of forbidden patterns in LRA

Notation for relations. Given a relation R, $dom(R)$ denotes its domain; $R^?$ and R^+ denote its reflexive and transitive closures; and R^{-1} denotes its inverse. The (left) composition of relations R_1, R_2 is denoted by $R_1 \mathbin{;} R_2$. We denote by $[A]$ the identity relation on a set A (e.g., $[A] \mathbin{;} R \mathbin{;} [B] = R \cap (A \times B)$).

First, we need a derived "happens-before" relation. For a given execution graph G, we define $G.\texttt{hb} \triangleq (G.\mathsf{po} \cup G.\texttt{rf})^+$. We require that $G.\texttt{hb}$ is a partial order, which results in our first constraint:

$$G.\texttt{hb} \text{ is irreflexive} \qquad \text{(irr-hb)}$$

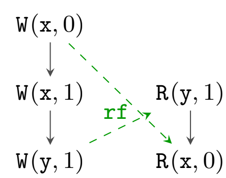
The next constraint intuitively makes sure that "a thread cannot read a value when it is aware of a later value written to the same location", where "aware" and "later" are interpreted using $G.\texttt{hb}$. Formally, we define $G.\texttt{hb}|_{\texttt{loc}} \triangleq \{\langle e_1, e_2\rangle \in G.\texttt{hb} \mid \texttt{loc}(e_1) = \texttt{loc}(e_2)\}$ (i.e., per-location restriction of the happens-before relation), and require the following:

$$G.\texttt{hb}|_{\texttt{loc}} \mathbin{;} [\mathsf{W}] \mathbin{;} G.\texttt{hb} \mathbin{;} G.\texttt{rf}^{-1} \text{ is irreflexive} \qquad \text{(weak-coherence)}$$

In particular, the following annotated outcome of the message-passing (MP) test is forbidden:

(MP)

```
x := 0     ||  a := y  // 1
x := 1     ||  b := x  // 0
y := 1     ||
```



An execution graph justifying this outcome must have $\texttt{rf}$-edges as depicted above. However, we have $\texttt{hb}|_{\texttt{loc}}$ from $\texttt{W}(\texttt{x}, 0)$ to $\texttt{W}(\texttt{x}, 1)$, $\texttt{hb}$ from $\texttt{W}(\texttt{x}, 1)$ to $\texttt{R}(\texttt{x}, 0)$, and $\texttt{rf}$ from $\texttt{W}(\texttt{x}, 0)$ to $\texttt{R}(\texttt{x}, 0)$, which is forbidden by weak-coherence.

The final condition that comes from WRA ensures that distinct RMW events never read from the same write event:

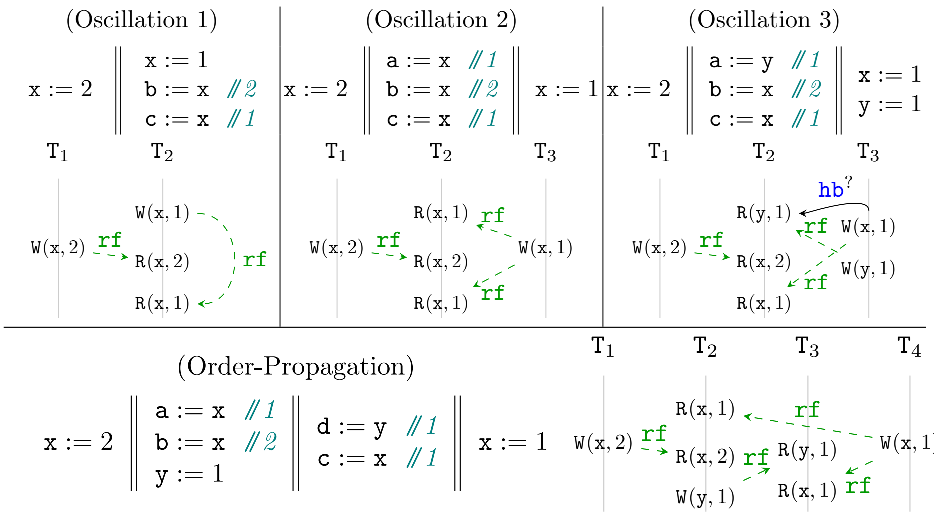
$$\forall \langle w_1, e_1\rangle, \langle w_2, e_2\rangle \in G.\texttt{rf} \mathbin{;} [\mathsf{RMW}].\ w_1 = w_2 \implies e_1 = e_2 \qquad \text{(weak-atomicity)}$$

This concludes the consistency constraints of WRA. As noted above, unlike RA, WRA admits behaviors in which threads oscillate between values that were concurrently written to the same location. Our proposed condition of LRA that prunes these behaviors is the following:

$$(G.\texttt{hb}|_{\texttt{loc}} \setminus G.\texttt{rf}) \mathbin{;} [\mathsf{R}] \mathbin{;} G.\texttt{hb} \mathbin{;} G.\texttt{rf}^{-1} \text{ is irreflexive} \qquad \text{(local-read-coherence)}$$

Intuitively, this constraint ensures that a thread cannot read from a certain write w if it is already aware of a read r' reading from the same location that is later than w and reads from some other write w'. Again, "aware" and "later" are interpreted using $G.\mathtt{hb}$.

The following examples demonstrate "oscillations" between observed values that are allowed in WRA but forbidden in LRA.



It can be checked that local-read-coherence forbids these execution graphs: in all of them we have (1) $G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}$ from $\mathtt{W}(\mathtt{x}, 1)$ to $\mathtt{R}(\mathtt{x}, 2)$; (2) $G.\mathtt{hb}$ from $\mathtt{R}(\mathtt{x}, 2)$ to the read $\mathtt{R}(\mathtt{x}, 1)$ that represents the read to c; and (3) $\mathtt{rf}$ from $\mathtt{W}(\mathtt{x}, 1)$ to that read.

Next, we establish the relation between LRA and RA (see [22] for a definition of RA).

Proposition 1. LRA is weaker than RA, that is: if a program state is reachable under RA, then it is also reachable under LRA.

Proof. We establish this result by recalling the following "read-coherence" consistency constraint of RA (see Figure 2 and [20] for more details). Note the use of modification order $G.\mathtt{mo}$ in RA to interpret one write being "later" than another, in the place of $G.\mathtt{hb}|_{\mathtt{loc}}$ in the "weak-coherence" in WRA. Here $G.\mathtt{mo}$ is disjoint union of relations $\{G.\mathtt{mo}_x\}_{x \in \mathsf{Loc}}$ where each $G.\mathtt{mo}_x$ is a strict total order on W_x.

$$G.\mathtt{mo} \,;\, G.\mathtt{hb} \,;\, G.\mathtt{rf}^{-1} \text{ is irreflexive} \qquad \text{(read-coherence)}$$

Since WRA is strictly weaker than RA, it suffices to show that the additional constraint "local-read-coherence" of LRA is also guaranteed in RA. The proof follows by contradiction. Assume otherwise, hence, for a given $x \in \mathsf{Loc}$, we have $w, w' \in \mathsf{W}_x$ and $r, r' \in \mathsf{R}_x$ where $\langle w, r' \rangle \in \mathtt{hb} \setminus \mathtt{rf}$, $\langle w', r' \rangle \in \mathtt{rf}$, $\langle w, r \rangle \in \mathtt{rf}$, and $w \neq w'$ (see right side of Figure 2). Since $\mathtt{loc}(w) = x = \mathtt{loc}(w')$, due to the RA semantics, we have one of the following cases:

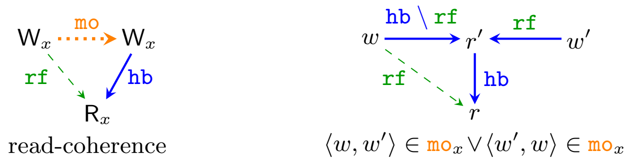


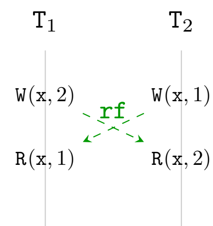
Fig. 2. Axiom read-coherence in RA and illustration for proof of Proposition 1

- $\langle w, w' \rangle \in \texttt{mo}_x$: In this case we have $\langle w, r \rangle \in \texttt{rf}$ while $\langle w, r \rangle \in \texttt{mo}_x \,;\texttt{hb}$, which contradicts the axiom read-coherence of RA.
- $\langle w', w \rangle \in \texttt{mo}_x$: In this case we have $\langle w', r' \rangle \in \texttt{rf}$ while $\langle w', r' \rangle \in \texttt{mo}_x \,;\texttt{hb}$, which again contradicts the axiom read-coherence of RA.

To see that LRA is *strictly* weaker than RA, we note that LRA does not provide full coherence. Indeed, as the next example shows, even programs with a single shared variable can exhibit weak behaviors:

(WW)

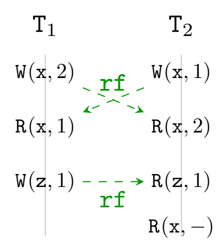
x := 2	x := 1
a := x // *1*	b := x // *2*



Interestingly, our final example shows the LRA model is possibly blocking: it may be the case that a thread simply cannot read from a certain location, since any option for reading would violate local-read-coherence.

(Blocking)

x := 2	x := 1
a := x // *1*	b := x // *2*
z := 1	c := z // *1*
	d := x // *nothing can be read*



Roughly speaking, the synchronization on z "joins" the threads and rules out both options. More formally, if the final read reads from W(x, 1), we violate local-read-coherence due to $G.\texttt{hb}|_{\texttt{loc}} \setminus G.\texttt{rf}$ from W(x, 1) to R(x, 2) and $G.\texttt{hb}$ from R(x, 2) to the final read. In turn, if the final read reads from W(x, 2), we violate local-read-coherence due to $G.\texttt{hb}|_{\texttt{loc}} \setminus G.\texttt{rf}$ from W(x, 2) to R(x, 1) and $G.\texttt{hb}$ from R(x, 1) to the final read.

It is important to note that the blocking aspect of LRA model does not affect the benefits of sound verification of the RA programs using LRA, since (due to Proposition 1) forbidden outcomes in LRA model (possibly due to a blocked run) are also forbidden in the RA model.

3.1 An Operational Presentation

Since LRA-consistency is "prefix-closed", it is straightforward to "operationalize" LRA's declarative presentation, which will help us below in relating the potential-model to LRA. To do so, we define a memory system, called opLRA, whose states are execution graphs, the only initial state is the empty execution graph, and the transitions are as follows:

$$\textsc{write}\quad \frac{e = \mathsf{NextEvent}(G.\mathtt{E}, \tau, \mathtt{W}(x, v_{\mathtt{W}})) \qquad G' = \langle G.\mathtt{E} \cup \{e\}, G.\mathtt{rf}\rangle}{G \xrightarrow{\tau, \mathtt{W}(x, v_{\mathtt{W}})}_{\text{opLRA}} G'}$$

$$\textsc{read/rmw}\quad \frac{\begin{array}{c} l = \mathtt{R}(x, v_{\mathtt{R}}) \lor l = \mathtt{RMW}(x, v_{\mathtt{R}}, v_{\mathtt{W}}) \\ e = \mathsf{NextEvent}(G.\mathtt{E}, \tau, l) \qquad G' = \langle G.\mathtt{E} \cup \{e\}, G.\mathtt{rf} \cup \{\langle w, e\rangle\}\rangle \\ w \in G.\mathsf{W}_x \qquad \mathtt{val}_{\mathtt{W}}(w) = v_{\mathtt{R}} \\ w \notin dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau]) \\ w \notin dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau]) \\ \mathtt{typ}(l) = \mathtt{RMW} \implies w \notin dom(G.\mathtt{rf} ; [\mathsf{RMW}]) \end{array}}{G \xrightarrow{\tau, l}_{\text{opLRA}} G'}$$

These transitions are enforcing consistency on every step, which allows us to establish the following relation.

Proposition 2. LRA is equivalent to opLRA, that is: a program state is reachable under LRA iff it is reachable under opLRA.

4 Lossy semantics for LRA

In this section, we present loLRA, a potential-based memory system that is equivalent to LRA and well suited for verification in the framework of WSTS.

The memory states of loLRA maintain a collection of "read/write-option" lists for each thread, called the *potential* of the thread. Concretely, a state of loLRA is a *potential mapping* $\mathcal{B}$ which maps each thread $\tau \in \mathsf{Tid}$ to its potential $\mathcal{B}(\tau)$. Potentials are finite sets of *option lists*, where each option list stands for a sequence of possible future reads (*read options*) and writes (*write options*) that ascribe possible operations the thread may perform in the order it may perform them. For instance, a list $o_1 \cdot o_2$ consisting of two read options, o_1 and o_2, allows the thread to read $\mathtt{val}(o_1)$ from location $\mathtt{loc}(o_1)$ and then $\mathtt{val}(o_2)$ from location $\mathtt{loc}(o_2)$. Thread potentials are explicitly "lossy"—a thread can non-deterministically lose whatever parts of its potential at any point. Initially, the loLRA memory system non-deterministically starts in a state where all potentials consist solely of write options.

Next, we present the full definitions (which, except for loLRA's transitions match precisely the definitions of the corresponding system for WRA in [22]).

Notation for sequences. We use ϵ to denote the empty sequence. The length of a sequence s is denoted by $|s|$ (in particular $|\epsilon| = 0$). We often identify a sequence s over Σ with its underlying function in $\{1, \dots, |s|\} \to \Sigma$, and write $s(k)$ for the symbol at position $1 \leq k \leq |s|$ in s. We write $\sigma \in s$ if the symbol σ appears in s, that is if $s(k) = \sigma$ for some $1 \leq k \leq |s|$. We use "$\cdot$" for the concatenation of sequences, and lift it to concatenation of sets S_1 and S_2 of sequences in the obvious way ($S_1 \cdot S_2 \triangleq \{s_1 \cdot s_2 \mid s_1 \in S_1, s_2 \in S_2\}$). We identify symbols with sequences of length 1 or their singletons when needed (e.g., in expressions like $\sigma \cdot S$ for $\sigma \in \Sigma$ and a set S of sequences over Σ).

Definition 3. Options, option lists, potentials, and potential mappings are defined as follows:

1. An *option* o is either $\langle \tau, x, v, \pi_{\mathtt{RMW}} \rangle$ (*read option*) or $\mathtt{O}_{\mathtt{W}}(x)$ (*write option*), where $\tau, \pi_{\mathtt{RMW}} \in \mathsf{Tid}$, $x \in \mathsf{Loc}$, and $v \in \mathsf{Val}$. The functions $\mathtt{typ}$, $\mathtt{tid}$, $\mathtt{loc}$, $\mathtt{val}$, and $\mathtt{rmw\text{-}tid}$ return (when applicable) the type ($\mathtt{R}/\mathtt{W}$), thread identifier (τ), location (x), value (v), and RMW thread identifier ($\pi_{\mathtt{RMW}}$) of a given option.
2. An *option list* L is a finite sequence of (read or write) options. For a given option list L, we define $\mathtt{loc}(L) \triangleq \{\mathtt{loc}(o) \mid o \in L\}$.
3. A *potential* B is a finite non-empty set of option lists.
4. A *potential mapping* $\mathcal{B}$ is a function assigning a potential to every $\tau \in \mathsf{Tid}$.

We define a (well quasi) ordering on option lists that naturally extends to potentials and to potential mappings.

Definition 4. The (overloaded) relation $\sqsubseteq$ is defined by:

1. on option lists: $L \sqsubseteq L'$ if L is a (not necessarily contiguous) subsequence of L';
2. on potentials: $B \sqsubseteq B'$ if $\forall L \in B.\, \exists L' \in B'.\, L \sqsubseteq L'$ (a.k.a. "Hoare ordering");
3. on potential mappings: $\mathcal{B} \sqsubseteq \mathcal{B}'$ if $\mathcal{B}(\tau) \sqsubseteq \mathcal{B}'(\tau)$ for every $\tau \in \mathsf{Tid}$ (componentwise order).

The memory system loLRA is formally defined as follows.

Definition 5. The memory system loLRA is defined by:

- loLRA.$\mathtt{Q}$ is the set of potential mappings.
- loLRA.$\mathtt{Q}_0 = \{\mathcal{B} \mid \forall \tau \in \mathsf{Tid}, L \in \mathcal{B}(\tau), o \in L.\, \mathtt{typ}(o) = \mathtt{W}\}$.
- The transitions of loLRA are given in Figure 3.

The transitions of loLRA are informally understood as follows:

- READ: For a thread τ to read v from x, all lists of τ should start with an option o with $\mathtt{val}(o) = v$ and $\mathtt{loc}(o) = x$ (since it is the same option o in the head of all lists, all lists of τ also start with the same thread identifier, which is important for the equivalence result; see [22, Example 5.5]). The read step consumes these options by discarding the first element from each of τ's lists.

WRITE

$$\frac{\begin{array}{l} o = \langle \tau, x, v_{\mathtt{W}}, \pi_{\mathtt{RMW}} \rangle \\ \forall \pi \in \mathsf{Tid}, L' \in \mathcal{B}'(\pi). \\ \quad ((\pi = \tau \implies \mathtt{O}_{\mathtt{W}}(x) \cdot L' \in \mathcal{B}(\tau)) \wedge (\pi \neq \tau \implies L' \in \mathcal{B}(\pi))) \vee \\ \quad (\exists n \geq 1, L_0, \dots, L_n. \\ \quad\quad L' = L_0 \cdot (o \cdot L_1) \cdot (o \cdot L_2) \cdot \ldots \cdot (o \cdot L_n) \wedge \\ \quad\quad \mathtt{O}_{\mathtt{W}}(x) \cdot (L_1 \cdot \ldots \cdot L_{n-1}) \cdot \mathtt{O}_{\mathtt{W}}(x) \cdot L_n \in \mathcal{B}(\tau) \wedge \\ \quad\quad (\pi = \tau \implies \mathtt{O}_{\mathtt{W}}(x) \cdot L_0 \cdot \ldots \cdot L_{n-1} \cdot \mathtt{O}_{\mathtt{W}}(x) \cdot L_n \in \mathcal{B}(\tau) \wedge x \notin \mathtt{loc}(L_0 \cdot \ldots \cdot L_{n-1})) \wedge \\ \quad\quad (\pi \neq \tau \implies L_0 \cdot \ldots \cdot L_{n-1} \cdot \mathtt{O}_{\mathtt{W}}(x) \cdot L_n \in \mathcal{B}(\pi) \wedge x \notin \mathtt{loc}(L_1 \cdot \ldots \cdot L_{n-1}))) \end{array}}{\mathcal{B} \xrightarrow{\tau, \mathtt{W}(x, v_{\mathtt{W}})}_{\text{loLRA}} \mathcal{B}'}$$

READ

$$\frac{\mathtt{loc}(o) = x \quad \mathtt{val}(o) = v_{\mathtt{R}} \quad \mathcal{B} = \mathcal{B}'[\tau \mapsto o \cdot \mathcal{B}'(\tau)]}{\mathcal{B} \xrightarrow{\tau, \mathtt{R}(x, v_{\mathtt{R}})}_{\text{loLRA}} \mathcal{B}'}$$

RMW

$$\frac{\mathtt{loc}(o) = x \quad \mathtt{val}(o) = v_{\mathtt{R}} \quad \mathtt{rmw\text{-}tid}(o) = \tau \quad \mathcal{B} = \mathcal{B}_{\text{mid}}[\tau \mapsto o \cdot \mathcal{B}_{\text{mid}}(\tau)] \quad \mathcal{B}_{\text{mid}} \xrightarrow{\tau, \mathtt{W}(x, v_{\mathtt{W}})}_{\text{loLRA}} \mathcal{B}'}{\mathcal{B} \xrightarrow{\tau, \mathtt{RMW}(x, v_{\mathtt{R}}, v_{\mathtt{W}})}_{\text{loLRA}} \mathcal{B}'}$$

LOWER

$$\frac{\mathcal{B}' \sqsubseteq \mathcal{B}}{\mathcal{B} \xrightarrow{\varepsilon}_{\text{loLRA}} \mathcal{B}'}$$

Fig. 3. Transitions of loLRA memory system

- WRITE: For a thread τ to write v to x, an option $\mathtt{O}_{\mathtt{W}}(x)$ must be the first in each of τ's lists. The WRITE consumes these options, discarding the first element from each of τ's lists. To allow future reads from the executed write, the write may add a read option o with $\mathtt{loc}(o) = x$, $\mathtt{val}(o) = v$, $\mathtt{tid}(o) = \tau$, and some $\mathtt{rmw\text{-}tid}(o)$ (possibly multiple times) in every existing list of every thread (including the writer itself). The WRITE step enforces carefully tailored conditions on *where* these new options are added:
  1. In the potential of the writer itself, a new option cannot be added after an existing write option to x (except for the write option that is consumed in this write step) and the last added read option should immediately precede an existing write option to x.
  2. In the potential of other threads the last added read option should immediately precede an existing write option to x that is to be consumed by the current write step.
  3. If more than one option is added, the added read options can never "surround" an existing read/write option with location x.
  4. New read options can be placed in a list L only if the suffix of L after the first occurrence of the newly added read options are present as an option list of the writing thread τ.
- RMW: The only additional requirement when performing an RMW compared to a non-interrupted execution of a read followed by a write is that two RMWs should never read from the same event. This is achieved by including *RMW thread identifiers* in read options, denoting the (unique) thread that may consume this option when executing an RMW. When a thread writes, it picks an (arbitrary) unique thread identifier ($\pi_{\mathtt{RMW}}$) for its added options; reads ignore this field; and RMWs by thread τ can only consume read options whose RMW thread identifier is τ.

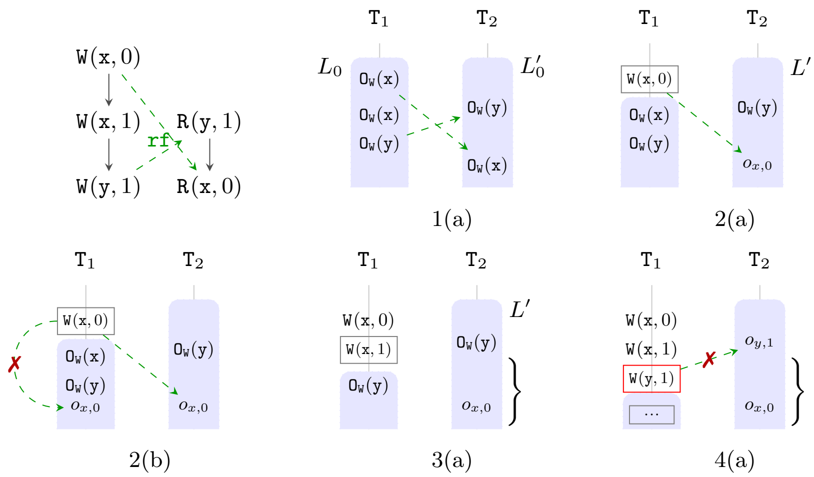


Fig. 4. This figure shows the loLRA transitions for MP program. Here the dashed line in 1(a) between $\mathtt{O}_\mathtt{W}(x)$ of $\mathtt{T}_1$ and $\mathtt{O}_\mathtt{W}(x)$ of $\mathtt{T}_2$ indicates that a future write $\mathtt{W}(x, 0)$ of $\mathtt{T}_1$ (see 2(a)) may replace the $\mathtt{O}_\mathtt{W}(x)$ of $\mathtt{T}_2$ with a read option $o_{x,0}$. We follow a similar depiction in all the remaining diagrams of the paper.

– LOWER: The step allows to remove read/write options as well as full option lists at any time.

We revisit the examples from §3 to illustrate that loLRA forbids those outcomes. In following discussions, shaded portions of the diagram for each thread correspond to its option lists. We write $o_{x,v}$ to represent a read option o with $\mathtt{loc}(o) = x$ and $\mathtt{val}(o) = v$.

Example 2. Recall the execution graph of MP from §3 (see Figure 4). Since no step in loLRA can introduce a write option, we observe the following facts about the option lists $L_0 \in \mathcal{B}_0(\mathtt{T}_1)$ and $L'_0 \in \mathcal{B}_0(\mathtt{T}_2)$ where $\mathcal{B}_0$ may lead to the annotated program state ($\mathtt{a} = 1$ and $\mathtt{b} = 0$) using a trace in which L_0 and L'_0 are not discarded by a LOWER step:

1. L_0 contains $\mathtt{O}_\mathtt{W}(\mathtt{x}) \cdot \mathtt{O}_\mathtt{W}(\mathtt{x}) \cdot \mathtt{O}_\mathtt{W}(\mathtt{y})$ as a sub-list to enable $\mathtt{W}(\mathtt{x}, 0)$, $\mathtt{W}(\mathtt{x}, 1)$, and $\mathtt{W}(\mathtt{y}, 1)$ in $\mathtt{T}_1$.
2. For the reads $\mathtt{R}(\mathtt{y}, 1)$ and $\mathtt{R}(\mathtt{x}, 0)$ to happen the corresponding writes $\mathtt{W}(\mathtt{y}, 1)$ and $\mathtt{W}(\mathtt{x}, 0)$ need to insert read options $o_{\mathtt{y},1}$ and $o_{\mathtt{x},0}$ at these locations (see READ step).
3. L'_0 contains $\mathtt{O}_\mathtt{W}(\mathtt{y})$ followed by $\mathtt{O}_\mathtt{W}(\mathtt{x})$ to enable future insertions of read options $o_{\mathtt{y},1}$ and $o_{\mathtt{x},0}$ by the writes $\mathtt{W}(\mathtt{y}, 1)$ and $\mathtt{W}(\mathtt{x}, 0)$ respectively (see condition 2 of WRITE step).

Starting in the state $\mathcal{B}_0$ (1(a) in Figure 4), one can reach state 3(a) through state 2(a) in two successive steps corresponding to execution of the first two writes, $\mathtt{W}(\mathtt{x}, 0)$ and $\mathtt{W}(\mathtt{x}, 1)$ of $\mathtt{T}_1$, where the first write $\mathtt{W}(\mathtt{x}, 0)$ replaces $\mathtt{O}_\mathtt{W}(\mathtt{x})$ in the option list of $\mathtt{T}_2$ with a read option $o_{\mathtt{x},0}$ resulting in $L' = L'_0[\mathtt{O}_\mathtt{W}(\mathtt{x}) \mapsto o_{\mathtt{x},0}]$.

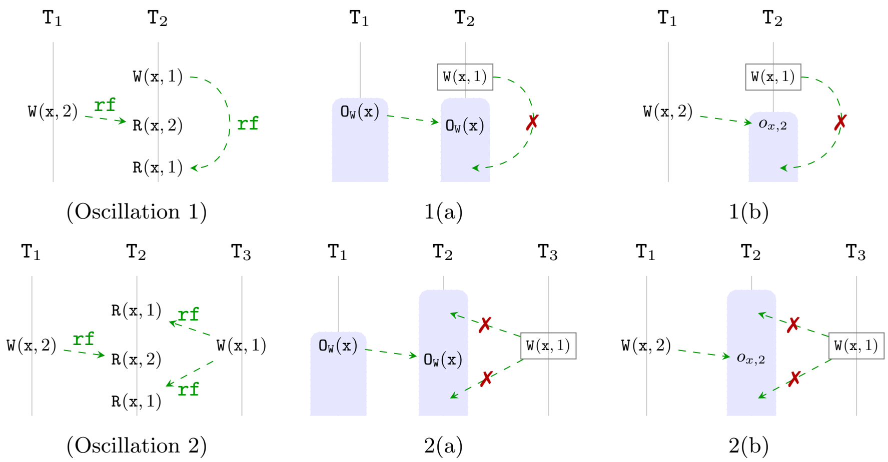
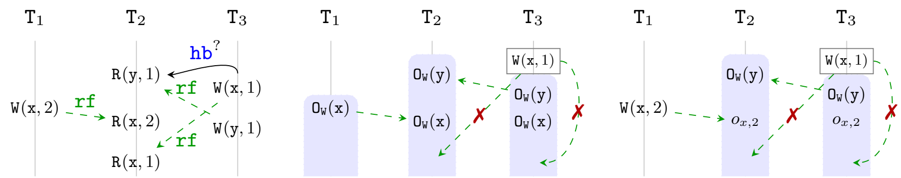


Fig. 5. loLRA transitions for Oscillation 1 and Oscillation 2 (Example 3).

In the next step (shown as 4(a)), we hope to perform the write $\mathtt{W}(\mathtt{y}, 1)$ in $\mathtt{T}_1$ and replace $\mathtt{O}_\mathtt{W}(\mathtt{y})$ in $\mathtt{T}_2$ with the read option $o_{\mathtt{y},1}$. However, the current write step requires that the suffix of L' after $\mathtt{O}_\mathtt{W}(\mathtt{y})$ (here, $o_{\mathtt{x},0}$) be present as an option list of thread $\mathtt{T}_1$ (due to condition 4 of the WRITE step). This is clearly not true and hence we can not continue with the current execution trace. To circumvent this blocking run the first write $\mathtt{W}(\mathtt{x}, 0)$ of $\mathtt{T}_1$ might want to non-deterministically insert a read option $o_{\mathtt{x},0}$ at the specified location (see 2(b)) in its option list. However, due to the presence of an earlier $\mathtt{O}_\mathtt{W}(\mathtt{x})$ in the option lists of $\mathtt{T}_1$ this is not allowed. Therefore, the loLRA semantics successfully forbids the annotated outcome of the message passing test.

Example 3. Recall the execution graphs of (Oscillation 1) and (Oscillation 2) from §3 (see Figure 5), where $\mathtt{T}_2$ oscillates between the observed values of x. Consider following two cases (and the corresponding execution graphs) to observe a contradiction for each possible trace of loLRA:

- $\mathtt{W}(\mathtt{x}, 1)$ executes before $\mathtt{W}(\mathtt{x}, 2)$: For (Oscillation 1) and (Oscillation 2) this is depicted as 1(a) and 2(a) of Figure 5 respectively. Note the presence of $\mathtt{O}_\mathtt{W}(\mathtt{x})$ at the specified locations in the option lists of thread $\mathtt{T}_2$ to mark the end of new read options due to the future write $\mathtt{W}(\mathtt{x}, 2)$. In the current state of (Oscillation 1), the write $\mathtt{W}(\mathtt{x}, 1)$ of thread $\mathtt{T}_2$ is not allowed to put a read option in its own option list due to the presence of an earlier $\mathtt{O}_\mathtt{W}(\mathtt{x})$ (see condition 1 of WRITE step). Similarly in the current state of (Oscillation 2), the write $\mathtt{W}(\mathtt{x}, 1)$ of thread $\mathtt{T}_3$ cannot place new read options in the list of thread $\mathtt{T}_2$ because $\mathtt{O}_\mathtt{W}(\mathtt{x})$ appears between the new read options (see condition 3 of WRITE step).
- $\mathtt{W}(\mathtt{x}, 1)$ executes after $\mathtt{W}(\mathtt{x}, 2)$: For (Oscillation 1) and (Oscillation 2) this is depicted as 1(b) and 2(b) of Figure 5 respectively. Note the presence of $o_{\mathtt{x},2}$ (instead of $\mathtt{O}_\mathtt{W}(\mathtt{x})$ in the previous case) at the specified location in the option



(Oscillation 3) 3(a) 3(b)

Fig. 6. The loLRA transitions for the program Oscillation 3 (Example 4).

lists of thread $\mathtt{T}_2$ to allow the read $\mathtt{R}(\mathtt{x}, 2)$ to read in future from the write $\mathtt{W}(\mathtt{x}, 2)$ of $\mathtt{T}_1$. Again in the states corresponding to 1(b) and 2(b), due to conditions 1 and 3 of WRITE step, $\mathtt{W}(\mathtt{x}, 1)$ is not allowed to put new read options at the specified locations.

Example 4. Recall the execution graph of (Oscillation 3) from §3, where $\mathtt{T}_2$ oscillates between the observed values of $\mathtt{x}$ (see Figure 6). We consider the following two cases and the resulting execution graphs, based on the order of execution between the write events $\mathtt{W}(\mathtt{x}, 1)$ and $\mathtt{W}(\mathtt{x}, 2)$, to observe a contradiction in each trace of loLRA:

– $\mathtt{W}(\mathtt{x}, 1)$ executes before $\mathtt{W}(\mathtt{x}, 2)$: This condition is depicted as 3(a). Note the presence of $\mathtt{O}_\mathtt{W}(\mathtt{y})$ and $\mathtt{O}_\mathtt{W}(\mathtt{x})$ at the specified location in the option list of $\mathtt{T}_2$ to mark the end of new read options due to the future writes $\mathtt{W}(\mathtt{y}, 1)$ and $\mathtt{W}(\mathtt{x}, 2)$ of $\mathtt{T}_3$ and $\mathtt{T}_1$ respectively. Also note the presence of $\mathtt{O}_\mathtt{W}(\mathtt{x})$ in the option lists of $\mathtt{T}_3$. We claim that this $\mathtt{O}_\mathtt{W}(\mathtt{x})$ is needed as justification for the future write $\mathtt{W}(\mathtt{y}, 1)$ of $\mathtt{T}_3$ (when the write $\mathtt{W}(\mathtt{y}, 1)$ will be replacing the write option $\mathtt{O}_\mathtt{W}(\mathtt{y})$ on $\mathtt{T}_2$ with the read option $o_{\mathtt{y},1}$). To justify the claim, assume otherwise (i.e., $\mathtt{O}_\mathtt{W}(\mathtt{x})$ is absent in the option list of $\mathtt{T}_3$), and we observe that $\mathtt{W}(\mathtt{y}, 1)$ of $\mathtt{T}_3$ can not continue in any of the following possible cases:
  - $\mathtt{W}(\mathtt{x}, 2)$ has not occurred when $\mathtt{W}(\mathtt{y}, 1)$ tries to execute: In this case $\mathtt{O}_\mathtt{W}(\mathtt{x})$ is still present in the option list of $\mathtt{T}_2$ and hence is required at the specified location in the option list of $\mathtt{T}_3$ as a justification for the current write $\mathtt{W}(\mathtt{y}, 1)$ (see condition 4 of the WRITE step). Therefore, the write $\mathtt{W}(\mathtt{y}, 1)$ can not continue in this case.
  - $\mathtt{W}(\mathtt{x}, 2)$ has occurred when $\mathtt{W}(\mathtt{y}, 1)$ tries to execute: In this case $\mathtt{O}_\mathtt{W}(\mathtt{x})$ on $\mathtt{T}_2$ has been replaced with a $o_{\mathtt{x},2}$ and hence $o_{\mathtt{x},2}$ is also expected in the option list of $\mathtt{T}_3$ (as justification for the current WRITE step $\mathtt{W}(\mathtt{y}, 1)$). However, the presence of $o_{\mathtt{x},2}$ in $\mathtt{T}_3$ can only be ensured (as insertion of new read option) by the corresponding write $\mathtt{W}(\mathtt{x}, 2)$. The write $\mathtt{W}(\mathtt{x}, 2)$ can not add a $o_{\mathtt{x},2}$ at the specified location due to the absence of $\mathtt{O}_\mathtt{W}(\mathtt{x})$ at the same location to mark the end of newly added read options (see condition 2 of WRITE step). Hence, in this case again the write $\mathtt{W}(\mathtt{y}, 1)$ can not continue.

  Assuming the presence of $\mathtt{O}_\mathtt{W}(\mathtt{x})$ in the option list of $\mathtt{T}_3$ (as shown in 3(a)) it is easy to see that $\mathtt{W}(\mathtt{x}, 1)$ of $\mathtt{T}_3$ can not put a read option in its own option

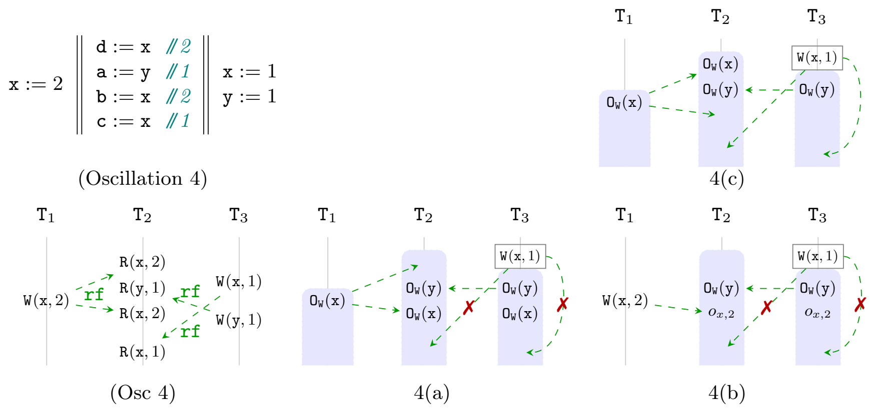


Fig. 7. The loLRA transitions for the program Oscillation 4 (Example 5).

list (see condition 1 of WRITE step) which is necessary as justification for the future write W(y, 1) of T_3 (again using similar arguments as discussed above). Therefore, the current case is forbidden by the lossy loLRA semantics.

- W(x, 1) executes after W(x, 2): This condition is depicted as 3(b), where the write W(x, 2) of T_1 has replaced the write option $O_W(x)$ in the option lists of T_2 with a read option $o_{x,2}$. Again, as discussed in the previous case (for justifying the future write W(y, 1) of T_3), the write W(x, 2) of T_1 should also place a read option $o_{x,2}$ in the option lists of T_3 at the specified location. Now, as shown in 3(b), the write W(x, 1) of T_3 can not put a read option in its own option list (due to the presence of an earlier $o_{x,2}$) which is necessary as justification for the future write W(y, 1) of T_3. Thus, the current case is also forbidden by the lossy loLRA semantics.

In the discussions so far (particularly related to cases 1(a), 2(a), and 3(a) of the previous examples), we observed that marking the end of newly added read options (using a pre-existing write option) is helpful in forbidding oscillations. In all of these cases it is easy to see (using exactly similar arguments) that we can also forbid these oscillatory behaviors by requiring (in conditions 1 and 2 of the WRITE step) that the beginning of newly added read options be marked using a pre-existing write option. Next example illustrates the distinctive advantage of marking the end over marking the beginning.

Example 5. Consider execution graph (Osc 4) corresponding to the annotated outcome of (Oscillation 4) shown in Figure 7. The constraint local-read-coherence forbids this execution graph since we have (1) $G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}$ from W(x, 1) to the third read R(x, 2) of T_2; (2) $G.\mathtt{hb}$ from the third read R(x, 2) of T_2 to the last read R(x, 1) of T_2; and (3) rf from W(x, 1) to the last read R(x, 1) of T_2.

Consider the following two possibilities (4(b) and 4(a) of Figure 7) corresponding to this outcome where: (1) W(x, 1) executes after W(x, 2); and (2) W(x, 1) executes before W(x, 2).

Assuming (1) and using arguments similar to Example 4, we land in configuration 4(b) which is not allowed by the lossy loLRA semantics. However, note that assuming (2) we get a contradiction only because $\mathtt{O_W(x)}$ is present at the specified location in 4(a) to mark the end of new read options in the option list of $\mathtt{T}_2$ (by the write $\mathtt{W}(\mathtt{x},2)$ of thread $\mathtt{T}_1$). Instead, if we choose to mark the beginning (and not the end) of new read options in the option list of $\mathtt{T}_2$ we result in the configuration of 4(c) resulting in the absence of any pre-existing $\mathtt{O_W(x)}$ at the end of the new entries. In this case, we observe that there is a trace of lossy loLRA (for the annotated outcome of (Oscillation 4)) in which $\mathtt{W}(\mathtt{x},1)$ and $\mathtt{W}(\mathtt{y},1)$ of $\mathtt{T}_3$ appears before $\mathtt{W}(\mathtt{x},2)$ of $\mathtt{T}_1$.

Next, we show that for a given program Pr, $Pr \bowtie$loLRA admits the required conditions of the WSTS framework that ensure decidability of the induced coverability problem (see, e.g., [9, 15]). In particular, the compatibility condition between the well-quasi-ordering on states and the transitions is trivial since we explicitly include the (LOWER) step in loLRA.

Lemma 1. *Given a program Pr, the LTS $Pr \bowtie lo$LRA equipped with the well-quasi-ordering $\sqsubseteq$ (lifted to states of $Pr \bowtie lo$LRA by defining $\langle \overline{p}, \mathcal{B}\rangle \sqsubseteq \langle \overline{p}', \mathcal{B}'\rangle$ iff $\overline{p} = \overline{p}'$ and $\mathcal{B} \sqsubseteq \mathcal{B}'$) is a WSTS that admits effective initialization and effective pred-basis.*

As a corollary, we obtain that state reachability under loLRA is decidable. We refer the reader to §B where we give more details and proofs (which generally follow those in [22]).

5 Equivalence of the Memory Systems for LRA

In this section we establish the equivalence between loLRA and opLRA by demonstrating a simulation between these systems. The states of loLRA and opLRA are related to each other using *write lists*, which match read options in loLRA's potentials with concrete write event in opLRA's execution graphs.

Definition 6. A *write list* is a sequence of write events and write options. Let G be an execution graph, L an option list, and $tid_{\mathtt{RMW}} : \mathsf{W} \to \mathsf{Tid}$. A write list W is a $\langle G, L, tid_{\mathtt{RMW}}\rangle$*-write-list* if $|L| = |W|$ and the following hold for every $1 \leq k \leq |W|$:

- If $L(k)$ is a write option, then $W(k) = L(k)$.
- If $L(k) = \langle \tau, x, v, \pi_{\mathtt{RMW}}\rangle$, then $W(k) \in G.\mathsf{W}$, $\mathtt{tid}(W(k)) = \tau$, $\mathtt{loc}(W(k)) = x$, $\mathtt{val_W}(W(k)) = v$, and $tid_{\mathtt{RMW}}(W(k)) = \pi_{\mathtt{RMW}}$.

In addition to the above, we require that weak-coherence and local-read-coherence are maintained by any extension of the execution graph G with a sequence of reads and writes of thread τ that are obtained by following the write list W. This is formalized in the following notion of $\langle G, \tau\rangle$*-consistency* of a write list W.

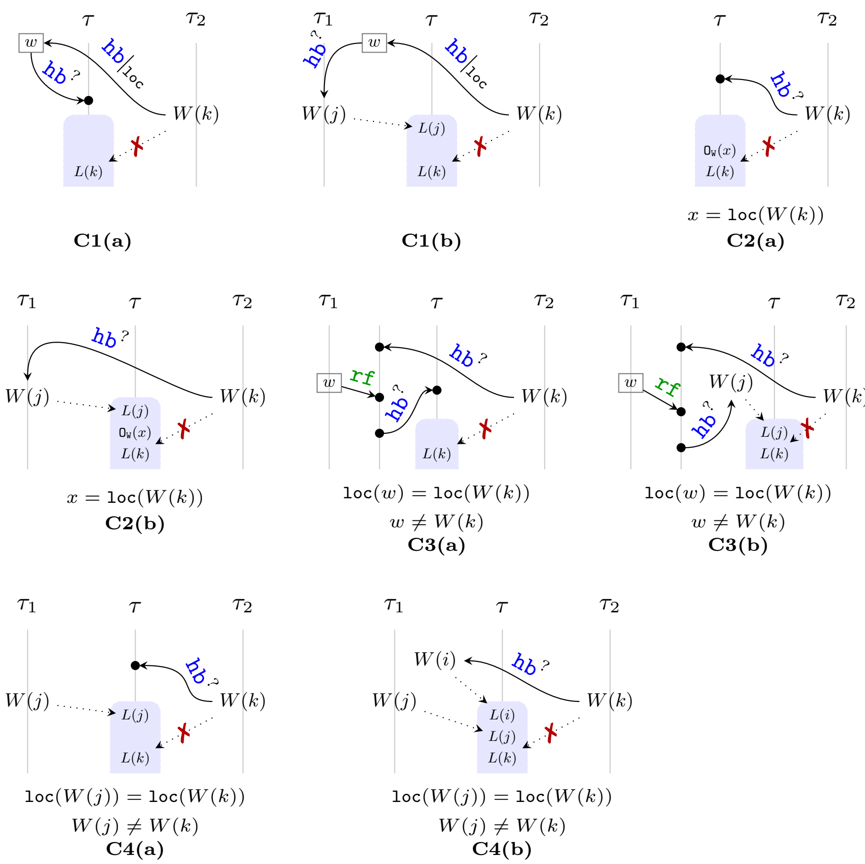


Fig. 8. Illustration of conditions in Definition 7 for the $\langle G, \tau\rangle$-*consistency* of W. Each condition is split into two cases (e.g., C1 is summarized using C1(a) or C1(b)).

Definition 7. A write list W is $\langle G, \tau\rangle$-*consistent* if for every $1 \leq k \leq |W|$ with $W(k) \in \mathsf{E}$:

C1 $W(k) \notin dom(G.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W(j) \mid 1 \leq j < k\}])$.
C2 If $W(i) = \mathtt{O_W}(\mathtt{loc}(W(k)))$ for some $i < k$,
then $W(k) \notin dom(G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W(j) \mid 1 \leq j < i\}])$.
C3 $W(k) \notin dom((G.\texttt{hb}|_{\texttt{loc}} \setminus G.\texttt{rf}) ; [\mathsf{R}] ; G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W(j) \mid 1 \leq j < k\}])$.
C4 If $\mathtt{loc}(W(j)) = \mathtt{loc}(W(k))$ and $W(k) \neq W(j)$ for some $j < k$,
then $W(k) \notin dom(G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W(i) \mid 1 \leq i < j\}])$.

Intuitively, for any future extension of execution graph with a sequence of events on τ, conditions C1 and C2 help in maintaining weak-coherence while C3 and C4 ensure that local-read-coherence is preserved. To assist readers, these conditions are depicted using diagrams in Figure 8 where the shaded area of τ represents a sequence of future events.

The simulation relation Υ is now defined as follows.

Definition 8. A state $\mathcal{B} \in$ loLRA.$\mathtt{Q}$ *matches* an execution graph G, denoted by $\mathcal{B} \curlyvee G$, if there exists a function $tid_{\mathtt{RMW}} : \mathsf{W} \to \mathsf{Tid}$, such that: (1) for every $\tau \in \mathsf{Tid}$ and $L \in \mathcal{B}(\tau)$, there exists a $\langle G, \tau\rangle$-consistent $\langle G, L, tid_{\mathtt{RMW}}\rangle$-write-list, and (2) for every $\langle w, e\rangle \in G.\mathtt{rf}\,;[\mathsf{RMW}]$, we have $\mathtt{tid}(e) = tid_{\mathtt{RMW}}(w)$.

Based on the simulation relation, we establish the equivalence of loLRA and opLRA. The proof, given in §A, shows that $\curlyvee$ constitutes a forward simulation from loLRA to opLRA, and $\curlyvee^{-1}$ constitutes a backward simulation from opLRA to loLRA.

Theorem 1. *The traces of lo*LRA *and the traces of op*LRA *coincide.*

6 Conclusion, Related and Future Work

We established the decidability of state reachability for finite-state programs under LRA, a memory model that lies strictly between WRA and RA. For that matter, we adapted the potential-based semantics of WRA from [22] to LRA, and showed that it meets the requirements for decidability of the WSTS framework.

In addition to the closely related work discussed in the introduction to this paper, the paper [14] studies the problem of verifying whether a given memory system provides causal consistency, which is a different verification problem than the one discussed in the current paper. The CC model in [14] (when restricted to single instruction transactions) is equivalent to (the RMW-free fragment of) WRA, whereas CCv from [14] is equivalent to SRA.

Another line of related work concerns *parametrized* programs, where one has an unknown number of threads but all of them run the same code. This arises a decidable verification problem under SC and TSO [5], but decidability of this problem is still unknown for WRA, SRA, and LRA. For the RMW-free fragment this problem is PSPACE for TSO [8] as well as for RA [19] (the latter result also allows a fixed number of distinguished threads running loop-free programs, possibly including RMWs).

An interesting direction for future work is to try to further close the gap between LRA and RA by introducing a restricted form of RA's modification order. A related problem that is still open (to the best of our knowledge) is whether the fragment of RA without RMWs induces a decidable verification problem. In addition, other models with undecidable reachability problems (such as the promising semantics [6] and the full POWER model [3]) may be bounded from below by decidable models.

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A Simulation Proofs

The following alternative formulation of the WRITE step will be convenient to use in our proofs. This formulation "works backwards"—choosing read options to omit from the target state for reaching the source state. Each such possibility is an "index choice":

Definition 9. *An* index choice *for a state $\mathcal{B}' \in lo\text{LRA}.\mathtt{Q}$ is a function $\mathcal{P}$ assigning a set $\mathcal{P}(\pi, L') \subseteq \{1, \dots, |L'|\}$ to every $\pi \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\pi)$. An index choice $\mathcal{P}$ for $\mathcal{B}' \in lo\text{LRA}.\mathtt{Q}$* justifies a $\langle \tau, \mathtt{W}(x, v_{\mathtt{W}}) \rangle$-step, *denoted by $\mathcal{P} \models \langle \tau, \mathtt{W}(x, v_{\mathtt{W}}) \rangle$, if the following hold:*

- *There exists $\pi_{\mathtt{RMW}} \in \mathsf{Tid}$ such that $L'(k) = \langle \tau, x, v_{\mathtt{W}}, \pi_{\mathtt{RMW}} \rangle$ for every $\pi \in \mathsf{Tid}$, $L' \in \mathcal{B}'(\pi)$ and $k \in \mathcal{P}(\pi, L')$.*
- *For every $\pi \in \mathsf{Tid}$, $L' \in \mathcal{B}'(\pi)$ and $k \in \{1, \dots, |L'|\} \setminus \mathcal{P}(\pi, L')$:*
  - *If $p_1 < k < p_2$ for some $p_1, p_2 \in \mathcal{P}(\pi, L')$, then $\mathtt{loc}(L'(k)) \neq x$.*
  - *If $\pi = \tau$ and $k < p$ for some $p \in \mathcal{P}(\pi, L')$, then $\mathtt{loc}(L'(k)) \neq x$.*

Now, a predecessor of $\mathcal{B}'$ with respect to a WRITE step intuitively satisfies two constraints: 1. For each list L' of $\mathcal{B}'$, there is a corresponding list L in $\mathcal{B}$ that possibly lack some read options of the form $\langle \tau, x, v_{\mathtt{W}}, \pi_{\mathtt{RMW}} \rangle$, corresponding to the new read options of $\mathcal{B}'$; and 2. If a list L' is different from the corresponding list L then there is a list of τ in $\mathcal{B}$ that justifies this difference. Notice that $\mathcal{B}$ may have arbitrary additional lists in addition to the above mandatory lists.

Note 1 (List operations). For an option list L, a location x, and a set $P \subseteq \{1, \dots, |L|\}$ of positions in L, we define (where $p_{\min} = \min P$ and $p_{\max} = \max P$ for $P \neq \emptyset$):

- $L_x \setminus P$ is the list derived from L by removing from it the positions in $P \setminus \{p_{\max}\}$ and replacing the value present at $L(p_{\max})$ with $\mathtt{O}_{\mathtt{W}}(x)$ (when $P = \emptyset$, we define $L_x \setminus P \triangleq L$). The mapping of the positions of L that are not in $P \setminus \{p_{\max}\}$ to their matching positions in $L_x \setminus P$ is denoted by $\mathsf{Map}_{\langle L,P \rangle}$. Formally, $\mathsf{Map}_{\langle L,P \rangle} \triangleq \lambda k \in ((\{1, \dots, |L|\} \setminus P) \cup \{p_{\max}\}).\, k - |\{j \in P \setminus \{p_{\max}\} \mid j < k\}|$ (and $\mathsf{Map}_{\langle L,P \rangle} \triangleq \lambda k \in \{1, \dots, |L|\}.\, k$ when $P = \emptyset$).
- $L_x \setminus\setminus P$ further removes from L the positions before the first position in P, namely $L_x \setminus\setminus P \triangleq \lambda k \in \{1, \dots, |L_x \setminus P| - (p_{\min} - 1)\}.\, (L_x \setminus P)(k + (p_{\min} - 1))$ (undefined if $P = \emptyset$). The mapping of the positions of L that are not in $P \setminus \{p_{\max}\}$ and not before the first position in P to their matching positions in $L_x \setminus\setminus P$ is denoted by $\mathsf{MMap}_{\langle L,P \rangle}$. Formally, $\mathsf{MMap}_{\langle L,P \rangle} \triangleq \lambda k \in (\{p_{\min}, \dots, |L|\} \setminus P) \cup \{p_{\max}\}.\, (\mathsf{Map}_{\langle L,P \rangle}(k) - (p_{\min} - 1))$.

For example, for the option list
$L = \langle \mathtt{T}_1, \mathtt{x}, 0, \mathtt{T}_4 \rangle \cdot \langle \mathtt{T}_1, \mathtt{x}, 1, \mathtt{T}_2 \rangle \cdot \langle \mathtt{T}_2, \mathtt{y}, 1, \mathtt{T}_3 \rangle \cdot \langle \mathtt{T}_1, \mathtt{x}, 1, \mathtt{T}_2 \rangle \cdot \langle \mathtt{T}_1, \mathtt{y}, 2, \mathtt{T}_4 \rangle \cdot \langle \mathtt{T}_1, \mathtt{x}, 1, \mathtt{T}_2 \rangle \cdot \langle \mathtt{T}_2, \mathtt{y}, 1, \mathtt{T}_1 \rangle$
and $P = \{2, 4, 6\}$, we have:

- $L_x \setminus P = \langle \mathtt{T}_1, \mathtt{x}, 0, \mathtt{T}_4 \rangle \cdot \langle \mathtt{T}_2, \mathtt{y}, 1, \mathtt{T}_3 \rangle \cdot \langle \mathtt{T}_1, \mathtt{y}, 2, \mathtt{T}_4 \rangle \cdot \mathtt{O}_{\mathtt{W}}(x) \cdot \langle \mathtt{T}_2, \mathtt{y}, 1, \mathtt{T}_1 \rangle$

- $\mathsf{Map}_{\langle L,P\rangle} = [1 \mapsto 1; 3 \mapsto 2; 5 \mapsto 3; 6 \mapsto 4; 7 \mapsto 5]$
- $L_x \setminus\!\!\setminus P = \langle \mathtt{T}_2, \mathtt{y}, 1, \mathtt{T}_3\rangle \cdot \langle \mathtt{T}_1, \mathtt{y}, 2, \mathtt{T}_4\rangle \cdot \mathtt{O}_{\mathtt{W}}(x) \cdot \langle \mathtt{T}_2, \mathtt{y}, 1, \mathtt{T}_1\rangle$
- $\mathsf{MMap}_{\langle L,P\rangle} = [3 \mapsto 1; 5 \mapsto 2; 6 \mapsto 3; 7 \mapsto 4]$

Definition 10. *The* source *of $\mathcal{B}'$ w.r.t. a thread τ, a location x, and an index choice $\mathcal{P}$ for $\mathcal{B}'$, denoted by $\mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})$, is given by:*

$$\mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P}) \triangleq \lambda\pi \in \mathsf{Tid}. \begin{cases} \{L'_x \setminus \mathcal{P}(\pi, L') \mid L' \in \mathcal{B}'(\pi)\} & \pi \neq \tau \\ \{L'_x \setminus \mathcal{P}(\tau, L') \mid L' \in \mathcal{B}'(\tau)\} \cup & \pi = \tau \\ \{L'_x \setminus\!\!\setminus \mathcal{P}(\eta, L') \mid \mathcal{P}(\eta, L') \neq \emptyset,\ \eta \in \mathsf{Tid} \text{ and } L' \in \mathcal{B}'(\eta)\} & \end{cases}$$

Proposition 3. $\mathcal{B} \xrightarrow{\tau, \mathtt{W}(x, v_{\mathtt{W}})}_{lo\mathrm{LRA}} \mathcal{B}'$ *iff there exists an index choice $\mathcal{P}$ for $\mathcal{B}'$ such that the following hold:*

- $\mathcal{P} \models \langle \tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle$*;*
- $\mathtt{O}_{\mathtt{W}}(x) \cdot \mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\tau) \subseteq \mathcal{B}(\tau)$*; and*
- $\mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\pi) \subseteq \mathcal{B}(\pi)$ *for every $\pi \in \mathsf{Tid} \setminus \{\tau\}$.*

Lemma 2. *For every trace of lo*LRA *there is an equivalent trace of op*LRA*.*

Proof. We show that $\curlyvee$ constitutes a forward simulation relation from loLRA to opLRA. First, the initial states clearly match: for every $\mathcal{B} \in \mathrm{loLRA}.\mathtt{Q}_0$ we have $\mathcal{B} \curlyvee G_0$, since (using any function $tid_{\mathtt{RMW}} : \mathsf{W} \to \mathsf{Tid}$) for every $\tau \in \mathsf{Tid}$ and $L \in \mathcal{B}(\tau)$, L itself, having only write options, is a $\langle G, \tau\rangle$-consistent $\langle G, L, \rangle$-write-list, regardless of what G is.

Now, suppose that $\mathcal{B} \curlyvee G$ and $\mathcal{B} \xrightarrow{\tau, l}_{\mathrm{loLRA}} \mathcal{B}'$. Let $tid_{\mathtt{RMW}} : \mathsf{W} \to \mathsf{Tid}$ that satisfies the conditions of Definition 8. We show that there exists G' such that $\mathcal{B}' \curlyvee G'$ and $G \xrightarrow{\tau, l}_{\mathrm{opLRA}} G'$. Consider the possible cases:

- WRITE-STEP $l = \mathtt{W}(x, v_{\mathtt{W}})$: Let $w = \mathsf{NextEvent}(G.\mathtt{E}, \tau, l)$. Let G' be the execution graph defined by $G'.\mathtt{E} = G.\mathtt{E} \cup \{w\}$ and $G'.\mathtt{rf} = G.\mathtt{rf}$. By definition, we have $G \xrightarrow{\tau, l}_{\mathrm{opLRA}} G'$. We show that $\mathcal{B}' \curlyvee G'$. First, since $\mathcal{B} \xrightarrow{\tau, l}_{\mathrm{loLRA}} \mathcal{B}'$, by Proposition 3, there exists an index choice $\mathcal{P}$ for $\mathcal{B}'$ that justifies a $\langle \tau, l\rangle$-step, such that $\mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\pi) \subseteq \mathcal{B}(\pi)$ for every $\pi \in \mathsf{Tid} \setminus \{\tau\}$ and $\mathtt{O}_{\mathtt{W}}(x) \cdot \mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\tau) \subseteq \mathcal{B}(\tau)$. Since $\mathcal{P}$ justifies a $\langle \tau, l\rangle$-step, there exists $\pi_{\mathtt{RMW}} \in \mathsf{Tid}$, such that $L'(k) = \langle \tau, x, v_{\mathtt{W}}, \pi_{\mathtt{RMW}}\rangle$ for every $\pi \in \mathsf{Tid}$, $L' \in \mathcal{B}'(\pi)$ and $k \in \mathcal{P}(\pi, L')$. Let $tid_{\mathtt{RMW}}{}' = tid_{\mathtt{RMW}}[w \mapsto \pi_{\mathtt{RMW}}]$. Since $w \notin dom(G'.\mathtt{rf})$, we vacuously have $\mathtt{tid}(e) = tid_{\mathtt{RMW}}{}'(w)$ for every $\langle w, e\rangle \in G'.\mathtt{rf}; [\mathsf{RMW}]$. It follows that for every $\langle w', e\rangle \in G'.\mathtt{rf}; [\mathsf{RMW}]$, we have $\mathtt{tid}(e) = tid_{\mathtt{RMW}}{}'(w')$. We show that for every $\pi \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\pi)$, there exists a $\langle G', \pi\rangle$-consistent $\langle G', L', tid_{\mathtt{RMW}}{}'\rangle$-write-list. Let $\pi \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\pi)$. We construct a $\langle G', \pi\rangle$-consistent $\langle G', L', tid_{\mathtt{RMW}}{}'\rangle$-write-list W'. Let $P \triangleq \mathcal{P}(\pi, L')$ and (the last two are only defined if $P \neq \emptyset$):
$$L \triangleq \begin{cases} L'_x \setminus P & \pi \neq \tau \\ \mathtt{O}_{\mathtt{W}}(x) \cdot (L'_x \setminus P) & \pi = \tau \end{cases} \quad f \triangleq \begin{cases} \mathsf{Map}_{\langle L', P\rangle} & \pi \neq \tau \\ \lambda k \in \{1, \ldots, |L'|\} \setminus P.\ \mathsf{Map}_{\langle L', P\rangle}(k) + 1 & \pi = \tau \end{cases}$$
$$L_\tau \triangleq \mathtt{O}_{\mathtt{W}}(x) \cdot L'_x \setminus\!\!\setminus P \quad f_\tau \triangleq \lambda k \in \{\min P, \ldots, |L'|\} \setminus P.\ \mathsf{MMap}_{\langle L', P\rangle}(k) + 1$$

Then, by definition, we have $L \in \mathcal{B}(\pi)$ and $L_\tau \in \mathcal{B}(\tau)$. Let W be a $\langle G, \pi\rangle$-consistent $\langle G, L, tid_{\texttt{RMW}}\rangle$-write-list, and W_τ be a $\langle G, \tau\rangle$-consistent $\langle G, L_\tau, tid_{\texttt{RMW}}\rangle$-write-list. Note that for every $k > \min P$ with $k \notin P$ and $\texttt{typ}(L'(k)) = \texttt{R}$, we have $\texttt{tid}(W(f(k))) = \texttt{tid}(L(f(k))) = \texttt{tid}(L_\tau(f_\tau(k))) = \texttt{tid}(W_\tau(f_\tau(k)))$, and so $G.\texttt{hb}$ must order the two write events, $W(f(k))$ and $W_\tau(f_\tau(k))$.
We define W' as follows:

$$W' \triangleq \lambda k \in \{1, ..., |L'|\}. \begin{cases} L'(k) & \texttt{typ}(L'(k)) = \texttt{W} \\ w & \texttt{typ}(L'(k)) = \texttt{R} \wedge k \in P \\ W(f(k)) & \texttt{typ}(L'(k)) = \texttt{R} \wedge k < \min P \\ \max_{G.\texttt{hb}}\{W(f(k)), W_\tau(f_\tau(k))\} & \text{otherwise} \end{cases}$$

It is easy to see that W' is a $\langle G', L', tid_{\texttt{RMW}}'\rangle$-write-list. We show that W' is $\langle G', \pi\rangle$-consistent. Let $1 \leq k \leq |L'|$ such that $W'(k) \in \mathsf{E}$. Let $y = \texttt{loc}(W'(k))$, $w_\pi = W(f(k))$ and $w_\tau = W_\tau(f_\tau(k))$ (the latter is only defined if $k > \min P$). We prove that each of the conditions in Definition 7 holds:

C1) $W'(k) \notin dom(G'.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G'.\texttt{hb}^? ; [\mathsf{E}^\pi \cup \{W'(j) \mid 1 \leq j < k\}])$. Suppose otherwise. First, note that we cannot have $k \in P$, since w is a maximal element in $G'.\texttt{hb}$. Consider the two possible cases:

(a) $W'(k) \in dom(G'.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G'.\texttt{hb}^? ; [\mathsf{E}^\pi])$: The definition of W' ensures $\langle w_\pi, W'(k)\rangle \in G'.\texttt{hb}|_{\texttt{loc}}{}^?$, and so $w_\pi \in dom(G'.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G'.\texttt{hb}^? ; [\mathsf{E}^\pi])$. Since W is $\langle G, \pi\rangle$-consistent, we have that $w_\pi \notin dom (G.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G.\texttt{hb}^? ; [\mathsf{E}^\pi])$, and therefore it must be the case that $\pi = \tau$ and $\langle w_\pi, w\rangle \in G'.\texttt{hb}|_{\texttt{loc}}$. It follows that $w_\pi \in dom(G.\texttt{hb}^? ; [\mathsf{E}^\tau])$. But, since $\pi = \tau$, we have $W(1) = L(1) = \texttt{0}_{\texttt{W}}(x) = \texttt{0}_{\texttt{W}}(\texttt{loc}(w_\pi))$, we obtain a contradiction to the fact that W is $\langle G, \tau\rangle$-consistent.

(b) $\langle W'(k), W'(j)\rangle \in G'.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G'.\texttt{hb}^?$ for some $1 \leq j < k$. Consider the two possible cases:

- $W'(j) = w$: In this case we must have $k \geq \min P$, and so $W'(k) = \max_{G.\texttt{hb}}\{w_\pi, w_\tau\}$. Hence, we have $\langle w_\tau, W'(k)\rangle \in G.\texttt{hb}|_{\texttt{loc}}{}^?$, and so $\langle w_\tau, w\rangle \in G'.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G'.\texttt{hb}^?$. Now, if $\langle w_\tau, w\rangle \in G'.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G'.\texttt{hb}$, then we also have $w_\tau \in dom(G.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G.\texttt{hb}^? ; [\mathsf{E}^\tau])$, which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent. Therefore, we have $\langle w_\tau, w\rangle \in G'.\texttt{hb}|_{\texttt{loc}}$. It follows that $w_\tau \in dom(G.\texttt{hb}^? ; [\mathsf{E}^\tau])$. But, since $W_\tau(1) = L_\tau(1) = \texttt{0}_{\texttt{W}}(x) = \texttt{0}_{\texttt{W}}(\texttt{loc}(w_\tau))$, we obtain a contradiction to the fact that W_τ is $\langle G, \tau\rangle$-consistent.
- $W'(j) \neq w$: In this case, we must have $\langle W'(k), W'(j)\rangle \in G.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G.\texttt{hb}^?$. The definition of W' ensures that $\langle w_\pi, W'(k)\rangle \in G.\texttt{hb}|_{\texttt{loc}}{}^?$, and so $\langle w_\pi, W'(j)\rangle \in G.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G.\texttt{hb}^?$. Now, since W is $\langle G, \pi\rangle$-consistent, we cannot have $W'(j) = W(f(j))$. Let $w'_\tau = W_\tau(f_\tau(j))$. Hence, $j \geq \min P$ and $W'(j) = w'_\tau$. It follows that $k \geq \min P$, and so $\langle w_\tau, W'(k)\rangle \in G.\texttt{hb}|_{\texttt{loc}}{}^?$. Hence, we have $\langle w_\tau, w'_\tau\rangle \in G.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G.\texttt{hb}^?$. This contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

C2) If $W'(i) = \texttt{0}_{\texttt{W}}(\texttt{loc}(W'(k)))$ for $i < k$, then $W'(k) \notin dom(G'.\texttt{hb}^? ; [\mathsf{E}^\pi \cup \{W'(j) \mid j < i\}])$. Suppose otherwise. Consider the two possible cases:

(a) There exists $i < k$ with $W'(i) = \mathtt{O}_{\mathtt{W}}(y)$ but $W'(k) \in \mathit{dom}(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. Note that the definition of W' ensures that $W'(i) = L'(i) = \mathtt{O}_{\mathtt{W}}(y)$, and since W is a $\langle G, L, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, it follows that $W(f(i)) = \mathtt{O}_{\mathtt{W}}(y)$. Consider the two possible cases:
   - $W'(k) = w$: In this case, we must have $y = x$, $\pi = \tau$ and $i < \max \mathcal{P}(\tau, L')$. Since $\mathcal{P}$ justifies a $\langle \tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle$-step, we cannot have $L'(i) = \mathtt{O}_{\mathtt{W}}(x)$.
   - $W'(k) \neq w$: In this case, the definition of W' ensures that $\langle w_\pi, W'(k)\rangle \in G'.\mathtt{hb}|_{\mathtt{loc}}{}^?$, and so $w_\pi \in \mathit{dom}(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. Since $w_\pi \neq w$ (as $w_\pi \in G.\mathtt{E}$), it follows that $w_\pi \in \mathit{dom}(G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. Since $W(f(i)) = \mathtt{O}_{\mathtt{W}}(y)$, this contradicts the fact that W is $\langle G, \pi\rangle$-consistent.

(b) There exists $j < i < k$ with $W'(i) = \mathtt{O}_{\mathtt{W}}(y)$ but $\langle W'(k), W'(j)\rangle \in G'.\mathtt{hb}^?$. Note that the definition of W' ensures that $W'(i) = L'(i) = \mathtt{O}_{\mathtt{W}}(y)$, and since W is a $\langle G, L, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, it follows that $W(f(i)) = \mathtt{O}_{\mathtt{W}}(y)$. In addition, since W_τ is $\langle G, L_\tau, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, it follows that $W_\tau(f_\tau(i)) = \mathtt{O}_{\mathtt{W}}(y)$ if $i > \min P$. Consider the possible cases:
   - $W'(k) = w$: In this case, we must have $y = x$ and $W'(j) = w$. It follows that $k, j \in P$, and since $\mathcal{P}$ justifies a $\langle \tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle$-step, we cannot have $L'(i) = \mathtt{O}_{\mathtt{W}}(x)$.
   - $W'(k) \neq w$ and $W'(j) = w$: In this case we must have $i, k > \min P$, and so $W'(k) = \max_{G.\mathtt{hb}}\{w_\pi, w_\tau\}$ and $W_\tau(f_\tau(i)) = \mathtt{O}_{\mathtt{W}}(y)$. Hence, we have $\langle w_\tau, W'(k)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$, and so $\langle w_\tau, w\rangle \in G'.\mathtt{hb}^?$. Since $w_\tau \neq w$ (as $w_\tau \in G.\mathtt{E}$), it follows that $w_\tau \in \mathit{dom}(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. Since $W_\tau(f_\tau(i)) = \mathtt{O}_{\mathtt{W}}(y)$, this contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.
   - $W'(k) \neq w$ and $W'(j) \neq w$: In this case, we have $\langle W'(k), W'(j)\rangle \in G.\mathtt{hb}^?$. Let $w^j_\pi = W(f(j))$ and $w^j_\tau = W_\tau(f_\tau(j))$ (the latter is only defined if $j > \min P$). Our construction ensures that one of the following holds:
     * $W'(j) = w^j_\pi$: Since $W'(k) \neq w$, the definition of W' ensures that $\langle w_\pi, W'(k)\rangle \in G'.\mathtt{hb}|_{\mathtt{loc}}{}^?$, and so $\langle w_\pi, w^j_\pi\rangle \in G.\mathtt{hb}^?$. This contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
     * $W'(j) = w^j_\tau$: In this case we have $j > \min P$, and so $k > \min P$. Since $W'(k) \neq w$, the definition of W' ensures that $\langle w_\tau, W'(k)\rangle \in G'.\mathtt{hb}|_{\mathtt{loc}}{}^?$, and so $\langle w_\tau, w^j_\tau\rangle \in G.\mathtt{hb}^?$. This contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

C3) $W'(k) \notin \mathit{dom}((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf}) ; [\mathsf{R}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\pi \cup \{W'(j) \mid 1 \leq j < k\}])$. Suppose otherwise. Consider the two possible cases:

(a) $W'(k) \in \mathit{dom}((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf}) ; [\mathsf{R}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. In this case we have $w' \in G'.\mathsf{W}_y$, $r \in G'.\mathsf{R}$ such that $\langle w', r\rangle \in G'.\mathtt{rf}$, $W'(k) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [r] ; G'.\mathtt{hb}^? ;[\mathsf{E}^\pi])$, and $w' \neq W'(k)$. Note that since w is a maximal element in $G'.\mathtt{hb}$, we have $r \in G.\mathsf{R}$, $w' \neq w$, and $W'(k) \neq w$. It follows that $w_\pi \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [r] ; G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. Consider the two possible cases:

- $w_\pi \neq w'$: In this case we have $w_\pi \in dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf})\,;[r]\,;G.\mathtt{hb}^?\,;[\mathsf{E}^\pi])$ which contradicts the fact that W is $\langle G,\pi\rangle$-consistent.
- $w_\pi = w'$: In this case we have $w_\pi \neq W'(k)$ (since $w' \neq W'(k)$). Therefore we have $w_\pi \in dom(G.\mathtt{hb}|_{\mathtt{loc}}\,;[W'(k)]\,;G.\mathtt{hb}^?\,;[\mathsf{E}^\pi])$ where $W'(k) \in G.\mathsf{W}$ which contradicts the fact that W is $\langle G,\pi\rangle$-consistent.

(b) $W'(k) \in dom((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf});[\mathsf{R}];G'.\mathtt{hb}^?;[\{W'(j) \in \mathsf{E} \mid 1 \leq j < k\}])$. In this case we have $w' \in G'.\mathsf{W}_y$, $r \in G'.\mathsf{R}$, $W'(j) \in G'.\mathsf{W}$ for some $j < k$ such that $\langle w', r\rangle \in G'.\mathtt{rf}$, $W'(k) \in dom(G'.\mathtt{hb}|_{\mathtt{loc}}\,;[r]\,;G'.\mathtt{hb}^?\,;[W'(j)])$ and $w' \neq W'(k)$. Note that, since w is a maximal element in $G'.\mathtt{hb}$, we have $r \in G.\mathsf{R}$, $w' \neq w$, and $W'(k) \neq w$. We claim that $w_\pi \neq w' \neq w_\tau$. Assume otherwise and we have $\langle w', r\rangle \in G.\mathtt{rf}$ where $w' \in dom(G.\mathtt{hb}|_{\mathtt{loc}}\,;[W'(k)]\,;G.\mathtt{hb}\,;[r])$ which contradicts the fact that G is LRA-consistent. For $j > \min P$, let $w_\pi^j = W(f(j))$ and $w_\tau^j = W_\tau(f_\tau(j))$. Consider the two possible cases:

- $j \in P$: In this case we have $w = W'(j)$ and $w_\tau \in dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf})\,;[r]\,;G.\mathtt{hb}^?;\,[\mathsf{E}^\tau])$ which contradicts the fact that W_τ is $\langle G,\tau\rangle$-consistent.
- $j \notin P$: If $j < \min P$ we have $W'(j) = w_\pi^j$ and $w_\pi \in dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf})\,;[r]\,;G.\mathtt{hb}^?\,;[w_\pi^j])$ which contradicts the fact that W is $\langle G,\pi\rangle$-consistent. If $j > \min P$, our construction ensures that one of the following holds:
  - $W'(j) = w_\pi^j$: In this case we have $w_\pi \in dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf})\,;[r]\,;G.\mathtt{hb}^?\,;[w_\pi^j])$ which contradicts the fact that W is $\langle G,\pi\rangle$-consistent.
  - $W'(j) = w_\tau^j$: In this case we have $w_\tau \in dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf})\,;[r]\,;G.\mathtt{hb}^?\,;[w_\tau^j])$ which contradicts the fact that W_τ is $\langle G,\tau\rangle$-consistent.

C4) If $\mathtt{loc}(W'(j)) = \mathtt{loc}(W'(k))$ and $W'(k) \neq W'(j)$ for some $j < k$, then $W'(k) \notin dom(G'.\mathtt{hb}^?\,;[\mathsf{E}^\pi \cup \{W'(i) \mid i < j\}])$. Suppose otherwise. Let $m = \min P$, $n = \max P$, $w_\pi^j = W(f(j))$, and $w_\tau^j = W_\tau(f_\tau(j))$. Consider the following possible cases:

(a) For some $j < k$, $\mathtt{loc}(W'(j)) = \mathtt{loc}(W'(k))$, $W'(k) \in dom(G'.\mathtt{hb}^?\,;[\mathsf{E}^\pi])$, and $W'(k) \neq W'(j)$. Note that $k \notin P$ (and hence $w \neq W'(k)$), otherwise, we will have $y = x$, $\pi = \tau$ and $j < \max \mathcal{P}(\tau, L')$ which contradicts the fact that $\mathcal{P}$ justifies a $\langle \tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle$-step (since $\mathtt{loc}(L'(j)) = x$). Consider the two possible cases:

- $W'(j) = w$: We claim that $\max P < k$, assuming otherwise (i.e., $j < k < \max P$), we have $\mathtt{loc}(L'(k)) = x = \mathtt{loc}(L(f(k)))$, which contradicts the fact that $\mathcal{P}$ justifies a $\langle\tau, l\rangle$-step (since, $f(j) < f(k) < f(n)$). If $\pi \neq \tau$, since $\mathcal{P}$ justifies a $\langle\tau, l\rangle$-step, we have $W(f(n)) = L(f(n)) = \mathtt{O}_{\mathtt{W}}(y)$, $f(n) < f(k)$ and $w_\pi \in dom(G.\mathtt{hb}^?\,;[\mathsf{E}^\pi])$. This contradicts the fact that W is $\langle G,\pi\rangle$-consistent. If $\pi = \tau$ we have $W(1) = L(1) = \mathtt{O}_{\mathtt{W}}(y)$ and $w_\pi \in$

$dom(G.\mathtt{hb}^? \,;[\mathsf{E}^\pi])$. This contradicts the fact that W is $\langle G, \pi\rangle$-consistent.

- $W'(j) \neq w$: In this case we have $j \neq m \neq k$. Consider the following three possible cases:
  - $j < k < m$: In this case we have $w_\pi = W'(k)$, $w^j_\pi = W'(j)$ and hence $w_\pi \in dom(G'.\mathtt{hb}^? \,;[\mathsf{E}^\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
  - $j < m < k$: In this case $w^j_\pi = W'(j)$. If $w^j_\pi \neq w_\pi$ we have $w_\pi \in dom(G'.\mathtt{hb}^? \,;[\mathsf{E}^\pi])$ where $\mathtt{loc}(w^j_\pi) = \mathtt{loc}(w_\pi)$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent. If $w^j_\pi = w_\pi$ we have $w_\pi \neq W'(k)$ and hence $w_\pi \in dom(G.\mathtt{hb}|_{\mathtt{loc}}\,;[W'(k)]\,;G.\mathtt{hb}^?\,;[\mathsf{E}^\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
  - $m < j < k$: If $w^j_\pi \neq w_\pi$ we have $w_\pi \in dom(G'.\mathtt{hb}^? \,;[\mathsf{E}^\pi])$ where $\mathtt{loc}(w^j_\pi) = \mathtt{loc}(w_\pi)$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent. If $w^j_\pi = w_\pi$ we consider the following two cases:
    - $W'(k) \neq w_\pi$: We have $w_\pi \in dom(G.\mathtt{hb}|_{\mathtt{loc}}\,;[W'(k)]\,;G.\mathtt{hb}^?\,;[\mathsf{E}^\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
    - $W'(k) = w_\pi$: In this case we have $W'(j) = w^j_\tau \neq w^j_\pi$. The definition of W' ensures that $\langle w_\tau, W'(k)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$ and hence $\langle w_\tau, w^j_\pi\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$ (since $W'(k) = w_\pi = w^j_\pi$). We also have $\langle w^j_\pi, W'(j)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$ and hence $\langle w^j_\pi, w^j_\tau\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}$ (since $w^j_\pi = W'(k) \neq W'(j) = w^j_\tau$). Therefore we have $w_\tau \in dom(G.\mathtt{hb}|_{\mathtt{loc}}\,;[w^j_\tau])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

(b) For some $1 \leq i < j < k$, $\mathtt{loc}(W'(j)) = \mathtt{loc}(W'(k))$, $W'(k) \in dom(G'.\mathtt{hb}^?\,;[W'(i)])$, and $W'(k) \neq W'(j)$. Note that $k \notin P$, otherwise, we will have $W'(k) = W'(i) = w$ and $\mathtt{loc}(L'(j)) = \mathtt{loc}(w)$ where $i < j < k$ and $i, k \in P$ which contradicts the fact that $\mathcal{P}$ justifies a $\langle \tau, l\rangle$-step. We consider following sub-cases:

- $i \in P$: In this case we have $W'(i) = w \neq W'(k)$ and hence $\langle W'(k), W'(i)\rangle \in G'.\mathtt{hb}$. Consider the following cases:
  - $W'(j) = W'(i)$: In this case we have $W'(j) = w = W'(i)$, $\mathtt{loc}(W'(k)) = \mathtt{loc}(w)$ and hence $W'(k) \in dom(\mathtt{hb}|_{\mathtt{loc}}\,;[w]\,;\mathtt{hb}^?\,;[W'(j)])$. Therefore the present case reduces to (item C1) and we observe a contradiction using same reasoning.
  - $W'(j) \neq W'(i)$: In this case we claim that $W'(k) = w_\tau$, assuming otherwise, we will have $w_\tau \in dom(\mathtt{hb}|_{\mathtt{loc}};[W'(k)];\mathtt{hb}^?;[\mathsf{E}^\tau])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent. Next consider following possibilities:
    - $w^j_\tau \neq w_\tau$: In this case $\mathtt{loc}(w^j_\tau) = \mathtt{loc}(w_\tau)$ and $w_\tau \in dom(G.\mathtt{hb}^?;[\mathsf{E}^\tau])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.
    - $w^j_\tau = w_\tau$: In this case $W'(k) = w_\tau = w^j_\tau$, $\langle w_\pi, W'(k)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$, and $\langle w^j_\tau, W'(j)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$. Since $w^j_\tau = W'(k) \neq W'(j) = w^j_\pi$, we have $w_\pi \in dom(G.\mathtt{hb}|_{\mathtt{loc}}\,;[w^j_\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.

- $i \notin P$: We consider the following cases:
  - $*$ $j \in P$: We claim that $\max P < k$, assuming otherwise (i.e., $j < k < \max P$), we have $\mathtt{loc}(L'(k)) = x = \mathtt{loc}(L(f(k)))$, which contradicts the fact that $\mathcal{P}$ justifies a $\langle \tau, l \rangle$-step (since, $f(j) < f(k) < f(n)$). Consider the following possible cases:
    - $\cdot$ $i < \min P$: Since $n = \max P$ and $\mathcal{P}$ justifies a $\langle \tau, l \rangle$-step, we have $W(f(n)) = L(f(n)) = \mathtt{O_W}(y)$ where $f(i) < f(n) < f(k)$ and $w_\pi \in dom(G.\mathtt{hb}^? ; [W(i)])$. This contradicts the fact that W is $\langle G, \pi \rangle$-consistent.
    - $\cdot$ $i > \min P$: Let $n_\pi = f(\max P)$ and $n_\tau = f_\tau(\max P)$. If $W'(i) = w^i_\pi$ then we have $w_\pi \in dom(G.\mathtt{hb}^? ; [w^i_\pi])$ where $W(n_\pi) = \mathtt{O_W}(\mathtt{loc}(w_\pi))$ (since $\mathcal{P}$ justifies a $\langle \tau, l \rangle$-step). This contradicts the fact that W is $\langle G, \pi \rangle$-consistent. If $W'(i) = w^i_\tau$ then we have $w_\tau \in dom(G.\mathtt{hb}^? ; [w^i_\tau])$ where $W_\tau(n_\tau) = \mathtt{O_W}(\mathtt{loc}(w_\tau))$. This contradicts the fact that W_τ is $\langle G, \tau \rangle$-consistent.
  - $*$ $j \notin P$: In this case we have $m \notin \{i, j, k\}$ and $i < j < k$. We consider the following possible cases:
    - $\cdot$ $\mathtt{tid}(W'(j)) = \mathtt{tid}(W'(k))$: In this case, since $W'(j) \neq W'(k)$, we have either (a) $\langle W'(j), W'(k) \rangle \in G'.\mathtt{hb}$ or (b) $\langle W'(k), W'(j) \rangle \in G'.\mathtt{hb}$. Assuming (a) we have $W'(j) \in dom(G'.\mathtt{hb}|_{\mathtt{loc}} ; [W'(k)] ; G'.\mathtt{hb}^? ; [W'(i)])$ where $i < j$ and hence the present case reduces to (item C1) resulting in a contradiction. Similarly, assuming (b) we have $W'(k) \in dom(G'.\mathtt{hb}|_{\mathtt{loc}} ; [W'(j)])$ where $j < k$ and hence again the present case reduces to (item C1) resulting in a contradiction.
    - $\cdot$ $\mathtt{tid}(W'(j)) \neq \mathtt{tid}(W'(k))$: In this case it is easy to see that $w^j_\pi \neq w_\pi$ and $w^j_\tau \neq w_\tau$. If $W'(i) = w^i_\pi$ then we have $w_\pi \in dom(G.\mathtt{hb}^? ; [w^i_\pi])$ and $w^j_\pi \neq w_\pi$ which contradicts the fact that W is $\langle G, \pi \rangle$-consistent. On the other hand, if $W'(i) = w^i_\tau$ then we have $w_\tau \in dom(G.\mathtt{hb}^? ; [w^i_\tau])$ and $w^j_\tau \neq w_\tau$ which contradicts the fact that W_τ is $\langle G, \tau \rangle$-consistent.

– READ-STEP $l = \mathtt{R}(x, v_\mathtt{R})$: By definition, since $\mathcal{B} \xrightarrow{\tau, l}_{\text{loLRA}} \mathcal{B}'$, there exists a read option o with $\mathtt{loc}(o) = x$ and $\mathtt{val}(o) = v_\mathtt{R}$ such that $\mathcal{B}(\tau) = o \cdot \mathcal{B}'(\tau)$. For every $L \in \mathcal{B}(\tau)$, let W_L be a $\langle G, \tau \rangle$-consistent $\langle G, L, tid_{\mathtt{RMW}} \rangle$-write-list. Let $A = \{W_L(1) \mid L \in \mathcal{B}(\tau)\}$. Since $\mathcal{B}(\tau)$ is non-empty, we know that A is not empty. Since each W_L is a $\langle G, L, tid_{\mathtt{RMW}} \rangle$-write-list, we have that $\mathtt{tid}(w) = \mathtt{tid}(o)$ for every $w \in A$. Hence, $G.\text{po}$ totally orders A. Let $w = \min_{G.\text{po}} A$ and let $L_{\min} \in \mathcal{B}(\tau)$ such that $w = W_{L_{\min}}(1)$. Let $r = \mathsf{NextEvent}(G.\mathtt{E}, \tau, l)$ and let G' be the execution graph given by $G'.\mathtt{E} = G.\mathtt{E} \cup \{r\}$ and $G'.\mathtt{rf} = G.\mathtt{rf} \cup \{\langle w, r \rangle\}$.

We show that $G \xrightarrow{\tau, l}_{\text{opLRA}} G'$. By definition, it suffices to show the following:

- $w \in G.\mathsf{W}_x$ and $\mathtt{val_W}(w) = v_\mathtt{R}$: We have $w = W_{L_{\min}}(1)$, and since $W_{L_{\min}}$ is a $\langle G, L_{\min}, tid_{\mathtt{RMW}} \rangle$-write-list, we have that $w \in G.\mathsf{W}$, $\mathtt{loc}(w) = \mathtt{loc}(W_{L_{\min}}(1)) = \mathtt{loc}(L_{\min}(1)) = \mathtt{loc}(o) = x$ and $\mathtt{val_W}(w) = \mathtt{val_W}(W_{L_{\min}}(1)) = \mathtt{val}(L_{\min}(1)) = \mathtt{val}(o) = v_\mathtt{R}$.

- $w \notin dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$: Since $W_{L_{\min}}$ is $\langle G, \tau\rangle$ -consistent and $w = W_{L_{\min}}(1)$, we cannot have $w \in dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$.
- $w \notin dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$: Since $W_{L_{\min}}$ is $\langle G, \tau\rangle$ -consistent and $w = W_{L_{\min}}(1)$, we cannot have $w \in dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$.

It remains to show that $\mathcal{B}' \curlyvee G'$. We use the same $tid_{\mathtt{RMW}}$ mapping and show that for every $\pi \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\pi)$, there exists a $\langle G', \pi\rangle$-consistent $\langle G', L', tid_{\mathtt{RMW}}\rangle$-write-list. Let $\pi \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\pi)$. We define a $\langle G', \pi\rangle$-consistent $\langle G', L', tid_{\mathtt{RMW}}\rangle$-write-list. Consider two cases:

- $\pi \neq \tau$: By definition, since $\mathcal{B} \xrightarrow{\tau, l}_{\text{loLRA}} \mathcal{B}'$, we have $L' \in \mathcal{B}(\pi)$. Let W be a $\langle G, \pi\rangle$-consistent $\langle G, L', tid_{\mathtt{RMW}}\rangle$-write-list. It is easy to see that W is also a $\langle G', L', tid_{\mathtt{RMW}}\rangle$-write-list. To see that W is $\langle G', \pi\rangle$-consistent, we note that if any one of the following holds in G' then the same holds in G,
  - $W(k) \in dom(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\pi \cup \{W(j) \mid 1 \leq j < k\}])$.
  - $W(k) \in dom(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi \cup \{W(j) \mid 1 \leq j < k\}])$.
  - $W(k) \in dom((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf});[\mathsf{R}];G'.\mathtt{hb}^?;[\mathsf{E}^\pi \cup \{W(j) \mid 1 \leq j < k\}])$.
  - $W(k) \in dom(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi \cup \{W(i) \mid 1 \leq i < j\}])$.

  Therefore, the $\langle G', \pi\rangle$-consistency of W directly follows from its $\langle G, \pi\rangle$-consistency.
- $\pi = \tau$: Let $L = o \cdot L'$. Then, $L \in \mathcal{B}(\tau)$. Let $W' = \lambda k \in \{1, \ldots, |L'|\}.\, W_L(1+k)$. It is easy to see that W' is a $\langle G', L', tid_{\mathtt{RMW}}\rangle$-write-list. We show that W' is $\langle G', \tau\rangle$-consistent. Let $1 \leq k \leq |W'|$ such that $W'(k) \in \mathsf{E}$. We prove that each of the conditions in Definition 7 holds:
  - C1) $W'(k) \notin dom(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\tau \cup \{W'(j) \mid 1 \leq j < k\}])$: Suppose otherwise. If $W'(k) \in dom(G.\mathtt{hb}|_{\mathtt{loc}};[\mathsf{W}];G.\mathtt{hb}^?;[\mathsf{E}^\tau \cup \{W'(j) \mid 1 \leq j < k\}])$, then $W_L(1+k) \in dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau \cup \{W_L(1+j) \mid 1 \leq j < k\}])$, which contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent. Hence, we must have $\langle W'(k), w\rangle \in G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^?$. Since $L(1) = o$, the definition of w ensures that $\langle w, W_L(1)\rangle \in G.\mathrm{po}^?$. It follows that $\langle W_L(1+k), W_L(1)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^?$, which again contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent.
  - C2) If $W'(i) = \mathtt{O_W}(\mathtt{loc}(W'(k)))$ for $i < k$, then $W'(k) \notin dom(G'.\mathtt{hb}^? ; [\mathsf{E}^\tau \cup \{W'(j) \mid j < i\}])$. Suppose otherwise. Consider the two possible cases:
    - (a) There exists $i < k$ with $W'(i) = \mathtt{O_W}(\mathtt{loc}(W'(k)))$ (i.e., $W_L(1+i) = \mathtt{O_W}(\mathtt{loc}(W_L(1+k))))$ but $W'(k) \in dom(G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. If $W'(k) \in dom(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, then $W_L(1+k) \in dom(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, which contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent. Hence, we must have $\langle W'(k), w\rangle \in G.\mathtt{hb}^?$. Since $L(1) = o$, the definition of w ensures that $\langle w, W_L(1)\rangle \in G.\mathrm{po}^?$. It follows that $\langle W_L(1+k), W_L(1)\rangle \in G.\mathtt{hb}$ while $W_L(1+i) = \mathtt{O_W}(\mathtt{loc}(W_L(1+k)))$ where $i < k$. Again, this contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent.
    - (b) There exists $j < i < k$ with $W'(i) = \mathtt{O_W}(\mathtt{loc}(W'(k)))$ (and so, $W_L(1+i) = \mathtt{O_W}(\mathtt{loc}(W_L(1+k))))$ but $\langle W'(k), W'(j)\rangle \in G'.\mathtt{hb}^?$.

In this case, since $W'(j) \in \mathsf{W}$, we must have $\langle W'(k), W'(j)\rangle \in G.\mathtt{hb}^?$. Hence, $\langle W_L(1+k)), W_L(1+j)\rangle \in G.\mathtt{hb}^?$ which contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent.

C3) $W'(k) \notin \mathit{dom}((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf});[\mathsf{R}];G'.\mathtt{hb}^?;[\mathsf{E}^\tau \cup \{W'(j) \mid 1 \leq j < k\}])$. Suppose otherwise. If $W'(k) \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf})\ ;[\mathsf{R}]\ ; G.\mathtt{hb}^?\ ; [\mathsf{E}^\tau \cup \{W'(j) \mid 1 \leq j < k\}])$ then $W_L(1+k) \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf})\ ; [\mathsf{R}]\ ; G.\mathtt{hb}^?\ ; [\mathsf{E}^\tau \cup \{W_L(1+j) \mid 1 \leq j+1 < k+1\}])$, which contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent. Hence, we must have $\langle W'(k), w\rangle \in (G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf})\ ; [\mathsf{R}]\ ; G.\mathtt{hb}^?$. Since $L(1) = o$, the definition of w ensures that $\langle w, W_L(1)\rangle \in G.\mathrm{po}^?$. It follows that $W_L(1+k) \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf})\ ; [\mathsf{R}];\ G.\mathtt{hb}^?\ ; [W_L(1)])$, which again contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent.

C4) If $\mathtt{loc}(W'(j)) = \mathtt{loc}(W'(k))$ and $W'(k) \neq W'(j)$ for some $j < k$, then $W'(k) \notin \mathit{dom}(G'.\mathtt{hb}^?\ ; [\mathsf{E}^\tau \cup \{W'(i) \mid 1 \leq i < j\}])$. Suppose otherwise. Consider the two possible cases:

(a) For $j < k$, $\mathtt{loc}(W'(j)) = \mathtt{loc}(W'(k))$, $W'(k) \in \mathit{dom}(G'.\mathtt{hb}^?\ ; [\mathsf{E}^\tau])$, and $W'(k) \neq W'(j)$. If $W'(k) \in \mathit{dom}(G.\mathtt{hb}^?\ ; [\mathsf{E}^\tau])$ then $W_L(1+k) \in \mathit{dom}(G.\mathtt{hb}^?\ ; [\mathsf{E}^\tau])$, which contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent. Hence, we must have $W_L(1+k) \in \mathit{dom}(G.\mathtt{hb}^?\ ; [w])$. Since $L(1) = o$, the definition of w ensures that $\langle w, W_L(1)\rangle \in G.\mathrm{po}^?$. It follows that $W_L(1+k) \in \mathit{dom}(G.\mathtt{hb}^?\ ; [W_L(1)])$, where $W_L(1+k) = W'(k) \neq W'(j) = W_L(1+j)$. Again, this contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent.

(b) For $1 \leq i < j < k$, $\mathtt{loc}(W'(j)) = \mathtt{loc}(W'(k))$, $W'(k) \in \mathit{dom}(G'.\mathtt{hb}^?;\ [W'(i)])$, and $W'(k) \neq W'(j)$. In this case we have $W_L(1+k) \in \mathit{dom}(G.\mathtt{hb}^?;[W_L(1+i)])$, where $\mathtt{loc}(W_L(1+j)) = \mathtt{loc}(W_L(1+k))$, $W_L(1+k) = W'(k) \neq W'(j) = W_L(1+j)$, and $1 < 1+i < 1+j < 1+k$. This contradicts the fact that W_L is $\langle G, \tau\rangle$-consistent.

– RMW-STEP $l = \mathtt{RMW}(x, v_{\mathtt{R}}, v_{\mathtt{W}})$: By definition and Proposition 3, since $\mathcal{B} \xrightarrow{\tau,l}_{\mathrm{loLRA}} \mathcal{B}'$, we have the following:

- There exists a read option o with $\mathtt{loc}(o) = x$, $\mathtt{val}(o) = v_{\mathtt{R}}$ and $\mathtt{rmw\text{-}tid}(o) = \tau$ such that $L(1) = o$ for every $L \in \mathcal{B}(\tau)$.
- There exists an index choice $\mathcal{P}$ for $\mathcal{B}'$ that justifies a $\langle \tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle$-step, such that $\mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\pi) \subseteq \mathcal{B}(\pi)$ for every $\pi \in \mathsf{Tid} \setminus \{\tau\}$ and $o \cdot \mathtt{O}_{\mathtt{W}}(x) \cdot \mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\tau) \subseteq \mathcal{B}(\tau)$.

For every $L \in \mathcal{B}(\tau)$, let W_L be a $\langle G, \tau\rangle$-consistent $\langle G, L, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list. Let $A = \{W_L(1) \mid L \in \mathcal{B}(\tau)\}$. Since $\mathcal{B}(\tau)$ is non-empty, we know that A is not empty. Since each W_L is a $\langle G, L, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, we have that $\mathtt{tid}(w) = \mathtt{tid}(o)$ for every $w \in A$. Hence, $G.\mathrm{po}$ totally orders A. Let $w = \min_{G.\mathrm{po}} A$ and let $L_{\min} \in \mathcal{B}(\tau)$ such that $w = W_{L_{\min}}(1)$. Let $e = \mathsf{NextEvent}(G.\mathtt{E}, \tau, l)$ and let G' be the execution graph given by $G'.\mathtt{E} = G.\mathtt{E} \cup \{e\}$ and $G'.\mathtt{rf} = G.\mathtt{rf} \cup \{\langle w, e\rangle\}$.

Note that $w = W_{L_{\min}}(1)$, and since $W_{L_{\min}}$ is a $\langle G, L_{\min}, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, we have:

- $w \in G.\mathsf{W}$.
- $\mathtt{loc}(w) = \mathtt{loc}(W_{L_{\min}}(1)) = \mathtt{loc}(L_{\min}(1)) = \mathtt{loc}(o) = x$.

- $\mathtt{val}_{\mathtt{W}}(w) = \mathtt{val}_{\mathtt{W}}(W_{L_{\min}}(1)) = \mathtt{val}(L_{\min}(1)) = \mathtt{val}(o) = v_{\mathtt{R}}$.
- $tid_{\mathtt{RMW}}(w) = tid_{\mathtt{RMW}}(W_{L_{\min}}(1)) = \mathtt{rmw\text{-}tid}(L_{\min}(1)) = \tau$.

Therefore, to show that $G \xrightarrow{\tau,l}_{\text{opLRA}} G'$, by definition, it suffices to show the following:

- $w \notin dom(G.\mathtt{hb}|_{\mathtt{loc}} \mathbin{;} [\mathsf{W}] \mathbin{;} G.\mathtt{hb}^? \mathbin{;} [\mathsf{E}^\tau])$: Since $W_{L_{\min}}$ is $\langle G, \tau\rangle$-consistent and $w = W_{L_{\min}}(1)$, we cannot have $w \in dom(G.\mathtt{hb}|_{\mathtt{loc}} \mathbin{;} [\mathsf{W}] \mathbin{;} G.\mathtt{hb}^? \mathbin{;} [\mathsf{E}^\tau])$.
- $w \notin dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) \mathbin{;} [\mathsf{R}] \mathbin{;} G.\mathtt{hb}^? \mathbin{;} [\mathsf{E}^\tau])$: Since $W_{L_{\min}}$ is $\langle G, \tau\rangle$- consistent and $w = W_{L_{\min}}(1)$, we cannot have $w \in dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) \mathbin{;} [\mathsf{R}] \mathbin{;} G.\mathtt{hb}^? \mathbin{;} [\mathsf{E}^\tau])$.
- $w \notin dom(G.\mathtt{rf}; [\mathsf{RMW}])$: Suppose otherwise, and let $e' \in \mathsf{RMW}$ such that $\langle w, e'\rangle \in G.\mathtt{rf}$. Then, since $tid_{\mathtt{RMW}}(w) = \tau$, the second condition in Definition 8 ensures that $\mathtt{tid}(e) = \tau$. Hence, $w \in dom(G.\mathtt{rf} \mathbin{;} [\mathsf{RMW} \cap \mathsf{E}^\tau]) \subseteq dom(G.\mathtt{hb}|_{\mathtt{loc}} \mathbin{;} [\mathsf{W}] \mathbin{;} G.\mathtt{hb}^? \mathbin{;} [\mathsf{E}^\tau])$, which contradicts the first item.

It remains to show that $\mathcal{B}' \curlyvee G'$. Since $\mathcal{P}$ justifies a $\langle \tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle$-step, there exists $\pi_{\mathtt{RMW}} \in \mathsf{Tid}$, such that $L'(k) = \langle \tau, x, v_{\mathtt{W}}, \pi_{\mathtt{RMW}}\rangle$ for every $\pi \in \mathsf{Tid}$, $L' \in \mathcal{B}'(\pi)$ and $k \in \mathcal{P}(\pi, L')$. Let $tid_{\mathtt{RMW}}{}' = tid_{\mathtt{RMW}}[w \mapsto \pi_{\mathtt{RMW}}]$. Since $e \notin dom(G'.\mathtt{rf})$, we vacuously have $\mathtt{tid}(e') = tid_{\mathtt{RMW}}{}'(e)$ for every $\langle e, e'\rangle \in G'.\mathtt{rf}; [\mathsf{RMW}]$. In addition, we have $\mathtt{tid}(e) = \tau = tid_{\mathtt{RMW}}(w) = tid_{\mathtt{RMW}}{}'(w)$. Since w is the unique event such that $\langle w, e\rangle \in G'.\mathtt{rf}$, it follows that for every $\langle w', e'\rangle \in G'.\mathtt{rf}; [\mathsf{RMW}]$, we have $\mathtt{tid}(e') = tid_{\mathtt{RMW}}{}'(w')$.

We show that for every $\pi \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\pi)$, there exists a $\langle G', \pi\rangle$-consistent $\langle G', L', tid_{\mathtt{RMW}}{}'\rangle$-write-list. Let $\pi \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\pi)$. We construct a $\langle G', \pi\rangle$-consistent $\langle G', L', tid_{\mathtt{RMW}}{}'\rangle$-write-list W'. Let $P \triangleq \mathcal{P}(\pi, L')$ and (the last two are only defined if $P \neq \emptyset$):

$$L \triangleq \begin{cases} L'_x \setminus P & \pi \neq \tau \\ o \cdot \mathtt{O}_{\mathtt{W}}(x) \cdot (L'_x \setminus P) & \pi = \tau \end{cases} \quad f \triangleq \begin{cases} \mathsf{Map}_{\langle L', P\rangle} & \pi \neq \tau \\ \lambda k \in \{1, \dots, |L'|\} \setminus P.\, \mathsf{Map}_{\langle L', P\rangle}(k) + 2 & \pi = \tau \end{cases}$$

$$L_\tau \triangleq o \cdot \mathtt{O}_{\mathtt{W}}(x) \cdot L'_x \setminus\setminus P \qquad f_\tau \triangleq \lambda k \in \{\min P, \dots, |L'|\} \setminus P.\, \mathsf{MMap}_{\langle L', P\rangle}(k) + 2$$

Then, by definition, we have $L \in \mathcal{B}(\pi)$ and $L_\tau \in \mathcal{B}(\tau)$. Let W be a $\langle G, \pi\rangle$-consistent $\langle G, L, tid_{\mathtt{RMW}}\rangle$-write-list, and W_τ be a $\langle G, \tau\rangle$-consistent $\langle G, L_\tau, tid_{\mathtt{RMW}}\rangle$-write-list. Note that for every $k > \min P$ with $k \notin P$ and $\mathtt{typ}(L'(k)) = \mathtt{R}$, we have $\mathtt{tid}(W(f(k))) = \mathtt{tid}(L(f(k))) = \mathtt{tid}(L_\tau(f_\tau(k))) = \mathtt{tid}(W_\tau(f_\tau(k)))$, and so $G.\mathtt{hb}$ must order the two write events, $W(f(k))$ and $W_\tau(f_\tau(k))$. We define W' as follows:

$$W' \triangleq \lambda k \in \{1, \dots, |L'|\}. \begin{cases} L'(k) & \mathtt{typ}(L'(k)) = \mathtt{W} \\ e & \mathtt{typ}(L'(k)) = \mathtt{R} \wedge k \in P \\ W(f(k)) & \mathtt{typ}(L'(k)) = \mathtt{R} \wedge k < \min P \\ \max_{G.\mathtt{hb}}\{W(f(k)), W_\tau(f_\tau(k))\} & \text{otherwise} \end{cases}$$

It is easy to see that W' is a $\langle G', L', tid_{\mathtt{RMW}}{}'\rangle$-write-list. We show that W' is $\langle G', \pi\rangle$-consistent. Let $1 \leq k \leq |L'|$ such that $W'(k) \in \mathsf{E}$. Let $y = \mathtt{loc}(W'(k))$, $w_\pi = W(f(k))$ and $w_\tau = W_\tau(f_\tau(k))$ (the latter is only defined if $k > \min P$). We prove that each of the conditions in Definition 7 holds:

C1) $W'(k) \notin dom(G'.\mathtt{hb}|_{\mathtt{loc}} \mathbin{;} [\mathsf{W}] \mathbin{;} G'.\mathtt{hb}^? \mathbin{;} [\mathsf{E}^\pi \cup \{W'(j) \mid 1 \leq j < k\}])$. Suppose otherwise. First, note that we cannot have $k \in P$, since e is a maximal element in $G'.\mathtt{hb}$. Consider the two possible cases:

(a) $W'(k) \in dom(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$: The definition of W' ensures that we have $\langle w_\pi, W'(k)\rangle \in G'.\mathtt{hb}|_{\mathtt{loc}}{}^?$, and so $w_\pi \in dom(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. Since W is $\langle G, \pi\rangle$-consistent, we have that $w_\pi \notin dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$, and therefore it must be the case that $\langle w_\pi, e\rangle \in G'.\mathtt{hb}|_{\mathtt{loc}};[\mathsf{W}];(G.\mathtt{hb}^?;G'.\mathtt{rf})^?$ and $\pi = \tau$. Now, if $w_\pi \in dom(G.\mathtt{hb}^?;[\mathsf{E}^\tau])$, then since $\pi = \tau$, we have $W(2) = L(2) = \mathtt{O}_\mathtt{W}(x) = \mathtt{O}_\mathtt{W}(\mathtt{loc}(w_\pi))$, and we obtain a contradiction to the fact that W is $\langle G, \pi\rangle$-consistent. Otherwise, we have $\langle w_\pi, w\rangle \in G.\mathtt{hb}|_{\mathtt{loc}};[\mathsf{W}];G.\mathtt{hb}^?$. Since $\pi = \tau$, we have $L(1) = o$, and the definition of w ensures that $\langle w, W(1)\rangle \in G.\mathrm{po}^?$. It follows that $\langle w_\pi, W(1)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}};[\mathsf{W}];G.\mathtt{hb}^?$, which again contradicts the fact that W is $\langle G, \pi\rangle$-consistent.

(b) $\langle W'(k), W'(j)\rangle \in G'.\mathtt{hb}|_{\mathtt{loc}};[\mathsf{W}];G'.\mathtt{hb}^?$ for some $1 \le j < k$. Consider the two possible cases:
- $W'(j) = e$: In this case we must have $k \ge \min P$, and so $W'(k) = \max_{G.\mathtt{hb}}\{w_\pi, w_\tau\}$. Hence, we have $\langle w_\tau, W'(k)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$. There are two possibilities:
  - $W'(k) \in dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$: In this case, since $\langle w_\tau, W'(k)\rangle \in G.\mathtt{hb}^?$, we get $w_\tau \in dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.
  - $W'(k) \notin dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$: In this case we have $W'(k) \in dom(\ G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [w])$. Since $\langle w_\tau, W'(k)\rangle \in G.\mathtt{hb}^?$ and $\langle w, W_\tau(1)\rangle \in G.\mathrm{po}^?$, it follows that $w_\tau \in dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [W_\tau(1)])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.
- $W'(j) \ne e$: In this case, we must have $\langle W'(k), W'(j)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^?$. The definition of W' ensures that $\langle w_\pi, W'(k)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$, and so $\langle w_\pi, W'(j)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^?$. Now, since W is $\langle G, \pi\rangle$-consistent, we cannot have $W'(j) = W(f(j))$. Let $w'_\tau = W_\tau(f_\tau(j))$. Hence, $j \ge \min P$ and $W'(j) = w'_\tau$. It follows that $k \ge \min P$, and so $\langle w_\tau, W'(k)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$. Hence, we have $\langle w_\tau, w'_\tau\rangle \in G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^?$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

C2) If $W'(i) = \mathtt{O}_\mathtt{W}(\mathtt{loc}(W'(k)))$ for $i < k$, then $W'(k) \notin dom(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi \cup \{W'(j) \mid j < i\}])$. Suppose otherwise. Consider two possible cases:

(a) There exists $i < k$ with $W'(i) = \mathtt{O}_\mathtt{W}(y)$ but $W'(k) \in dom(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. Note that the definition of W' ensures that $W'(i) = L'(i) = \mathtt{O}_\mathtt{W}(y)$, and since W is a $\langle G, L, tid_{\mathtt{RMW}}\rangle$-write-list, it follows that $W(f(i)) = \mathtt{O}_\mathtt{W}(y)$. We claim that $W'(k) \ne e$. Assuming otherwise we will have $y = x$, $\pi = \tau$, and $i < \max \mathcal{P}(\tau, L')$, which contradicts the fact that $\mathcal{P}$ justifies a $\langle \tau, \mathtt{W}(x, v_\mathtt{W})\rangle$-step (since $L'(i) = \mathtt{O}_\mathtt{W}(x)$). Consider the two possible cases:
- $W'(k) \in dom(G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$: In this case, since $\langle w_\pi, W'(k)\rangle \in G.\mathtt{hb}^?$, we get $w_\pi \in dom(G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
- $W'(k) \notin dom(G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$: In this case we have $\tau = \pi$ and $W'(k) \in dom(G.\mathtt{hb}^?;[w])$. Since $\langle w_\pi, W'(k)\rangle \in G.\mathtt{hb}^?$ and $\langle w, W(1)\rangle$

$\in G.\mathrm{po}^?$, it follows that $w_\pi \in \mathit{dom}(G.\mathtt{hb}^? ; [W(1)])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.

(b) There exists $j < i < k$ with $W'(i) = \mathtt{O}_\mathtt{W}(y)$ and $\langle W'(k), W'(j)\rangle \in G'.\mathtt{hb}^?$. Note that the definition of W' ensures that $W'(i) = L'(i) = \mathtt{O}_\mathtt{W}(y)$, and since W is a $\langle G, L, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, it follows that $W(f(i)) = \mathtt{O}_\mathtt{W}(y)$. In addition, since W_τ is $\langle G, L_\tau, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, it follows that $W_\tau(f_\tau(i)) = \mathtt{O}_\mathtt{W}(y)$ if $i > \min P$. Let $m = \min P$, $w_\pi^j = W(f(j))$, and $w_\tau^j = W_\tau(f_\tau(j))$. Consider the possible cases:

- $\langle W'(k), W'(j)\rangle \in G.\mathtt{hb}^?$: If $W'(j) = w_\pi^j$ we have $\langle w_\pi, w_\pi^j\rangle \in G.\mathtt{hb}^?$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent. If $W'(j) = w_\tau^j$ we have $j > m$ and $\langle w_\tau, w_\tau^j\rangle \in G.\mathtt{hb}^?$. This contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.
- $\langle W'(k), W'(j)\rangle \notin G.\mathtt{hb}^?$: In this case we have $W'(j) = e$ and $W'(k) \in \mathit{dom}(G.\mathtt{hb}^? ; [w])$. Since $\langle w_\tau, W'(k)\rangle \in G.\mathtt{hb}^?$ and $\langle w, W_\tau(1)\rangle \in G.\mathrm{po}^?$, it follows that $w_\tau \in \mathit{dom}(G.\mathtt{hb}^? ; [W_\tau(1)])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

C3) $W'(k) \notin \mathit{dom}((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf}) ; [\mathsf{R}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\pi \cup \{W'(j) \mid 1 \le j < k\}])$. Suppose otherwise. Consider two possible cases:

(a) $W'(k) \in \mathit{dom}((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf}) ; [\mathsf{R}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. In this case we have $w' \in G'.\mathsf{W}_y$, $r \in G'.\mathsf{R}$ such that $\langle w', r\rangle \in G'.\mathtt{rf}$, $W'(k) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [r] ; G'.\mathtt{hb}^? ;[\mathsf{E}^\pi])$, and $w' \ne W'(k)$. Note that since e is a maximal element in $G'.\mathtt{hb}$, we have $r \in G.\mathsf{R}$, $w' \ne e$, and $W'(k) \ne e$. Consider the two possible cases:

- $W'(k) \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [r] ; G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$: In this case we have $w_\pi \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}};[r];G.\mathtt{hb}^?;[\mathsf{E}^\pi])$. We claim that $w' \ne w_\pi$. Assuming otherwise, we will have $w_\pi \ne W'(k)$ (since $w' \ne W'(k)$). Therefore we have $w_\pi \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [W'(k)] ; G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$ where $W'(k) \in G.\mathsf{W}$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent. Since $w' \ne w_\pi$, we have $w_\pi \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [r] ; G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
- $W'(k) \notin \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [r] ; G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$: In this case we have $W'(k) \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [r])$, $r \in \mathit{dom}(G.\mathtt{hb} ; [w])$, and $e \in \mathsf{E}^\pi$ (i.e., $\tau = \pi$). Since $\langle w_\tau, W'(k)\rangle \in G.\mathtt{hb}^?$ and $\langle w, W_\tau(1)\rangle \in G.\mathrm{po}^?$, it follows that $w_\tau \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [r] ; G.\mathtt{hb}^? ; [W_\tau(1)])$. If $w_\tau = w'$ we have $w_\tau \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [W'(k)] ; G.\mathtt{hb}^? ; [W_\tau(1)])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent. On the other hand if $w_\tau \ne w'$ we have $w_\tau \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [r] ; G.\mathtt{hb}^? ; [W_\tau(1)])$ which again contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

(b) $W'(k) \in \mathit{dom}((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf});[\mathsf{R}];G'.\mathtt{hb}^?;[\{W'(j) \in \mathsf{E} \mid 1 \le j < k\}])$. In this case we have $w' \in G'.\mathsf{W}_y$, $r \in G'.\mathsf{R}$, $W'(j) \in G'.\mathsf{W}$ for some $j < k$ such that $\langle w', r\rangle \in G'.\mathtt{rf}$, $W'(k) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [r] ; G'.\mathtt{hb}^? ; [W'(j)])$ and $w' \ne W'(k)$. Note that, since e is a maximal element in $G'.\mathtt{hb}$, we have $r \in G.\mathsf{R}$, $w' \ne e$, and $W'(k) \ne e$. We claim that $w_\pi \ne w' \ne w_\tau$. Assume otherwise and we have $\langle w', r\rangle \in G.\mathtt{rf}$ where

$w' \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [W'(k)] ; G.\mathtt{hb} ; [r])$ which contradicts the fact that G is LRA-consistent. For $j > \min P$, let $w_\pi^j = W(f(j))$ and $w_\tau^j = W_\tau(f_\tau(j))$. Consider the two possible cases:

- $j \in P$: In this case we have $e = W'(j)$ and $w_\tau \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [r] ; G.\mathtt{hb}^? ; [w])$. Since $\langle w, W_\tau(1)\rangle \in G.\mathtt{po}$ we have $w_\tau \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [r] ; G.\mathtt{hb}^? ; [W_\tau(1)])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.
- $j \notin P$: If $j < \min P$ we have $W'(j) = w_\pi^j$ and $w_\pi \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [r] ; G.\mathtt{hb}^? ;[w_\pi^j])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent. If $j > \min P$, our construction ensures that one of the following holds:
  - $W'(j) = w_\pi^j$: In this case we have $w_\pi \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [r] ; G.\mathtt{hb}^? ; [w_\pi^j])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
  - $W'(j) = w_\tau^j$: In this case we have $w_\tau \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [r] ; G.\mathtt{hb}^? ; [w_\tau^j])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

C4) If $\mathtt{loc}(W'(j)) = \mathtt{loc}(W'(k))$ and $W'(k) \neq W'(j)$ for some $j < k$, then $W'(k) \notin \mathit{dom}(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi \cup \{W'(i) \mid i < j\}])$. Suppose otherwise. Let $m = \min P$, $n = \max P$, $w_\pi^j = W(f(j))$, and $w_\tau^j = W_\tau(f_\tau(j))$. Consider the following two possible cases:

(a) For some $j < k$, $\mathtt{loc}(W'(j)) = \mathtt{loc}(W'(k))$, $W'(k) \in \mathit{dom}(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$, and $W'(k) \neq W'(j)$. Note that $k \notin P$ (and hence $e \neq W'(k)$). Assuming otherwise, we must have $y = x$, $\pi = \tau$ and $j < \max \mathcal{P}(\tau, L')$ which contradicts the fact that $\mathcal{P}$ justifies a $\langle \tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle$-step (since $\mathtt{loc}(L'(j)) = x$). Consider the two possible cases:

- $W'(j) = e$: If $\pi \neq \tau$, since $\mathcal{P}$ justifies a $\langle \tau, l\rangle$-step, we have $W(n) = L(n) = L'(n) = \mathtt{O}_{\mathtt{W}}(y)$ and $w_\pi \in \mathit{dom}(G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. This contradicts the fact that W is $\langle G, \pi\rangle$-consistent. If $\pi = \tau$ we have $W(1) = L(1) = \mathtt{O}_{\mathtt{W}}(y)$ and $w_\pi \in \mathit{dom}(G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$. This contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
- $W'(j) \neq e$: In this case we have $j \neq m \neq k$. Consider the following three possible cases:
  - $j < k < m$: In this case we have $w_\pi = W'(k)$, $w_\pi^j = W'(j)$ and hence $w_\pi \in \mathit{dom}(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
  - $j < m < k$: In this case $w_\pi^j = W'(j)$. If $w_\pi^j \neq w_\pi$ we have $w_\pi \in \mathit{dom}(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$ where $\mathtt{loc}(w_\pi^j) = \mathtt{loc}(w_\pi)$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent. If $w_\pi^j = w_\pi$ we have $w_\pi \neq W'(k)$ and hence $w_\pi \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [W'(k)] ; G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
  - $m < j < k$: If $w_\pi^j \neq w_\pi$ we have $w_\pi \in \mathit{dom}(G'.\mathtt{hb}^? ; [\mathsf{E}^\pi])$ where $\mathtt{loc}(w_\pi^j) = \mathtt{loc}(w_\pi)$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent. If $w_\pi^j = w_\pi$ we consider the following two cases:

- $W'(k) \neq w_\pi$: We have $w_\pi \in dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [W'(k)] ; G.\mathtt{hb}^? ; [\mathsf{E}^\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
- $W'(k) = w_\pi$: In this case we have $W'(j) = w^j_\tau \neq w^j_\pi$. The definition of W' ensures that $\langle w_\tau, W'(k)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$ and hence $\langle w_\tau, w^j_\pi\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$ (since $W'(k) = w_\pi = w^j_\pi$). We also have $\langle w^j_\pi, W'(j)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$ and hence $\langle w^j_\pi, w^j_\tau\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}$ (since $w^j_\pi = W'(k) \neq W'(j) = w^j_\tau$). Therefore we have $w_\tau \in dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [w^j_\tau])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

(b) For some $1 \leq i < j < k$, $\mathtt{loc}(W'(j)) = \mathtt{loc}(W'(k))$, $W'(k) \in dom(G'.\mathtt{hb}^? ; [W'(i)])$, and $W'(k) \neq W'(j)$. We claim that $k \notin P$. Assuming otherwise we will have $W'(k) = W'(i) = e$ and $\mathtt{loc}(L'(j)) = \mathtt{loc}(e)$ where $i < j < k$ and $i, k \in P$ which contradicts the fact that $\mathcal{P}$ justifies a $\langle \tau, l\rangle$-step. We further consider the following sub-cases:

- $i \in P$: In this case we have $W'(i) = e \neq W'(k)$ and hence $\langle W'(k), W'(i)\rangle \in G'.\mathtt{hb}$. Consider the following two cases:
  - ∗ $W'(j) = W'(i)$: In this case we have $W'(j) = e = W'(i)$, $\mathtt{loc}(W'(k)) = \mathtt{loc}(e)$ and hence $W'(k) \in dom(\mathtt{hb}|_{\mathtt{loc}} ; [e] ; \mathtt{hb}^? ; [W'(j)])$ where $e \in \mathsf{W}$. Therefore the present case reduces to (A) and we can observe a contradiction using the same reasoning.
  - ∗ $W'(j) \neq W'(i)$: We claim that $W'(k) = w_\tau$. Assuming otherwise, we will have $w_\tau \in dom(\mathtt{hb}|_{\mathtt{loc}} ; [W'(k)] ; \mathtt{hb}^? ; [\mathsf{E}^\tau])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent. We consider the following possible cases:
    - $w^j_\tau \neq w_\tau$: In this case $\mathtt{loc}(w^j_\tau) = \mathtt{loc}(w_\tau)$ and $w_\tau \in dom(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.
    - $w^j_\tau = w_\tau$: In this case $W'(k) = w_\tau = w^j_\tau$, $\langle w_\pi, W'(k)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$, and $\langle w^j_\tau, W'(j)\rangle \in G.\mathtt{hb}|_{\mathtt{loc}}{}^?$. Since $w^j_\tau = W'(k) \neq W'(j) = w^j_\pi$, we have $w_\pi \in dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [w^j_\pi])$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
- $i \notin P$: We consider the following two cases:
  - ∗ $j \in P$: We claim that $\max P < k$. Assuming otherwise (i.e., $j < k < \max P$), we will have $\mathtt{loc}(L'(k)) = x$, which contradicts the fact that $\mathcal{P}$ justifies a $\langle \tau, l\rangle$-step. Consider the following possible cases:
    - $i < \min P$: Since $\mathcal{P}$ justifies a $\langle \tau, l\rangle$-step, we have $W(n) = L(n) = L'(n) = \mathtt{O_W}(y)$ where $f(i) < n$ and $w_\pi \in dom(G.\mathtt{hb}^? ; [W(i)])$. This contradicts the fact that W is $\langle G, \pi\rangle$-consistent.
    - $i > \min P$: Let $n_\pi = f(\max P)$ and $n_\tau = f_\tau(\max P)$. If $W'(i) = w^i_\pi$ then we have $w_\pi \in dom(G.\mathtt{hb}^? ; [w^i_\pi])$ where $W(n_\pi) = \mathtt{O_W}(\mathtt{loc}(w_\pi))$ (since $\mathcal{P}$ justifies a $\langle \tau, l\rangle$-step). This contradicts the fact that W is $\langle G, \pi\rangle$-consistent. If $W'(i) = w^i_\tau$ then we have $w_\tau \in dom(G.\mathtt{hb}^? ; [w^i_\tau])$ where $W_\tau(n_\tau) = \mathtt{O_W}(\mathtt{loc}(w_\tau))$. This contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

  - $*$ $j \notin P$: In this case we have $m \notin \{i, j, k\}$ and $i < j < k$. We consider the following possible cases:
    - · $\texttt{tid}(W'(j)) = \texttt{tid}(W'(k))$: In this case, since $W'(j) \neq W'(k)$, we have either (a) $\langle W'(j), W'(k)\rangle \in G'.\texttt{hb}$ or (b) $\langle W'(k), W'(j)\rangle \in G'.\texttt{hb}$. Assuming (a) we have $W'(j) \in \mathit{dom}(G'.\texttt{hb}|_{\texttt{loc}} ; [W'(k)] ; G'.\texttt{hb}^? ; [W'(i)])$ where $i < j$ and hence the present case reduces to (A) resulting in a contradiction. Similarly, assuming (b) we have $W'(k) \in \mathit{dom}(G'.\texttt{hb}|_{\texttt{loc}} ; [W'(j)])$ where $j < k$ and hence the present case reduces to (A) again resulting in a contradiction.
    - · $\texttt{tid}(W'(j)) \neq \texttt{tid}(W'(k))$: In this case it is easy to see that $w^j_\pi \neq w_\pi$ and $w^j_\tau \neq w_\tau$. If $W'(i) = w^i_\pi$ then we have $w_\pi \in \mathit{dom}(G.\texttt{hb}^? ; [w^i_\pi])$ and $w^j_\pi \neq w_\pi$ which contradicts the fact that W is $\langle G, \pi\rangle$-consistent. On the other hand, if $W'(i) = w^i_\tau$ then we have $w_\tau \in \mathit{dom}(G.\texttt{hb}^? ; [w^i_\tau])$ and $w^j_\tau \neq w_\tau$ which contradicts the fact that W_τ is $\langle G, \tau\rangle$-consistent.

Finally, suppose that $\mathcal{B} \curlyvee G$ and $\mathcal{B} \xrightarrow{\varepsilon}_{\text{loLRA}} \mathcal{B}'$ (using the LOWER step). Let $\mathit{tid}_{\texttt{RMW}} : \mathsf{W} \to \mathsf{Tid}$ that satisfies the conditions of Definition 8. To show that $\mathcal{B}' \curlyvee G$, we use the same $\mathit{tid}_{\texttt{RMW}}$ mapping and show that for every $\tau \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\tau)$, there exists a $\langle G, \tau\rangle$-consistent $\langle G, L', \mathit{tid}_{\texttt{RMW}}\rangle$-write-list.

Let $\tau \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\tau)$. We define a $\langle G, \tau\rangle$-consistent $\langle G, L', \mathit{tid}_{\texttt{RMW}}\rangle$-write-list W'. By definition, since $\mathcal{B} \xrightarrow{\varepsilon}_{\text{loLRA}} \mathcal{B}'$, there exists $L \in \mathcal{B}(\tau)$ such that $L' \sqsubseteq L$. Let W be a $\langle G, \tau\rangle$-consistent $\langle G, L, \mathit{tid}_{\texttt{RMW}}\rangle$-write-list, and let $f : \{1, \dots, |L'|\} \to \mathbb{N}$ be an increasing function such that $L'(k) = L(f(k))$ for every $k \in \mathit{dom}(f)$. It is easy to see that $W' = \lambda k \in \{1, \dots, |L'|\}.\, W(f(k))$ is a $\langle G, L', \mathit{tid}_{\texttt{RMW}}\rangle$-write-list. To see that W' is $\langle G, \tau\rangle$-consistent, note that:

- $W'(k) \in \mathit{dom}(G.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W'(j) \mid 1 \leq j < k\}])$ implies $W(f(k)) \in \mathit{dom}(G.\texttt{hb}|_{\texttt{loc}} ; [\mathsf{W}] ; G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W(f(j)) \mid 1 \leq f(j) < f(k)\}])$.
- $W'(k) \in \mathit{dom}(G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W'(j) \mid 1 \leq j < k\}])$ implies $W(f(k)) \in \mathit{dom}(G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W(f(j)) \mid 1 \leq f(j) < f(k)\}])$.
- $W'(k) \in \mathit{dom}((G.\texttt{hb}|_{\texttt{loc}} \setminus G.\texttt{rf}) ; [\mathsf{R}] ; G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W'(j) \mid 1 \leq j < k\}])$ implies $W(f(k)) \in \mathit{dom}((G.\texttt{hb}|_{\texttt{loc}} \setminus G.\texttt{rf}) ; [\mathsf{R}] ; G.\texttt{hb}^? ; [\mathsf{E}^\tau \cup \{W(f(j)) \mid 1 \leq f(j) < f(k)\}])$.

Therefore, the $\langle G, \tau\rangle$-consistency of W' directly follows from the $\langle G, \tau\rangle$-consistency of W.

Lemma 3. *For every trace of op*LRA *there is an equivalent trace of lo*LRA*.*

Proof. We show that $\curlyvee^{-1}$ constitutes a backward simulation from opLRA to loLRA. The two first requirements of a backward simulation clearly hold for $\curlyvee$: 1. $\curlyvee^{-1}$ is total, as for every state G of opLRA, we have $(\lambda\tau \in \mathsf{Tid}.\ \{\epsilon\}) \curlyvee G$. 2. Consider a state $\mathcal{B}$ of loLRA, such that $\mathcal{B} \curlyvee G_0$. By the definition of $\curlyvee$, it should be possible to link every read option of $\mathcal{B}$ to some write event of G_0. Since there are no write events in G_0, there cannot be read options in $\mathcal{B}$, implying that $\mathcal{B} \in \text{loLRA}.\mathtt{Q}_0$.

We move to the third requirement. Suppose that $G \xrightarrow{\tau,l}_{\text{opLRA}} G'$ and $\mathcal{B}' \curlyvee G'$, witnessed by a function $tid_{\mathtt{RMW}} : \mathsf{W} \to \mathsf{Tid}$. We construct a state $\mathcal{B}$ such that $\mathcal{B} \xrightarrow{\tau,l}_{\text{loLRA}} \mathcal{B}'$ and $\mathcal{B} \curlyvee G$. Consider the possible cases:

– WRITE-STEP $l = \mathtt{W}(x, v_{\mathtt{W}})$: Let $w = \mathsf{NextEvent}(G.\mathtt{E}, \tau, l)$. Since $G \xrightarrow{\tau,l}_{\text{opLRA}} G'$, we have $G'.\mathtt{E} = G.\mathtt{E} \cup \{w\}$ and $G'.\mathtt{rf} = G.\mathtt{rf}$. Let $\mathcal{P}$ be the index choice for $\mathcal{B}'$ that assigns the set of "new" positions in $\mathcal{B}'$:

$$\mathcal{P} \triangleq \lambda\pi \in \mathsf{Tid}, L' \in \mathcal{B}'(\pi).\ \{1 \leq k \leq |L'| \mid W'_{\langle\pi,L'\rangle}(k) = w\}.$$

Then, we define

$$\mathcal{B} \triangleq \lambda\pi \in \mathsf{Tid}.\ \begin{cases} \mathtt{O}_{\mathtt{W}}(x) \cdot \mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\tau) & \pi = \tau \\ \mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\pi) & \pi \neq \tau \end{cases}$$

By Proposition 3, to show that $\mathcal{B} \xrightarrow{\tau,l}_{\text{loLRA}} \mathcal{B}'$, it suffices to prove that $\mathcal{P}$ justifies a $\langle\tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle$-step. Let $\pi_{\mathtt{RMW}} = tid_{\mathtt{RMW}}(w)$. Thus, we show that the following hold for every $\pi \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\pi)$, where $P = \mathcal{P}(\pi, L')$ and $W' = W'_{\langle\pi,L'\rangle}$:

- Let $k \in P$. Then, we have $W'(k) = w$, and thus $L'(k) = \langle\tau, x, v_{\mathtt{W}}, \pi_{\mathtt{RMW}}\rangle$.
- Let $k \in \{1, \ldots, |L'|\} \setminus P$ such that $p_1 < k < p_2$ for some $p_1, p_2 \in P$. We show that $L'(k) \neq \mathtt{O}_{\mathtt{W}}(x)$. Since $p_1, p_2 \in P$, we have $W'(p_1) = W'(p_2) = w$, and so $\langle W'(p_1), W'(p_2)\rangle \in G'.\mathtt{hb}^?$. Since W' is $\langle G', \pi\rangle$-consistent, we *cannot* have $W'(k) = \mathtt{O}_{\mathtt{W}}(\mathtt{loc}(W'(p_2)))$, and so $L'(k) \neq \mathtt{O}_{\mathtt{W}}(x)$.
- Suppose that $\pi = \tau$ and let $k \in \{1, \ldots, |L'|\} \setminus P$ such that $k < p$ for some $p \in P$. We show that $L'(k) \neq \mathtt{O}_{\mathtt{W}}(x)$. Since $p \in P$, we have $W'(p) = w$, and so $W'(p) \in dom(G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. Since W' is $\langle G', \tau\rangle$-consistent, we *cannot* have $W'(k) = \mathtt{O}_{\mathtt{W}}(\mathtt{loc}(W'(p)))$, and so $L'(k) \neq \mathtt{O}_{\mathtt{W}}(x)$.

Next, we prove that $\mathcal{B} \curlyvee G$, by showing that for every $\pi \in \mathsf{Tid}$ and $L \in \mathcal{B}(\pi)$, there exists a $\langle G, \pi\rangle$-consistent $\langle G, L, tid_{\mathtt{RMW}}\rangle$-write-list. Since $G.\mathtt{rf} \subseteq G'.\mathtt{rf}$, the second condition of $\curlyvee$ (Definition 8), namely that for every $\langle w, e\rangle \in G.\mathtt{rf}; [\mathsf{RMW}]$, we have $\mathtt{tid}(e) = tid_{\mathtt{RMW}}(w)$, trivially holds. Let $\pi \in \mathsf{Tid}$ and $L \in \mathcal{B}(\pi)$. Following the construction of $\mathcal{B}$, one of the following holds:

- $\pi \neq \tau$ and $L = L'_x \backslash \mathcal{P}(\pi, L')$ for some $L' \in \mathcal{B}'(\pi)$. Let $P = \mathcal{P}(\pi, L')$, $W' = W'_{\langle\pi,L'\rangle}$ and $f = \mathsf{Map}^{-1}_{\langle L',P\rangle}$. We define $W \triangleq \lambda k \in \{1, \ldots, |L|\}.\ W'(f(k))$. Using the fact that W' is a $\langle G', L', tid_{\mathtt{RMW}}\rangle$-write-list, it is easy to see that W is a $\langle G, L, tid_{\mathtt{RMW}}\rangle$-write-list.

It remains to show that W is $\langle G, \pi\rangle$-consistent, namely the conditions of Definition 7 hold. Indeed, the construction of W and the fact that $G.\mathtt{hb} \subseteq G'.\mathtt{hb}$ directly ensure that these conditions follows from the $\langle G', \pi\rangle$-consistency of W'.

- $\pi = \tau$ and $L = \mathtt{O_W}(x) \cdot (L'_x \setminus \mathcal{P}(\tau, L'))$ for some $L' \in \mathcal{B}'(\tau)$. Let $P = \mathcal{P}(\tau, L')$, $W' = W'_{\langle \tau, L'\rangle}$ and $f = \lambda k \in \{2, \dots, |L|\}.\ \mathsf{Map}^{-1}_{\langle L', P\rangle}(k-1)$. We define:
$$W \triangleq \lambda k \in \{1, \dots, |L|\}. \begin{cases} \mathtt{O_W}(x) & k = 1 \\ W'(f(k)) & k > 1 \end{cases}$$
Using the fact that W' is a $\langle G', L', \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, it is easy to see that W is a $\langle G, L, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list. It remains to show that W is $\langle G, \tau\rangle$-consistent. The condition C2(b) follow directly from the $\langle G', \tau\rangle$-consistency of W'. Condition C2(a), however, deserves more attention, as we added $\mathtt{O_W}(x)$ at the start of the list. Assume toward contradiction some k, such that $W(k) \in \mathsf{E}$, $\mathtt{loc}(W(k)) = x$ and $W(k) \in \mathit{dom}(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. Then since $W'(f(k)) = W(k)$, $w = \max_{G'.\mathrm{po}} G'.\mathsf{E}^\tau$ and $\mathtt{loc}(W(k)) = \mathtt{loc}(w)$, we have $\langle W'(f(k)), w\rangle \in G'.\mathtt{hb}|_{\mathtt{loc}};[\mathsf{W}] ; G'.\mathtt{hb}^?$, contradicting (C1(a) in) the $\langle G', \tau\rangle$-consistency of W'.
- $\pi = \tau$ and $L = \mathtt{O_W}(x) \cdot (L'_x \setminus\!\!\setminus \mathcal{P}(\eta, L'))$ for some $\eta \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\eta)$. Let $P = \mathcal{P}(\eta, L')$, $m = \min(P)$, $W' = W'_{\langle \eta, L'\rangle}$ and $f = \lambda k \in \{2, \dots, |L|\}.\ \mathsf{MMap}^{-1}_{\langle L', P\rangle}(k-1)$. We define:
$$W \triangleq \lambda k \in \{1, \dots, |L|\}. \begin{cases} \mathtt{O_W}(x) & k = 1 \\ W'(f(k)) & k > 1 \end{cases}$$
By the fact that W' is a $\langle G', L', \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, we get that W is a $\langle G, L, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list. It remains to show that W is $\langle G, \tau\rangle$-consistent. We consider all the $\langle G, \tau\rangle$-consistency conditions for W:
  - C1) The existence of some k, such that $W(k) \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\{W(j) \mid 1 \leq j < k\}])$ directly contradicts the same condition in the $\langle G', \eta\rangle$-consistency of W'. Now, assume toward contradiction some k, such that $W(k) \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. Then, since $W'(f(k)) = W(k)$, $f(k) > m$, $W'(m) = w$ and $w = \max_{G'.\mathrm{po}} G'.\mathsf{E}^\tau$, we have $W'(f(k)) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\{W'(j) \mid 1 \leq j < f(k)\}])$, contradicting (C1(b) in) the $\langle G', \eta\rangle$- consistency of W'.
  - C2) The existence of some $i < k$ such that $W(k) \in \mathit{dom}(G.\mathtt{hb}^?;[\{W(j) \mid 1 \leq j < i\}])$ directly contradicts the same condition in the $\langle G', \eta\rangle$ -consistency of W'. Now, assume toward contradiction the existence of some $i < k$, such that $W(i) = \mathtt{O_W}(\mathtt{loc}(W(k)))$ and $W(k) \in \mathit{dom}(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. First if $i = 1$, then $\mathtt{loc}(W(k)) = x$, and as above, since $W'(f(k)) = W(k)$, $f(k) > m$, $W'(m) = w$ and $w = \max_{G'.\mathrm{po}} G'.\mathsf{E}^\tau$, we have $W'(f(k)) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\{W'(j) \mid 1 \leq j < f(k)\}])$, contradicting (C1(b) in) the $\langle G', \eta\rangle$-consistency of W'. Now, suppose that $i > 1$. Then, again, since $W'(f(k)) = W(k)$,

$f(k) > f(i) > m$, $W'(m) = w$ and $w = \max_{G'.\mathrm{po}} G'.\mathsf{E}^\tau$, we have $\langle W'(f(k)), W'(m)\rangle \in G'.\mathtt{hb}^?$, contradicting (C2(a) in) the $\langle G', \eta\rangle$-consistency of W'.

C3) The existence of some k such that $W(k) \in dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [\{W(j) \mid 1 \leq j < k\}])$ directly contradicts the same condition in the $\langle G', \eta\rangle$-consistency of W'. Now, assume toward contradiction some k, such that $W(k) \in dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. Then, since $W'(f(k)) = W(k)$, $f(k) > m$, $W'(m) = w$ and $w = \max_{G'.\mathrm{po}} G'.\mathsf{E}^\tau$, we have $W'(f(k)) \in dom((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf}) ; [\mathsf{R}] ; G'.\mathtt{hb}^? ; [\{W'(j) \mid 1 \leq j < k\}])$, contradicting (C3(b) in) the $\langle G', \eta\rangle$-consistency of W'.

C4) The existence of some $j < k$ such that $\mathtt{loc}(W(j)) = \mathtt{loc}(W(k))$, $W(k) \neq W(j)$, and $W(k) \in dom(G.\mathtt{hb}^? ; [\{W(i) \mid 1 \leq i < j\}])$ directly contradicts the same condition in the $\langle G', \eta\rangle$-consistency of W'. Now, assume toward contradiction the existence of some $j < k$, such that $\mathtt{loc}(W(j)) = \mathtt{loc}(W(k))$, $W(k) \neq W(j)$, and $W(k) \in dom(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. Then, since $W'(f(k)) = W(k)$, $f(k) > f(j) > m$, and $W'(m) = w = \max_{G'.\mathrm{po}} G'.\mathsf{E}^\tau$, we have $\langle W'(f(k)), W'(m)\rangle \in G'.\mathtt{hb}^?$, contradicting (C4(b) in) the $\langle G', \eta\rangle$-consistency of W'.

– READ-STEP $l = \mathtt{R}(x, v_{\mathtt{R}})$: Let $r = \mathsf{NextEvent}(G.\mathtt{E}, \tau, l)$. Since $G \xrightarrow{\tau,l}_{\mathrm{opLRA}} G'$, we have that $G'.\mathtt{E} = G.\mathtt{E} \cup \{r\}$ and $G'.\mathtt{rf} = G.\mathtt{rf} \cup \{\langle w, r\rangle\}$ for some write event $w \in G.\mathsf{W}_x$ such that $\mathtt{val}_{\mathtt{W}}(w) = v_{\mathtt{R}}$, $w \notin dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, and $w \notin dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$.
Let $o = \langle \mathtt{tid}(w), x, v_{\mathtt{R}}, tid_{\mathtt{RMW}}(w)\rangle$. We define $\mathcal{B}$ by:

$$\mathcal{B} \triangleq \lambda\pi \in \mathsf{Tid}.\ \begin{cases} o \cdot \mathcal{B}'(\tau) & \pi = \tau \\ \mathcal{B}'(\pi) & \pi \neq \tau \end{cases}$$

By definition, we have $\mathcal{B} \xrightarrow{\tau,l}_{\mathrm{loLRA}} \mathcal{B}'$. We show that $\mathcal{B} \curlyvee G$. Note that the second condition of $\curlyvee$ (Definition 8) trivially holds. It remains to be shown that for every $\pi \in \mathsf{Tid}$ and $L \in \mathcal{B}(\pi)$, there exists a $\langle G, \pi\rangle$-consistent $\langle G, L, tid_{\mathtt{RMW}}\rangle$-write-list.
For $\pi \neq \tau$ and $L \in \mathcal{B}(\pi)$, observe that $L \in \mathcal{B}'(\pi)$, and since $G.\mathtt{hb} \subseteq G'.\mathtt{hb}$, we have that $W'_{\langle \pi, L'\rangle}$ is also a $\langle G, \pi\rangle$-consistent $\langle G, L, tid_{\mathtt{RMW}}\rangle$-write-list.
For $\pi = \tau$ consider an option list $L \in \mathcal{B}(\tau)$. Let $L' \in \mathcal{B}'(\tau)$ such that $L = o \cdot L'$. Let $W' = W'_{\langle \tau, L'\rangle}$. We define $W \triangleq w \cdot W'$. By the fact that W' is a $\langle G', L', tid_{\mathtt{RMW}}\rangle$-write-list, we get that W is a $\langle G, L, tid_{\mathtt{RMW}}\rangle$-write-list. It remains to show that W is $\langle G, \tau\rangle$-consistent. We prove below the $\langle G, \tau\rangle$-consistency conditions for W:

C1) Given that W' is $\langle G', \tau\rangle$-consistent we only need to show that $w \notin dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, which is guaranteed by the properties of w as stated above (it follows from the preconditions of the READ step in opLRA).

C2) The condition C2(a) for W directly follows from $\langle G', \tau\rangle$-consistency of W'. For C2(b), given the $\langle G', \tau\rangle$-consistency of W', it suffices to handle the case that $j = 1$. Thus, assume toward contradiction some $1 < k \leq |L|$ and $1 < i < k$, such that $W(i) = \mathtt{O_W}(\mathtt{loc}(W(k)))$ and $\langle W(k), w\rangle \in G.\mathtt{hb}^?$. Then, since $r \in G'.\mathsf{E}^\tau$ and $\langle w, r\rangle \in G'.\mathtt{rf}$, we get that $W'(k-1) \in dom(G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, while $W'(i-1) = \mathtt{O_W}(\mathtt{loc}(W'(k-1)))$, contradicting (C2(a) in) the $\langle G', \tau\rangle$-consistency of W'.

C3) Given that W' is $\langle G', \tau\rangle$-consistent we only need to show that $w \notin dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, which is guaranteed by the properties of w as stated above (it follows from the preconditions of the READ step in opLRA).

C4) For condition C4(a), given the $\langle G', \tau\rangle$-consistency of W', it suffices to handle the case that $j = 1$. Thus, assume toward contradiction some $1 < k \leq |L|$ where $w = W(1) \neq W(k)$, $\mathtt{loc}(w) = \mathtt{loc}(W(k))$, and $W(k) \in dom(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. Then, since $r \in G'.\mathsf{E}^\tau$ and $\langle w, r\rangle \in G'.\mathtt{rf}$, we get that $W'(k-1) \in dom((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf}) ; [r] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, contradicting (C3(a) in) the $\langle G', \tau\rangle$-consistency of W'. Similarly for condition C4(b), given the $\langle G', \tau\rangle$-consistency of W', it suffices to handle the case that $i = 1$. Assume toward contradiction some $1 < j < k \leq |L|$ where $w = W(1)$, $W(j) \neq W(k)$, $\mathtt{loc}(W(j)) = \mathtt{loc}(W(k))$, and $W(k) \in dom(G.\mathtt{hb}^? ; [w])$. Then, since $r \in G'.\mathsf{E}^\tau$ and $\langle w, r\rangle \in G'.\mathtt{rf}$, we get $W'(k-1) \in dom(G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$ where $W'(j-1) \neq W'(k-1)$ and $\mathtt{loc}(W'(j-1)) = \mathtt{loc}(W'(k-1))$, contradicting (C4(a) in) the $\langle G', \tau\rangle$-consistency of W'.

– RMW-STEP $l = \mathtt{RMW}(x, v_{\mathtt{R}}, v_{\mathtt{W}})$: Let $e = \mathsf{NextEvent}(G.\mathtt{E}, \tau, l)$. Since $G \xrightarrow{\tau,l}_{\text{opLRA}} G'$, we have $G'.\mathtt{E} = G.\mathtt{E} \cup \{e\}$, $G'.\mathtt{rf} = G.\mathtt{rf} \cup \{\langle w, e\rangle\}$ and $\mathtt{val_W}(w) = v_{\mathtt{R}}$, for some $w \in \mathsf{W}_x$, such that $w \notin dom(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, $w \notin dom((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, and $w \notin dom(G.\mathtt{rf}; [\mathsf{RMW}])$.
Let $\mathcal{P}$ be the index choice for $\mathcal{B}'$ that assigns the set of "new" positions in $\mathcal{B}'$:

$$\mathcal{P} \triangleq \lambda\pi \in \mathsf{Tid}, L' \in \mathcal{B}'(\pi).\ \{1 \leq k \leq |L'| \mid W'_{\langle\pi, L'\rangle}(k) = e\}.$$

Then, we define:

$$\mathcal{B} \triangleq \lambda\pi \in \mathsf{Tid}.\ \begin{cases} o \cdot \mathtt{O_W}(x) \cdot \mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\tau) & \pi = \tau \\ \mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\pi) & \pi \neq \tau \end{cases}$$

where o is the read option given by $o \triangleq \langle \mathtt{tid}(w), x, v_{\mathtt{R}}, \tau\rangle$.
Using Proposition 3, to show that $\mathcal{B} \xrightarrow{\tau,l}_{\text{loLRA}} \mathcal{B}'$, it suffices to prove that $\mathcal{P}$ justifies a $\langle \tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle$-step. This is done as in the write case, together with the following observation: Since $e \in G'.\mathtt{E}^\tau$, $e \in \mathsf{RMW}$ and $\langle w, e\rangle \in G'.\mathtt{rf}$, the fact that $tid_{\mathtt{RMW}}$ witnesses $\mathcal{B}' \curlyvee G'$, guarantees that $tid_{\mathtt{RMW}}(w) = \tau$.
It remains to show that $\mathcal{B} \curlyvee G$. We show that for every $\pi \in \mathsf{Tid}$ and $L \in \mathcal{B}(\pi)$, there exists a $\langle G, \pi\rangle$-consistent $\langle G, L, tid_{\mathtt{RMW}}\rangle$-write-list. (The second condition of $\curlyvee$ (Definition 8) trivially holds.) Let $\pi \in \mathsf{Tid}$ and $L \in \mathcal{B}(\pi)$. Following the construction of $\mathcal{B}$, one of the following holds:

- $\pi \neq \tau$ and $L = L'_x \setminus \mathcal{P}(\pi, L')$ for some $L' \in \mathcal{B}'(\pi)$. This case is exactly the same as the analogous case in the WRITE step.

- $\pi = \tau$ and $L = o \cdot \mathtt{O_W}(x) \cdot (L'_x \setminus \mathcal{P}(\tau, L'))$ for some $L' \in \mathcal{B}'(\tau)$. Let $P = \mathcal{P}(\tau, L')$, $W' = W'_{\langle \tau, L' \rangle}$ and $f = \lambda k \in \{3, \dots, |L|\}.\ \mathsf{Map}^{-1}_{\langle L', P \rangle}(k-2)$. We define:

$$W \triangleq \lambda k \in \{1, \dots, |L|\}. \begin{cases} w & k = 1 \\ \mathtt{O_W}(x) & k = 2 \\ W'(f(k)) & k > 2 \end{cases}$$

By the fact that W' is a $\langle G', L', \mathit{tid}_{\mathtt{RMW}} \rangle$-write-list, we get that W is a $\langle G, L, \mathit{tid}_{\mathtt{RMW}} \rangle$-write-list. It remains to show that W is $\langle G, \tau \rangle$-consistent. We prove below the $\langle G, \tau \rangle$-consistency conditions for W:

C1) Observe that $W(1) = w$ and $w \notin \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$ is guaranteed by the properties of w as stated above (it follows from the preconditions of the RMW step in opLRA). The condition C1(a) for W directly follows from $\langle G', \tau \rangle$-consistency of W'. For C1(b), given the $\langle G', \tau \rangle$-consistency of W', it suffices to handle the case that $j = 1$ (i.e., $W(j) = w$) and $k > 2$. Thus, assume toward contradiction some $2 < k \leq |L|$ such that $\langle W(k), w \rangle \in G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^?$. Since $\langle w, e \rangle \in G'.\mathtt{rf}$ and $e \in \mathsf{E}^\tau$, we would have $W'(f(k)) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, contradicting (C1(a) in) the $\langle G', \tau \rangle$-consistency of W'.

C2) To ensure the condition C2(a), since we added $W(2) = \mathtt{O_W}(x)$, which is not present in W', we need to show that $W(k) \notin \mathit{dom}(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$ whenever $\mathtt{loc}(W(k)) = x$ and $2 < k \leq |L|$. Assume otherwise and we have $e = \max_{G'.\mathtt{po}} G'.\mathsf{E}^\tau$, $\mathtt{loc}(W(k)) = \mathtt{loc}(e)$, $e \in \mathsf{W}$, $W'(f(k)) = W(k)$, and $W'(f(k)) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, which contradicts (C1 in) the fact that W' is $\langle G', \tau \rangle$-consistent. For the condition C2(b), since we added $W(1) = w$ and $W(2) = \mathtt{O_W}(x)$, we should ensure that for every $2 < k \leq |L|$, if $\mathtt{loc}(W(k)) = x$ then $\langle W(k), w \rangle \notin G.\mathtt{hb}^?$. Assume otherwise and we have $e = \max_{G'.\mathtt{po}} G'.\mathsf{E}^\tau$, $\mathtt{loc}(W(k)) = \mathtt{loc}(e)$, $e \in \mathsf{W}$, $W'(f(k)) = W(k) \neq e$. Hence $W'(f(k)) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$ which contradicts (C1(a) in) the fact that W' is $\langle G', \tau \rangle$-consistent.

C3) For condition C3(a), knowing that W' is $\langle G', \tau \rangle$-consistent we only need to show that $w \notin \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, which is guaranteed by the properties of w as stated above (it follows from the preconditions of the READ step in opLRA). For the condition C3(b), given the $\langle G', \tau \rangle$-consistency of W', it suffices to handle the case that $j = 1$. Thus assume toward contradiction some $2 < k \leq |L|$ where $w = W(1)$ and $W(k) \in \mathit{dom}((G.\mathtt{hb}|_{\mathtt{loc}} \setminus G.\mathtt{rf}) ; [\mathsf{R}] ; G.\mathtt{hb}^? ; [w])$. Since $W'(f(k)) = W(k)$, $\langle w, e \rangle \in G'.\mathtt{rf}$, and $e = \max_{G'.\mathtt{po}} G'.\mathsf{E}^\tau$ it follows that $W'(f(k)) \in \mathit{dom}((G'.\mathtt{hb}|_{\mathtt{loc}} \setminus G'.\mathtt{rf}) ; [\mathsf{R}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$ which contradicts (C3(a) in) the $\langle G', \tau \rangle$-consistency of W'.

C4) For the condition C4(a), given the $\langle G', \tau \rangle$-consistency of W', it suffices to handle the case that $j = 1$. Thus assume toward contradiction some $2 < k \leq |L|$ where $w = W(1)$, $w \neq W(k)$, $\mathtt{loc}(e) =$

$\mathtt{loc}(w) = \mathtt{loc}(W(k))$, and $W(k) \in \mathit{dom}(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. Then since $e = \max_{G'.\mathrm{po}} G'.\mathsf{E}^\tau$, $e \in \mathsf{W}$, $\mathtt{loc}(e) = \mathtt{loc}(W(k))$ and $W(k) = W'(f(k))$, we have $W'(f(k)) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$, contradicting (C1(a) in) the $\langle G', \tau\rangle$-consistency of W'. Similarly for condition C4(b), given the $\langle G', \tau\rangle$-consistency of W', it suffices to handle the case that $i = 1$. Assume toward contradiction some $1 < j < k \leq |L|$ where $w = W(1)$, $W(j) \neq W(k)$, $\mathtt{loc}(W(j)) = \mathtt{loc}(W(k))$, and $W(k) \in \mathit{dom}(G.\mathtt{hb}^? ; [w])$. Then, since $e \in G'.\mathsf{E}^\tau$ and $\langle w, e\rangle \in G'.\mathtt{rf}$, we get $W'(f(k)) \in \mathit{dom}(G'.\mathtt{hb}^? ; [\mathsf{E}^\tau])$ where $W'(f(j)) \neq W'(f(k))$ and $\mathtt{loc}(W'(f(j))) = \mathtt{loc}(W'(f(k)))$, contradicting (C4(a) in) the $\langle G', \tau\rangle$-consistency of W'.

- $\pi = \tau$ and $L = o \cdot \mathtt{O}_{\mathtt{W}}(x) \cdot (L'_x \setminus\setminus \mathcal{P}(\eta, L'))$ for some $\eta \in \mathsf{Tid}$ and $L' \in \mathcal{B}'(\eta)$. Let $P = \mathcal{P}(\eta, L')$, $W' = W'_{\langle \eta, L'\rangle}$, $m = \min(P)$ and $f = \lambda k \in \{3, \dots, |L|\}.\ \mathsf{MMap}^{-1}_{\langle L', P\rangle}(k-2)$. We define:

$$W \triangleq \lambda k \in \{1, \dots, |L|\}. \begin{cases} w & k = 1 \\ \mathtt{O}_{\mathtt{W}}(x) & k = 2 \\ W'(f(k)) & k > 2 \end{cases}$$

By the fact that W' is a $\langle G', L', \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list, we get that W is a $\langle G, L, \mathit{tid}_{\mathtt{RMW}}\rangle$-write-list. It remains to show that W is $\langle G, \tau\rangle$-consistent. We prove below the $\langle G, \tau\rangle$-consistency conditions for W:

C1) The difference from the previous case is that we have the $\langle G', \eta\rangle$-consistency of $W'_{\langle \eta, L'\rangle}$ rather than of $W'_{\langle \tau, L'\rangle}$. Hence, we should make sure that for every $2 < k \leq |L|$, we still have $W(k) \notin \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau \cup \{w\}])$. Assume first toward contradiction some k, such that $W(k) \in \mathit{dom}(G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$. Then, since $W'(f(k)) = W(k)$, $f(k) > m$, $W'(m) = e$ and $e = \max_{G'.\mathrm{po}} G'.\mathsf{E}^\tau$, we have $W'(f(k)) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\{W'(j) \mid 1 \leq j < f(k)\}])$, contradicting (C1(b) in) the $\langle G', \eta\rangle$- consistency of W'. Next, assume toward contradiction some k, such that $\langle W(k), w\rangle \in G.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G.\mathtt{hb}^?$. Then, we reach an analogous contradiction, since $\langle w, e\rangle \in G'.\mathtt{rf}$.

C2) For condition C2(a), knowing the $\langle G', \eta\rangle$-consistency of W' it suffices to handle the case that $i = 2$. Assume towards contradiction some $k > 2$, such that $W(k) \in \mathit{dom}(G.\mathtt{hb}^? ; [\mathsf{E}^\tau])$ and $\mathtt{loc}(e) = x = \mathtt{loc}(W(k))$. Since $W'(f(k)) = W(k)$, $f(k) > m$, $W'(m) = e$ and $e = \max_{G'.\mathrm{po}} G'.\mathsf{E}^\tau$, we have $W'(f(k)) \in \mathit{dom}(G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^? ; [\{W'(j) \mid 1 \leq j < f(k)\}])$, contradicting (C1(b) in) the $\langle G', \eta\rangle$-consistency of W'. For the condition C2(b), since we added $W(1) = w$ and $W(2) = \mathtt{O}_{\mathtt{W}}(x)$, we should ensure that for every $2 < k \leq |L|$, if $\mathtt{loc}(W(k)) = x$ then $\langle W(k), w\rangle \notin G.\mathtt{hb}^?$. Indeed, assume toward contradiction that $\langle W(k), w\rangle \in G.\mathtt{hb}^?$. Then, since $\langle w, e\rangle \in G'.\mathtt{rf}$, $W'(m) = e$ and $e \in \mathsf{W}$, we get that $\langle W'(f(k)), W'(m)\rangle \in G'.\mathtt{hb}|_{\mathtt{loc}} ; [\mathsf{W}] ; G'.\mathtt{hb}^?$. Since $f(k) > m$, this contradicts (C1(b) in) the $\langle G', \eta\rangle$-consistency of W'.

C3) For condition C3(a), knowing that W' is $\langle G', \eta\rangle$-consistent we only need to show that $W(1) \notin dom((G.\texttt{hb}|_{\texttt{loc}} \setminus G.\texttt{rf})\,;[\mathsf{R}]\,;G.\texttt{hb}^?\,;[\mathsf{E}^\tau])$, which is guaranteed by the properties of w as stated above (it follows from the preconditions of the READ step in opLRA and the fact that $w = W(1)$). For the condition C3(b), given the $\langle G', \eta\rangle$-consistency of W', it suffices to handle the case that $j = 1$. Thus assume toward contradiction some $2 < k \leq |L|$ where $w = W(1)$ and $W(k) \in dom((G.\texttt{hb}|_{\texttt{loc}} \setminus G.\texttt{rf})\,;[\mathsf{R}]\,;G.\texttt{hb}^?\,;[w])$. Since $W'(f(k)) = W(k)$, $\langle w, e\rangle \in G'.\texttt{rf}$, and $e = W'(m)$, it follows that $W'(f(k)) \in dom((G'.\texttt{hb}|_{\texttt{loc}} \setminus G'.\texttt{rf})\,;[\mathsf{R}]\,;G'.\texttt{hb}^?\,;[W'(m)])$. Since $f(k) > m$, this contradicts (C3(b) in) the $\langle G', \eta\rangle$-consistency of W'.

C4) For the condition C4(a), given the $\langle G', \eta\rangle$-consistency of W', it suffices to handle the case that $j = 1$. Thus assume toward contradiction some $2 < k \leq |L|$ where $w = W(1)$, $w \neq W(k)$, $\texttt{loc}(e) = \texttt{loc}(w) = \texttt{loc}(W(k))$, and $W(k) \in dom(G.\texttt{hb}^?\,;[\mathsf{E}^\tau])$. Then since $e = \max_{G'.\texttt{po}} G'.\mathsf{E}^\tau = W'(m)$, $e \in \mathsf{W}$, $\texttt{loc}(e) = \texttt{loc}(W(k))$, and $W(k) = W'(f(k))$, we have $W'(f(k)) \in dom(G'.\texttt{hb}|_{\texttt{loc}};[\mathsf{W}];G'.\texttt{hb}^?;[W'(m)])$, contradicting (C1(b) in) the $\langle G', \eta\rangle$-consistency of W' (since $f(k) > m$). Similarly for condition C4(b), given the $\langle G', \eta\rangle$-consistency of W', it suffices to handle the case that $i = 1$. Assume toward contradiction some $1 < j < k \leq |L|$ where $w = W(1)$, $W(j) \neq W(k)$, $\texttt{loc}(W(j)) = \texttt{loc}(W(k))$, and $W(k) \in dom(G.\texttt{hb}^?\,;[w])$. Then, since $e = \max_{G'.\texttt{po}} G'.\mathsf{E}^\tau = W'(m)$ and $\langle w, e\rangle \in G'.\texttt{rf}$, we get $W'(f(k)) \in dom(G'.\texttt{hb}^?\,;[W'(m)])$ where $W'(f(j)) \neq W'(f(k))$ and $\texttt{loc}(W'(f(j))) = \texttt{loc}(W'(f(k)))$, contradicting (C4(b) in) the $\langle G', \eta\rangle$-consistency of W' (since $f(k) > f(j) > m$).

B Decidability of State Reachability under loLRA

We establish the decidability of the reachability problem under the loLRA model. We start by recalling the framework of well-structured transition systems.

Preliminaries. A *well-quasi-ordering* (wqo) on a set S is a reflexive and transitive relation $\precsim$ on S such that for every infinite sequence $s_1, s_2, \ldots$ of elements of S, we have $s_i \precsim s_j$ for some $i < j$. In a context of a set S and a wqo $\precsim$ on S, the *upward closure* of a set $U \subseteq S$, denoted by $\uparrow U$, is given by $\{s \in S \mid \exists u \in U.\ u \precsim s\}$; a set $U \subseteq S$ is called *upward closed* if $U = \uparrow U$; and a set $B \subseteq U$ is called a *basis* of U if $U = \uparrow B$. The properties of a wqo ensure that every upward closed set has a *finite* basis.

A *well-structured transition system* (WSTS) is an LTS A equipped with a wqo $\precsim$ on $A.\mathsf{Q}$ that is *compatible* with A, that is: if $q_1 \to_A q_2$ and $q_1 \precsim q_3$, then there exists $q_4 \in A.\mathsf{Q}$ such that $q_3 \to_A^* q_4$ and $q_2 \precsim q_4$. The *coverability problem* for $\langle A, \precsim \rangle$ asks whether an input state $q \in A.\mathsf{Q}$ is coverable, namely: is some state q' with $q \precsim q'$ reachable in A?

Coverability is decidable (see, e.g., [9,15]) for a WSTS $\langle A, \precsim \rangle$ provided that $\precsim$ is decidable and the following hold:

(i) *effective initialization*: there exists an algorithm that accepts a state $q \in A.\mathsf{Q}$ and decides whether $\uparrow\{q\} \cap A.\mathsf{Q}_0 = \emptyset$.
(ii) *effective pred-basis*: there exists an algorithm that accepts a state $q \in A.\mathsf{Q}$ and returns a finite basis of $\uparrow \mathsf{pred}_A(\uparrow\{q\})$.

For the latter we define the set of *predecessors* of a set $S \subseteq A.\mathsf{Q}$ *w.r.t. a symbol* $\sigma \in \Sigma$, denoted by $\mathsf{pred}_A^\sigma(S)$, by $\{q \in A.\mathsf{Q} \mid \exists q' \in S.\ q \xrightarrow{\sigma}_A q'\}$. The set of *predecessors* of a set $S \subseteq A.\mathsf{Q}$, denoted by $\mathsf{pred}_A(S)$, is given by $\bigcup_{\sigma \in \Sigma} \mathsf{pred}_A^\sigma(S)$.

*lo*LRA *as a Well-Structured Transition System.* The $\sqsubseteq$ ordering on the states of loLRA is clearly decidable and also forms a wqo. Indeed, by Higman's lemma, $\sqsubseteq$ is a wqo on the set of all option lists. In turn, its lifting to potentials (which are finite by definition) is a wqo on the set of all potentials (see [31]). Finally, by Dickson's lemma, the pointwise lifting of $\sqsubseteq$ to functions assigning a potential to every $\tau \in \mathsf{Tid}$ (i.e., states of loLRA) is also a wqo.

Now, let Pr be a program. The $\sqsubseteq$ ordering is naturally lifted to states of the concurrent system $Pr \bowtie$loLRA (that is, pairs $\langle \overline{p}, \mathcal{B} \rangle \in Pr.\mathsf{Q} \times$ loLRA.Q) by defining $\langle \overline{p}, \mathcal{B} \rangle \sqsubseteq \langle \overline{p}', \mathcal{B}' \rangle$ iff $\overline{p} = \overline{p}'$ and $\mathcal{B} \sqsubseteq \mathcal{B}'$.

Lemma 4. *$Pr \bowtie lo$LRA equipped with $\sqsubseteq$ is a WSTS that admits effective initialization and effective pred-basis.*

Proof. First, since $Pr.\mathsf{Q}$ is (by definition) finite and $\sqsubseteq$ is a wqo on loLRA.Q, we have that $\sqsubseteq$ is a wqo of $Pr \bowtie$loLRA.Q.

Second, since LOWER is explicitly included in loLRA, $\sqsubseteq$ is clearly compatible with $Pr \bowtie$loLRA. Indeed, given $q_1 = \langle \overline{p}_1, \mathcal{B}_1 \rangle$, $q_2 = \langle \overline{p}_2, \mathcal{B}_2 \rangle$ and $q_3 = \langle \overline{p}_3, \mathcal{B}_3 \rangle$ such that $q_1 \to_{Pr \bowtie \text{loLRA}} q_2$ and $q_1 \sqsubseteq q_3$ (so $\overline{p}_1 = \overline{p}_3$), for $q_4 = q_2$, we have $q_3 \to^*_{Pr \bowtie \text{loLRA}} q_4$ (since $\mathcal{B}_3 \xrightarrow{\varepsilon}_{\text{loLRA}} \mathcal{B}_1$ using the LOWER step) and $q_2 \sqsubseteq q_4$.

Next, $Pr\bowtie$loLRA trivially admits effective initialization. Indeed, the states $\langle\overline{p},\mathcal{B}\rangle$ for which $\uparrow\{\langle\overline{p},\mathcal{B}\rangle\} \cap Pr\bowtie\text{loLRA}.\mathtt{Q_0} \neq \emptyset$ are exactly the initial states themselves—$Pr.\mathtt{Q_0} \times \{\lambda\tau.\ \{\epsilon\}\}$.

It remains to show the effective pred-basis for $Pr\bowtie$loLRA. For this matter, we demonstrate how to calculate a finite basis of $\uparrow\mathsf{pred}^{\alpha}_{\text{loLRA}}(\uparrow\{\mathcal{B}'\})$ for α of the form $\langle\tau,\mathtt{W}(x,v_{\mathtt{W}})\rangle$, $\langle\tau,\mathtt{R}(x,v_{\mathtt{R}})\rangle$, $\langle\tau,\mathtt{RMW}(x,v_{\mathtt{R}},v_{\mathtt{W}})\rangle$ or ε. Then, a finite basis of $\uparrow\mathsf{pred}^{\alpha}_{Pr\bowtie\text{loLRA}}(\uparrow\{\langle\overline{p}',\mathcal{B}'\rangle\})$ is $\mathsf{pred}^{\alpha}_{Pr}(\{\overline{p}'\}) \times Q^{\alpha}$ for $\alpha \neq \varepsilon$; and $\{\overline{p}'\} \times Q^{\alpha}$ for $\alpha = \varepsilon$ (silent memory step). In addition, for a silent program step, a finite basis of $\uparrow\mathsf{pred}^{\langle\tau,\varepsilon\rangle}_{Pr\bowtie\text{loLRA}}(\uparrow\{\langle\overline{p}',\mathcal{B}'\rangle\})$ is given by $\mathsf{pred}^{\langle\tau,\varepsilon\rangle}_{Pr}(\{\overline{p}'\}) \times \{\mathcal{B}'\}$.

Silent memory step The set of predecessors of $\mathcal{B}'$ with respect to a silent memory step (i.e., using LOWER) is very simple—it contains any state $\mathcal{B}$ such that $\mathcal{B}' \sqsubseteq \mathcal{B}$. Thus, $\{\mathcal{B}'\}$ is a finite basis of $\uparrow\mathsf{pred}^{\varepsilon}_{\text{loLRA}}(\{\mathcal{B}'\})$, as well as of $\uparrow\mathsf{pred}^{\varepsilon}_{\text{loLRA}}(\uparrow\{\mathcal{B}'\})$.

Read A predecessor $\mathcal{B}$ of $\mathcal{B}'$ with respect to a READ step is similar to $\mathcal{B}'$, except for having in each read-option list of τ an additional first read option of the form $\langle\tau_{\mathtt{W}},x,v_{\mathtt{R}},\pi_{\mathtt{RMW}}\rangle$. Hence, for $\alpha = \langle\tau,\mathtt{R}(x,v_{\mathtt{R}})\rangle$, the set $\{\mathcal{B}'[\tau \mapsto \langle\tau_{\mathtt{W}},x,v_{\mathtt{R}},\pi_{\mathtt{RMW}}\rangle\cdot\mathcal{B}'(\tau)] \mid \tau_{\mathtt{W}},\pi_{\mathtt{RMW}} \in \mathsf{Tid}\}$ is a finite basis of $\uparrow\mathsf{pred}^{\alpha}_{\text{loLRA}}(\{\mathcal{B}'\})$. It is also a basis of $\uparrow\mathsf{pred}^{\alpha}_{\text{loLRA}}(\uparrow\{\mathcal{B}'\})$: For a state $\mathcal{B}''$ with $\mathcal{B}' \sqsubseteq \mathcal{B}''$, a corresponding read option $\langle\tau_{\mathtt{W}},x,v_{\mathtt{R}},\pi_{\mathtt{RMW}}\rangle$ appears in the lists of τ in $\mathsf{pred}^{\alpha}_{\text{loLRA}}(\{\mathcal{B}''\})$ before some additional read options, ensuring that $\mathsf{pred}^{\alpha}_{\text{loLRA}}(\{\mathcal{B}'\}) \sqsubseteq \mathsf{pred}^{\alpha}_{\text{loLRA}}(\{\mathcal{B}''\})$.

Write We construct the basis of the predecessors w.r.t. a WRITE step by considering all (finitely many) possibilities of omitting read options from lists of $\mathcal{B}'$. By Proposition 3 and the following Lemma 5, we get a finite basis of $\uparrow\mathsf{pred}^{\langle\tau,\mathtt{W}(x,v_{\mathtt{W}})\rangle}_{\text{loLRA}}(\uparrow\{\mathcal{B}'\})$, as:
$\{\mathsf{src}_x(\mathcal{B}',\tau,\mathcal{P})[\tau \mapsto \mathtt{O_W}(x)\cdot\mathsf{src}_x(\mathcal{B}',\tau,\mathcal{P})(\tau)] \mid \mathcal{P} \text{ is an index choice for } \mathcal{B}' \text{ and } \mathcal{P} \models \langle\tau,\mathtt{W}(x,v_{\mathtt{W}})\rangle\}$.

Lemma 5. *Let $\mathcal{P}$ be an index choice for $\mathcal{B}' \in$ loLRA.$\mathtt{Q}$ such that $\mathcal{P} \models \langle\tau,\mathtt{W}(x,v_{\mathtt{W}})\rangle$. If $\mathcal{B}'_0 \sqsubseteq \mathcal{B}'$, then there exists an index choice $\mathcal{P}_0$ for $\mathcal{B}'_0$ such that $\mathcal{P}_0 \models \langle\tau,\mathtt{W}(x,v_{\mathtt{W}})\rangle$ and $\mathsf{src}_x(\mathcal{B}'_0,\tau,\mathcal{P}_0) \sqsubseteq \mathsf{src}_x(\mathcal{B}',\tau,\mathcal{P})$.*

Proof. Since $\mathcal{B}'_0 \sqsubseteq \mathcal{B}'$, for every $\pi \in \mathsf{Tid}$, there exists a function $F_\pi : \mathcal{B}'_0(\pi) \to \mathcal{B}'(\pi)$ such that for every $L'_0 \in \mathcal{B}'_0(\pi)$, we have $L'_0 \sqsubseteq F_\pi(L'_0)$, witnessed by a strictly increasing function $f_{\langle\pi,L'_0\rangle} : \{1,\dots,|L'_0|\} \to \{1,\dots,|F_\pi(L'_0)|\}$, such that $L'_0(k) = (F_\pi(L'_0))(f_{\langle\pi,L'_0\rangle}(k))$ for every $k \in \{1,\dots,|L'_0|\}$.
We define $\mathcal{P}_0$ to be the positions in $\mathcal{P}$ that originated in $\mathcal{B}'_0$, according to the $f_{\langle\pi,L'_0\rangle}$ functions. That is,

$$\mathcal{P}_0 \triangleq \lambda\pi \in \mathsf{Tid}, L'_0 \in \mathcal{B}'_0(\pi).\ \{k \in \{1,\dots,|L'_0|\} \mid f_{\langle\pi,L'_0\rangle}(k) \in \mathcal{P}(\pi,F_\pi(L'_0))\}.$$

It is easy to verify that $\mathcal{P}_0$ justifies a $\langle\tau,\mathtt{W}(x,v_{\mathtt{W}})\rangle$-step. Let $\mathcal{B}_0 = \mathsf{src}_x(\mathcal{B}'_0,\tau,\mathcal{P}_0)$. We show that $\mathcal{B}_0 \sqsubseteq \mathsf{src}_x(\mathcal{B}',\tau,\mathcal{P})$.
Recall that for every thread $\pi \in \mathsf{Tid}$, we have that every list $L_0 \in \mathcal{B}_0(\pi)$ is equal to $L'_0 \setminus \mathcal{P}_0(\pi,L'_0)$ (or resp. to $L'_0 \setminus\!\!\setminus \mathcal{P}_0(\eta,L'_0)$) for some list L'_0 of $\mathcal{B}'_0(\pi)$ (resp. for some list L'_0 of $\mathcal{B}'_0(\eta)$ for some $\eta \in \mathsf{Tid}$). Hence, we can define a function $H_\pi : \mathcal{B}_0(\pi) \to \mathsf{src}_x(\mathcal{B}',\tau,\mathcal{P})(\pi)$, by setting $H_\pi(L_0) = F_\pi(L'_0) \setminus$

$\mathcal{P}(\pi, F_\pi(L'_0))$. Observe that for every $L_0 \in \mathcal{B}_0(\pi)$, we have $L_0 \sqsubseteq H_\pi(L_0)$, witnessed by the function $h_{\langle\pi,L_0\rangle} : \{1,\ldots,|L_0|\} \to \{1,\ldots,|H_\pi(L_0)|\}$, defined by

$$h_{\langle\pi,L_0\rangle}(k) \triangleq \mathsf{Map}_{\langle F_\pi(L'_0),\mathcal{P}(\pi,F_\pi(L'_0))\rangle}(f_{\langle\pi,L'_0\rangle}(\mathsf{Map}^{-1}_{\langle L'_0,\mathcal{P}_0(\pi,L'_0)\rangle}(k))),$$

for every $k \in \{1,\ldots,|L_0|\}$. (Respectively, we define $H_\pi(L_0) = F_\eta(L'_0) \setminus\setminus \mathcal{P}(\eta, F_\eta(L'_0)))$, witnessed analogously.)

RMW The predecessor with respect to an RMW step intuitively combines the predecessors with respect to the read and write steps. By Proposition 3 and Lemma 5, we get that the set

$$\{\mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})[\tau \mapsto \langle\tau_{\mathtt{W}}, x, v_{\mathtt{R}}, \tau\rangle\cdot\mathtt{O}_{\mathtt{W}}(x)\cdot\mathsf{src}_x(\mathcal{B}', \tau, \mathcal{P})(\tau)] \mid \tau_{\mathtt{W}} \in \mathsf{Tid} \text{ and } \mathcal{P} \text{ is an index choice for } \mathcal{B}' \text{ such that } \mathcal{P} \models \langle\tau, \mathtt{W}(x, v_{\mathtt{W}})\rangle\}$$

is a finite basis of $\uparrow\mathsf{pred}^{\langle\tau,\mathtt{RMW}(x,v_{\mathtt{R}},v_{\mathtt{W}})\rangle}_{\text{loLRA}}(\uparrow\{\mathcal{B}'\})$.

It is now easy to establish the decidability of reachability under loLRA.

Theorem 2 (loLRA **reachability).** *Given a program Pr and a state $\overline{p} \in Pr.\mathtt{Q}$, it is decidable to check whether $\overline{p}$ is reachable under the memory system lo*LRA*.*

Proof. Since the first component (the program state) in $\sqsubseteq$-ordered pairs of $Pr \bowtie$loLRA's states is equal, reachability under loLRA is reduced to coverability in $\langle Pr \bowtie \text{loLRA}, \sqsubseteq\rangle$, which is decidable by Lemma 4 and the framework of [9].