Canonical Constructive Systems*

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Abstract. We define the notions of a canonical inference rule and a canonical constructive system in the framework of strict single-conclusion Gentzen-type systems (or, equivalently, natural deduction systems), and develop a corresponding general non-deterministic Kripke-style semantics. We show that every constructive canonical system induces a class of non-deterministic Kripke-style frames, for which it is strongly sound and complete. This non-deterministic semantics is used to show that such a system always admits a strong form of the cut-elimination theorem, and for providing a decision procedure for such systems.

1 Introduction

The standard intuitionistic connectives ($\supset, \wedge, \vee$, and $\bot$) are of great importance in theoretical computer science, especially in type theory, where they correspond to basic operations on types (via the formulas-as-types principle and Curry-Howard isomorphism). Now a natural question is: what is so special about these connectives? The standard answer is that they are all *constructive* connectives. But then what exactly is a constructive connective, and can we define other basic constructive connectives beyond the four intuitionistic ones? And what does the last question mean anyway: how do we "define" new (or old) connectives?

Concerning the last question there is a long tradition starting from [10] (see e.g. [14] for discussions and references) according to which the meaning of a connective is determined by the introduction and elimination rules which are associated with it. Here one usually has in mind natural deduction systems of an ideal type, where each connective has its own introduction and elimination rules, and these rules should meet the following conditions: in a rule for some connective this connective should be mentioned exactly once, and no other connective should be involved. The rule should also be pure in the sense of [1] (i.e., there should be no side conditions limiting its application), and its active formulas should be immediate subformulas of its principal formula.

Unfortunately, already the handling of negation requires rules which are not ideal in this sense. For intuitionistic logic this problem has been solved by not taking negation as a basic constructive connective, but defining it instead in terms of more basic connectives that can be characterized by "ideal" rules ($\neg\varphi$

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is defined as $\varphi \rightarrow \perp$). For classical logic the problem was solved by Gentzen himself by moving to what is now known as Gentzen-type systems or sequential calculi. These calculi employ single-conclusion sequents in their intuitionistic version, and multiple-conclusion sequents in their classical version. Instead of introduction and elimination rules they use left introduction rules and right introduction rules. The intuitive notions of an "ideal rule" can be adapted to such systems in a straightforward way, and it is well known that the usual classical connectives and the basic intuitionistic connectives can indeed be fully characterized by "ideal" Gentzen-type rules. Moreover: although this can be done in several ways, in all of them the cut-elimination theorem obtains.

For the multiple-conclusion framework these facts were considerably generalized in [5,6] by defining "multiple-conclusion canonical propositional Gentzen-type rules and systems" in precise terms. A constructive necessary and sufficient *coherence* criterion for the non-triviality of such systems was then provided, and it was shown that a system of this kind admits cut-elimination iff it is coherent. It was further proved that the semantics of such systems is provided by two-valued non-deterministic matrices (two-valued Nmatrices) — a natural generalization of the classical truth-tables. In fact, a characteristic two-valued Nmatrix was constructed for every coherent canonical propositional system. That work shows that there is a large family of what may be called semi-classical connectives (which includes all the classical connectives), each of which has both a proof-theoretical characterization in terms of a coherent set of canonical (= "ideal") rules, and a semantic characterization using two-valued Nmatrices.

In this paper we develop a similar theory for the constructive propositional framework. We define the notions of a canonical rule and a canonical system in the framework of strict single-conclusion Gentzen-type systems (or, equivalently, natural deduction systems). We prove that here too a canonical system is non-trivial iff it is coherent (where coherence is a constructive condition, defined like in the multiple-conclusion case). We develop a general non-deterministic Kripke-style semantics for such systems, and show that every constructive canonical system (i.e. coherent canonical single-conclusion system) induces a class of non-deterministic Kripke-style frames for which it is strongly sound and complete. We use this non-deterministic semantics to show that all constructive canonical systems admit a strong form of the cut-elimination theorem. We also use it for providing decision procedures for all such systems. These results again identify a large family of basic constructive connectives, each having both a proof-theoretical characterization in terms of a coherent set of canonical rules, and a semantic characterization using non-deterministic frames. The family includes the standard intuitionistic connectives ($\supset, \wedge, \vee$, and $\perp$), as well as many other independent connectives.

2 Canonical Constructive Systems

In what follows $\mathcal{L}$ is a propositional language, $\mathcal{F}$ is its set of wffs, p, q, r denote atomic formulas, ψ, φ, θ denote arbitrary formulas (of $\mathcal{L}$), T, S denote subsets of $\mathcal{F}$, and $\Gamma, \Delta, \Sigma, \Pi$ denote finite subsets of $\mathcal{F}$. We assume that the atomic formulas of $\mathcal{L}$ are $p_1, p_2, \dots$ (in particular: $\{p_1, p_2, \dots, p_n\}$ are the first n atomic formulas of $\mathcal{L}$).

Definition 1. A *Tarskian consequence relation* (*tcr* for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ and formulas of $\mathcal{L}$ that satisfies the following conditions:

strong reflexivity: if $\varphi \in T$ then $T \vdash \varphi$.
monotonicity: if $T \vdash \varphi$ and $T \subseteq T'$ then $T' \vdash \varphi$.
transitivity (*cut*): if $T \vdash \psi$ and $T, \psi \vdash \varphi$ then $T \vdash \varphi$.

Definition 2. A *substitution* in $\mathcal{L}$ is a function σ from the atomic formulas to the set of formulas of $\mathcal{L}$. σ is extended to formulas and sets of formulas in the obvious way.

Definition 3. A tcr $\vdash$ for $\mathcal{L}$ is *structural* if for every substitution σ and every T and φ, if $T \vdash \varphi$ then $\sigma(T) \vdash \sigma(\varphi)$. $\vdash$ is *finitary* if the following condition holds for all T and φ: if $T \vdash \varphi$ then there exists a finite $\Gamma \subseteq T$ such that $\Gamma \vdash \varphi$. $\vdash$ is *consistent* (or *non-trivial*) if $p_1 \not\vdash p_2$.

It is easy to see (see [6]) that there are exactly two inconsistent structural tcrs in any given language[1]. These tcrs are obviously trivial, so we exclude them from our definition of a *logic*:

Definition 4. A propositional *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a tcr for $\mathcal{L}$ which is structural, finitary, and consistent.

Since a finitary consequence relation $\vdash$ is determined by the set of pairs $\langle \Gamma, \varphi \rangle$ such that $\Gamma \vdash \varphi$, it is natural to base proof systems for logics on the use of such pairs. This is exactly what is done in natural deduction systems and in (strict) single-conclusion Gentzen-type systems (both introduced in [10]). Formally, such systems manipulate objects of the following type:

Definition 5. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas, and Δ is either a singleton or empty. A sequent of the form $\Gamma \Rightarrow \{\varphi\}$ is called *definite*, and we shall denote it by $\Gamma \Rightarrow \varphi$. A sequent of the form $\Gamma \Rightarrow \{\}$ is called *negative*, and we shall denote it by $\Gamma \Rightarrow$. A Horn *clause* is a sequent which consists of atomic formulas only.

Note. Natural deduction systems, and the strict single-conclusion Gentzen-type systems investigated in this paper, manipulate only definite sequents in their derivations. However, negative sequents may be used in the formulations of their rules (in the form of negative Horn clauses).

The following definitions formulate in exact terms the idea of an "ideal rule" which was described in the introduction:

[1] In one $T \vdash \varphi$ for every T and φ, in the other $T \vdash \varphi$ for every nonempty T and φ.

Definition 6

1. A *canonical introduction rule* is an expression of the form:

$$\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} / \ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$$

where $m \geq 0$, $\diamond$ is a connective of arity n, and for all $1 \leq i \leq m$, $\Pi_i \Rightarrow \Sigma_i$ is a *definite* Horn clause such that $\Pi_i \cup \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.
2. A *canonical elimination rule*[2] is an expression of the form

$$\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} / \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$$

where $m \geq 0$, $\diamond$ is a connective of arity n, and for all $1 \leq i \leq m$, $\Pi_i \Rightarrow \Sigma_i$ is a Horn clause (either definite or negative) such that $\Pi_i \cup \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.
3. An *application* of the rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} / \ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ is any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i)\}_{1\leq i\leq m}}{\Gamma \Rightarrow \diamond(\sigma(p_1), \ldots, \sigma(p_n))}$$

where Γ is a finite set of formulas and σ is a substitution in $\mathcal{L}$.
4. An *application* of the rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} / \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i), E_i\}_{1\leq i\leq m}}{\Gamma, \diamond(\sigma(p_1), \ldots, \sigma(p_n)) \Rightarrow \theta}$$

where Γ and σ are as above, θ is a formula, and for all $1 \leq i \leq m$: $E_i = \theta$ in case Σ_i is empty, and E_i is empty otherwise.

Note. We formulated the definition above in terms of Gentzen-type systems. However, we could have formulated them instead in terms of natural deduction systems. The definition of an application of an introduction rule is defined in this context exactly as above, while an application of an elimination rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} / \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ is in the context of natural deduction any inference step of the form:

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i), E_i\}_{1\leq i\leq m} \quad \Gamma \Rightarrow \diamond(\sigma(p_1), \ldots, \sigma(p_n))}{\Gamma \Rightarrow \theta}$$

where Γ, σ, θ and E_i are as above.

Here are some examples of well-known canonical rules:

[2] The introduction/elimination terminology is due to the natural deduction context. For the Gentzen-type context the names "right introduction rule" and "left introduction rule" might be more appropriate, but we prefer to use a uniform terminology.

Conjunction. The two usual rules for conjunction are:

$$\{p_1, p_2 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow \quad \text{and} \quad \{\ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\} \ / \ \ \Rightarrow p_1 \wedge p_2$$

In the Gentzen-type context applications of these rules have the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta}{\Gamma, \psi \wedge \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \psi \quad \Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \wedge \varphi}$$

In natural deduction systems applications of the first have the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta \quad \Gamma \Rightarrow \psi \wedge \varphi}{\Gamma \Rightarrow \theta}$$

The above elimination rule can easily be shown to be equivalent to the combination of the two more usual elimination rules for conjunction.

Implication. The two usual rules for implication are:

$$\{\Rightarrow p_1\ ,\ \ p_2 \Rightarrow\} \ / \ p_1 \supset p_2 \Rightarrow \quad \text{and} \quad \{p_1 \Rightarrow p_2\} \ / \ \ \Rightarrow p_1 \supset p_2$$

In the Gentzen-type context applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \supset \varphi \Rightarrow \theta} \qquad \frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \psi \supset \varphi}$$

In natural-deduction systems applications of the first have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta \quad \Gamma \Rightarrow \psi \supset \varphi}{\Gamma \Rightarrow \theta}$$

Again this form of the rule is obviously equivalent to the more usual one (from $\Gamma \Rightarrow \psi$ and $\Gamma \Rightarrow \psi \supset \varphi$ infer $\Gamma \Rightarrow \varphi$).

Absurdity. In intuitionistic logic there is no introduction rule for the absurdity constant $\perp$, and there is exactly one elimination rule for it: $\{\} \ / \ \perp\Rightarrow$. In the Gentzen-type context applications of this rule provide new *axioms*: $\Gamma, \perp\Rightarrow \varphi$. In natural-deduction systems applications of the same rule allow us to infer $\Gamma \Rightarrow \varphi$ from $\Gamma \Rightarrow\perp$.

Semi-implication. Consider the "semi-implication" $\leadsto$ with the following two rules:[3]

$$\{\Rightarrow p_1\ ,\ \ p_2 \Rightarrow\} \ / \ p_1 \leadsto p_2 \Rightarrow \quad \text{and} \quad \{\Rightarrow p_2\} \ / \ \ \Rightarrow p_1 \leadsto p_2$$

In the Gentzen-type context applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \leadsto \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \leadsto \varphi}$$

Again in natural-deduction systems applications of the first rule are equivalent to MP for $\leadsto$ (from $\Gamma \Rightarrow \psi$ and $\Gamma \Rightarrow \psi \leadsto \varphi$ infer $\Gamma \Rightarrow \varphi$).

[3] This connective was introduced in [11] for different purposes.

From now on we shall concentrate on single-conclusion Gentzen-type systems (translating our notions and results to natural deduction systems is easy).

Definition 7. A single-conclusion Gentzen-type system is called *canonical* if its axioms are the sequents of the form $\varphi \Rightarrow \varphi$, cut (from $\Gamma \Rightarrow \varphi$ and $\Delta, \varphi \Rightarrow \psi$ infer $\Gamma, \Delta \Rightarrow \psi$) and weakening (from $\Gamma \Rightarrow \psi$ infer $\Gamma, \Delta \Rightarrow \psi$) are among its rules, and each of its other rules is either a canonical introduction rule or a canonical elimination rule.

Definition 8. Let **G** be a canonical Gentzen-type system.

1. $\mathcal{S} \vdash_{\mathbf{G}}^{seq} s$ (where s is a sequent and $\mathcal{S}$ is a set of sequents) if there is a derivation in **G** of s from $\mathcal{S}$.
2. The tcr $\vdash_{\mathbf{G}}$ between *formulas* which is induced by **G** is defined by: $T \vdash_{\mathbf{G}} \varphi$ iff there exists a finite $\Gamma \subseteq T$ such that $\vdash_{\mathbf{G}}^{seq} \Gamma \Rightarrow \varphi$.

Proposition 1. *$T \vdash_{\mathbf{G}} \varphi$ iff $\{\Rightarrow \psi \mid \psi \in T\} \vdash_{\mathbf{G}}^{seq} \Rightarrow \varphi$.*

Proposition 2. *If **G** is canonical then $\vdash_{\mathbf{G}}$ is a structural and finitary tcr.*

The last proposition does not guarantee that every canonical system induces a *logic* (see Definition 4). For this the system should satisfy one more condition:

Definition 9. A set $\mathcal{R}$ of canonical rules for an n-ary connective $\diamond$ is called *coherent* if $S_1 \cup S_2$ is classically inconsistent (and so the empty clause can be derived from it using cuts) whenever $\mathcal{R}$ contains both $S_1 / \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ and $S_2 / \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$.

Examples

- All the sets of rules for the connectives $\wedge, \supset, \perp$, and $\leadsto$ which were introduced in the examples above are coherent. For example, for the two rules for conjunction we have $S_1 = \{p_1, p_2 \Rightarrow\ \}$, $S_2 = \{\ \Rightarrow p_1\ ,\ \Rightarrow p_2\}$, and $S_1 \cup S_2$ is the classically inconsistent set $\{p_1, p_2 \Rightarrow\ ,\ \Rightarrow p_1\ ,\ \Rightarrow p_2\}$ (from which the empty sequent can be derived using two cuts).
- In [13] Prior introduced a "connective" T (which he called "Tonk") with the following rules: $\{p_1 \Rightarrow\ \} \ /\ p_1 T p_2 \Rightarrow$ and $\{\ \Rightarrow p_2\} \ /\ \Rightarrow p_1 T p_2$. Prior then used "Tonk" to infer everything from everything (trying to show by this that rules alone cannot define a connective). Now the union of the sets of premises of these two rules is $\{p_1 \Rightarrow\ ,\ \Rightarrow p_2\}$, and this is a classically consistent set of clauses. It follows that Prior's set of rules for Tonk is incoherent.

Definition 10. A canonical single-conclusion Gentzen-type system, **G**, is called *coherent* if every primitive connective of the language of **G** has in **G** a coherent set of rules.

Theorem 1. *Let **G** be a canonical Gentzen-type system. $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ is a logic (i.e. $\vdash_{\mathbf{G}}$ is structural, finitary and consistent) iff **G** is coherent.*

Proof. Proposition 2 ensures that $\vdash_{\mathbf{G}}$ is a structural and finitary tcr.

That the coherence of **G** implies the consistency of the *multiple* conclusion consequence relation which is naturally induced by **G** was shown in [5,6]. That consequence relation extends $\vdash_{\mathbf{G}}$, and therefore also the latter is consistent.

For the converse, assume that **G** is incoherent. This means that **G** includes two rules $S_1/\diamond(p_1,\ldots,p_n)\Rightarrow$ and $S_2/\ \Rightarrow\diamond(p_1,\ldots,p_n)$, such that the set of clauses $S_1\cup S_2$ is classically satisfiable. Let v be an assignment in $\{t,f\}$ that satisfies all the clauses in $S_1\cup S_2$. Define a substitution σ by:

$$\sigma(p)=\begin{cases}p_{n+1} & v(p)=f\\ p & v(p)=t\end{cases}$$

Let $\Pi\Rightarrow q\in S_1\cup S_2$. Then $\vdash_G^{seq} p_1,\ldots,p_n,\sigma(\Pi)\Rightarrow\sigma(q)$. This is trivial in case $v(q)=t$, since in this case $\sigma(q)=q\in\{p_1,\ldots,p_n\}$. On the other hand, if $v(q)=f$ then $v(p)=f$ for some $p\in\Pi$ (since v satisfies the clause $\Pi\Rightarrow q$). Therefore in this case $\sigma(p)=\sigma(q)=p_{n+1}$, and so again $p_1,\ldots,p_n,\sigma(\Pi)\Rightarrow\sigma(q)$ is trivially derived from an axiom. We can similarly prove that $\vdash_{\mathbf{G}}^{seq} p_1,\ldots,p_n,\sigma(\Pi)\Rightarrow p_{n+1}$ in case $\Pi\Rightarrow\ \in S_1\cup S_2$. Now by applying $S_1/\diamond(p_1,\ldots,p_n)\Rightarrow$ and $S_2/\ \Rightarrow\diamond(p_1,\ldots,p_n)$ to these provable sequents we get proofs in **G** of $p_1,\ldots,p_n\Rightarrow\diamond(\sigma(p_1),\ldots,\sigma(p_n))$ and of $p_1,\ldots,p_n,\diamond(\sigma(p_1),\ldots,\sigma(p_n))\Rightarrow p_{n+1}$. That $\vdash_{\mathbf{G}}^{seq} p_1,\ldots,p_n\Rightarrow p_{n+1}$ then follows using a cut. This easily entails that $p_1\vdash_{\mathbf{G}} p_2$, and hence $\vdash_{\mathbf{G}}$ is not consistent. □

Note. The last theorem implies that coherence is a minimal demand from any acceptable canonical system **G**. It follows that not every set of such rules is legitimate for defining constructive connectives - only coherent ones do (and this is what is wrong with "Tonk"). Accordingly we define:

Definition 11. A *canonical constructive system* is a coherent canonical single-conclusion Gentzen-type system.

The following definition will be needed in the sequel:

Definition 12. Let $\mathcal{S}$ be a set of sequents.

1. A cut is called an $\mathcal{S}$-cut if the cut formula occurs in $\mathcal{S}$.
2. We say that there exists in a system **G** an $\mathcal{S}$-cut-free proof of a sequent s from a set of sequents $\mathcal{S}$ iff there exists a proof of s from $\mathcal{S}$ in **G** where all cuts are $\mathcal{S}$-cuts.
3. ([2]) A system **G** admits strong cut-elimination iff whenever $\mathcal{S}\vdash_{\mathbf{G}}^{seq} s$, there exists an $\mathcal{S}$-cut-free proof of s from $\mathcal{S}$.[4]

[4] By cut-elimination we mean here just the existence of proofs without (certain forms of) cuts, rather than an algorithm to transform a given proof to a cut-free one (for the assumptions-free case the term "cut-admissibility" is sometimes used).

3 Semantics for Canonical Constructive Systems

The most useful semantics for propositional intuitionistic logic (the paradigmatic constructive logic) is that of Kripke frames. In this section we generalize this semantics to arbitrary canonical constructive systems. For this we should introduce *non-deterministic* Kripke frames.[5]

Definition 13. A generalized $\mathcal{L}$-*frame* is a triple $\mathcal{W} = \langle W, \leq, v\rangle$ such that:

1. $\langle W, \leq\rangle$ is a nonempty partially ordered set.
2. v is a function from $\mathcal{F}$ to the set of persistent functions from W into $\{t, f\}$ (A function $h : W \to \{t, f\}$ is *persistent* if $h(a) = t$ implies that $h(b) = t$ for every $b \in W$ such that $a \leq b$).

Notation: We shall usually write $v(a, \varphi)$ instead of $v(\varphi)(a)$.

Definition 14. A generalized $\mathcal{L}$-frame $\langle W, \leq, v\rangle$ is a *model* of a formula φ if $v(\varphi) = \lambda a \in W.t$ (i.e.: $v(a, \varphi) = t$ for every $a \in W$). It is a model of a theory T if it is a model of every $\varphi \in T$.

Definition 15. Let $\mathcal{W} = \langle W, \leq, v\rangle$ be a generalized $\mathcal{L}$-frame, and let $a \in W$.

1. A sequent $\Gamma \Rightarrow \varphi$ is *locally true* in a if either $v(a, \psi) = f$ for some $\psi \in \Gamma$, or $v(a, \varphi) = t$.
2. A sequent $\Gamma \Rightarrow \varphi$ is *true* in a if it is locally true in every $b \geq a$.
3. A sequent $\Gamma \Rightarrow$ is *(locally) true* in a if $v(a, \psi) = f$ for some $\psi \in \Gamma$.
4. $\mathcal{W}$ is a *model* of a sequent s (either of the form $\Gamma \Rightarrow \varphi$ or $\Gamma \Rightarrow$) if s is true in every $a \in W$ (iff s is locally true in every $a \in W$). It is a model of a set of sequents $\mathcal{S}$ if it is a model of every $s \in \mathcal{S}$.

Note. $\mathcal{W}$ is a model of a formula φ iff it is a model of the sequent $\Rightarrow \varphi$.

Definition 16. Let $\langle W, \leq, v\rangle$ be a generalized $\mathcal{L}$-frame. A substitution σ in $\mathcal{L}$ satisfies a Horn clause $\Pi \Rightarrow \Sigma$ in $a \in W$ if $\sigma(\Pi) \Rightarrow \sigma(\Sigma)$ is true in a.

Note. Because of the persistence condition, a definite Horn clause of the form $\Rightarrow q$ is satisfied in a by σ iff $v(a, \sigma(q)) = t$.

Definition 17. Let $\mathcal{W} = \langle W, \leq, v\rangle$ be a generalized $\mathcal{L}$-frame, and let $\diamond$ be an n-ary connective of $\mathcal{L}$.

1. $\mathcal{W}$ *respects* an introduction rule r for $\diamond$ if $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$ whenever all the premises of r are satisfied in a by a substitution σ such that $\sigma(p_i) = \psi_i$ for $1 \leq i \leq n$ (The values of $\sigma(q)$ for $q \notin \{p_1, \ldots, p_n\}$ are immaterial here).
2. $\mathcal{W}$ *respects* an elimination rule r for $\diamond$ if $v(a, \diamond(\psi_1, \ldots, \psi_n)) = f$ whenever all the premises of r are satisfied in a by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$.

[5] Another type of non-deterministic (intuitionistic) Kripke frames, based on 3-valued and 4-valued non-deterministic matrices, was used in [3,4]. Non-deterministic modal Kripke frames were recently used in [9].

3. Let **G** be a canonical Gentzen-type system for $\mathcal{L}$. $\mathcal{W}$ is **G**-legal if it respects all the rules of **G**.

Examples

- By definition, a generalized $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\supset\Rightarrow)$ iff for every $a \in W$, $v(a, \varphi \supset \psi) = f$ whenever $v(b, \varphi) = t$ for every $b \geq a$ and $v(a, \psi) = f$. Because of the persistence condition, this is equivalent to: $v(a, \varphi \supset \psi) = f$ whenever $v(a, \varphi) = t$ and $v(a, \psi) = f$. Again by the persistence condition, this is equivalent to: $v(a, \varphi \supset \psi) = f$ whenever there exists $b \geq a$ such that $v(b, \varphi) = t$ and $v(b, \psi) = f$. $\mathcal{W}$ respects $(\Rightarrow\supset)$ iff for every $a \in W$, $v(a, \varphi \supset \psi) = t$ whenever for every $b \geq a$, either $v(b, \varphi) = f$ or $v(b, \psi) = t$. Hence the two rules together impose exactly the well-known Kripke semantics for intuitionistic implication ([12]).
- A generalized $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\rightsquigarrow\Rightarrow)$ under the same conditions it respects $(\supset\Rightarrow)$. $\mathcal{W}$ respects $(\Rightarrow\rightsquigarrow)$ iff for every $a \in W$, $v(a, \varphi \rightsquigarrow \psi) = t$ whenever $v(a, \psi) = t$ (recall that this is equivalent to: $v(b, \psi) = t$ for every $b \geq a$). Note that in this case the two rules for $\rightsquigarrow$ do not always determine the value assigned to $\varphi \rightsquigarrow \psi$: if $v(a, \psi) = f$, and there is no $b \geq a$ such that $v(b, \varphi) = t$ and $v(b, \psi) = f$, then $v(a, \varphi \rightsquigarrow \psi)$ is free to be either t or f. So the semantics of this connective is non-deterministic.
- A generalized $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(T \Rightarrow)$ (see second example after Definition 9) if $v(a, \varphi T \psi) = f$ whenever $v(a, \varphi) = f$. It respects $(\Rightarrow T)$ if $v(a, \varphi T \psi) = t$ whenever $v(a, \psi) = t$. The two constraints contradict each other in case both $v(a, \varphi) = f$ and $v(a, \psi) = t$. This is a semantic explanation why Prior's "connective" T ("Tonk") is meaningless.

Definition 18. Let **G** be a canonical constructive system.

1. $\mathcal{S} \models^{seq}_{\mathbf{G}} s$ (where $\mathcal{S}$ is a set of sequents and s is a sequent) iff every **G**-legal model of $\mathcal{S}$ is also a model of s.
2. The semantic tcr $\models_{\mathbf{G}}$ between *formulas* which is induced by **G** is defined by: $\mathcal{T} \models_{\mathbf{G}} \varphi$ if every **G**-legal model of $\mathcal{T}$ is also a model of φ.

Again we have:

Proposition 3. *$\mathcal{T} \models_{\mathbf{G}} \varphi$ iff $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \models^{seq}_{\mathbf{G}} \Rightarrow \varphi$.*

4 Soundness, Completeness, Cut-Elimination

In this section we show that the two logics induced by a canonical constructive system **G** ($\vdash_{\mathbf{G}}$ and $\models_{\mathbf{G}}$) are identical. Half of this identity is given in the following theorem:

Theorem 2. *Every canonical constructive system **G** is strongly sound with respect to the semantics of **G**-legal generalized frames. In other words:*

1. *If* $T \vdash_{\mathbf{G}} \varphi$ *then* $T \models_{\mathbf{G}} \varphi$.
2. *If* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *then* $\mathcal{S} \models^{seq}_{\mathbf{G}} s$.

Proof. We prove the second part first. Assume that $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$, and $\mathcal{W} = \langle W, \leq, v\rangle$ is a **G**-legal model of $\mathcal{S}$. We show that s is locally true in every $a \in W$. Since the axioms of G and the premises of $\mathcal{S}$ trivially have this property, and the cut and weakening rules obviously preserve it, it suffices to show that the property of being locally true is preserved also by applications of the logical rules of **G**.

- First we deal with the elimination rules of **G**. Suppose $\Gamma, \diamond(\psi_1, \dots, \psi_n) \Rightarrow \theta$ is derived from $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i)\}_{1\leq i\leq m_1}$ and $\{\Gamma, \sigma(\Pi_i) \Rightarrow \theta\}_{m_1+1\leq i\leq m}$, using the elimination rule $r = \{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/\ \diamond(p_1, p_2, \dots, p_n) \Rightarrow$ (where Σ_i is empty for $m_1 + 1 \leq i \leq m$, and σ is a substitution such that $\sigma(p_j) = \psi_j$ for $1 \leq j \leq n$). Assume that all the premises of this application have the required property. Let $a \in W$. If $v(a, \psi) = f$ for some $\psi \in \Gamma$ or $v(a, \theta) = t$, then we are done. Assume otherwise. Then $v(a, \theta) = f$, and (by the persistence condition) $v(b, \psi) = t$ for every $\psi \in \Gamma$ and $b \geq a$. Hence our assumption concerning $\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i)\}_{1\leq i\leq m_1}$ entails that for every $b \geq a$ and $1 \leq i \leq m_1$, either $v(b, \psi) = f$ for some $\psi \in \sigma(\Pi_i)$, or $v(b, \sigma(\Sigma_i)) = t$. This immediately implies that every definite premise of the rule is satisfied in a by σ. Since $v(a, \theta) = f$, our assumption concerning $\{\Gamma, \sigma(\Pi_i) \Rightarrow \theta\}_{m_1+1\leq i\leq m}$ entails that for every $m_1 + 1 \leq i \leq m$, $v(a, \psi) = f$ for some $\psi \in \sigma(\Pi_i)$. Hence the negative premises of the rule are also satisfied in a by σ. Since $\mathcal{W}$ respects r, it follows that $v(a, \diamond(\psi_1, \dots, \psi_n)) = f$, as required.
- Dealing with the introduction rules is easier, and it is left for the reader.

The first part follows from the second by Propositions 1 and 3. □

For the converse, we first prove the following key result.

Theorem 3. *Let* **G** *be a canonical constructive system in* $\mathcal{L}$*, and let* $\mathcal{S} \cup \{s\}$ *be a set of sequents in* $\mathcal{L}$*. Then either there is an* $\mathcal{S}$*-cut-free proof of* s *from* $\mathcal{S}$*, or there is a* **G***-legal model of* $\mathcal{S}$ *which is not a model of* s*.*

Proof. **(outline)** Assume that $s = \Gamma_0 \Rightarrow \varphi_0$ does not have an $\mathcal{S}$-cut-free proof in **G**. Let $\mathcal{F}'$ be the set of subformulas of $\mathcal{S} \cup \{s\}$. Given a formula $\varphi \in \mathcal{F}'$, call a theory $\mathcal{T} \subseteq \mathcal{F}'$ φ*-maximal* if there is no finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \varphi$ has an $\mathcal{S}$-cut-free-proof from $\mathcal{S}$, but every proper extension $\mathcal{T}' \subseteq \mathcal{F}'$ of $\mathcal{T}$ contains such a finite subset Γ. Obviously, if $\Gamma \subseteq \mathcal{F}'$, $\varphi \in \mathcal{F}'$ and $\Gamma \Rightarrow \varphi$ has no $\mathcal{S}$-cut-free-proof from $\mathcal{S}$, then Γ can be extended to a theory $\mathcal{T} \subseteq \mathcal{F}'$ which is φ-maximal. In particular: Γ_0 can be extended to a φ_0-maximal theory $\mathcal{T}_0$.

Now let $\mathcal{W} = \langle W, \subseteq, v\rangle$, where:

- W is the set of all extensions of $\mathcal{T}_0$ in $\mathcal{F}'$ which are φ-maximal for some $\varphi \in \mathcal{F}'$.
- v is defined inductively as follows. For atomic formulas:

$$v(\mathcal{T}, p) = \begin{cases} t & p \in \mathcal{T} \\ f & p \notin \mathcal{T} \end{cases}$$

Suppose $v(\mathcal{T}, \psi_i)$ has been defined for all $\mathcal{T} \in W$ and $1 \leq i \leq n$. We let $v(\mathcal{T}, \diamond(\psi_1, \dots, \psi_n)) = t$ iff at least one of the following holds:

1. There exists an introduction rule for $\diamond$ whose set of premises is satisfied in $\mathcal{T}$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$.
2. $\diamond(\psi_1, \dots, \psi_n) \in \mathcal{T}$ and there does not exist $\mathcal{T}' \in W, \mathcal{T} \subseteq \mathcal{T}'$, and an elimination rule for $\diamond$ whose set of premises is satisfied in $\mathcal{T}'$ by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$.[6]

First we prove that $\mathcal{W}$ is a generalized $\mathcal{L}$-frame:

- W is not empty because $\mathcal{T}_0 \in W$.
- That v is persistent is proved by structural induction.

Next we prove that $\mathcal{W}$ is **G**-legal:

1. The introduction rules are directly respected by the first condition in v's definition.
2. Let r be an elimination rule for $\diamond$, and suppose all its premises are satisfied in some $\mathcal{T} \in W$ by a substitution σ such that $\sigma(p_i) = \psi_i$. Then neither of the conditions under which $v(\mathcal{T}, \diamond(\psi_1, \dots, \psi_n)) = t$ can hold: the second by definition, and the first because of **G**'s coherence.

It remains to prove that $\mathcal{W}$ is a model of $\mathcal{S}$ but not of s. For this we first prove that the following hold for every $\mathcal{T} \in W$ and every formula $\psi \in \mathcal{F}'$:

(a) If $\psi \in \mathcal{T}$ then $v(\mathcal{T}, \psi) = t$.
(b) If $\mathcal{T}$ is ψ-maximal then $v(\mathcal{T}, \psi) = f$.

(a) and **(b)** are proved together by a simultaneous induction on the complexity of ψ. We omit the details here.

Next we note that **(b)** can be strengthened as follows:

(c) If $\psi \in \mathcal{F}'$, $\mathcal{T} \in W$ and there is no finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \psi$ has an $\mathcal{S}$-cut-free-proof from $\mathcal{S}$, then $v(\mathcal{T}, \psi) = f$.

Indeed, under these conditions $\mathcal{T}$ can be extended to a ψ-maximal theory $\mathcal{T}'$. Now $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and by **(b)**, $v(\mathcal{T}', \psi) = f$. Hence also $v(\mathcal{T}, \psi) = f$.

Now **(a)** and **(b)** together imply that $v(\mathcal{T}_0, \psi) = t$ for every $\psi \in \Gamma_0 \subseteq \mathcal{T}_0$, and $v(\mathcal{T}_0, \varphi_0) = f$. Hence $\mathcal{W}$ is not a model of s. We end the proof by showing that $\mathcal{W}$ is a model of $\mathcal{S}$. So let $\psi_1, \dots, \psi_n \Rightarrow \theta \in \mathcal{S}$ and let $\mathcal{T} \in W$, where $\mathcal{T}$ is φ-maximal. Assume by way of contradiction that $v(\mathcal{T}, \psi_i) = t$ for $1 \leq i \leq n$, while $v(\mathcal{T}, \theta) = f$. By **(c)**, for every $1 \leq i \leq n$ there is a finite $\Gamma_i \subseteq \mathcal{T}$ such that $\Gamma_i \Rightarrow \psi_i$ has an $\mathcal{S}$-cut-free-proof from $\mathcal{S}$. On the other hand $v(\mathcal{T}, \theta) = f$ implies (by **(a)**)

[6] This inductive definition isn't totally formal, since satisfaction by a substitution is defined for a generalized $\mathcal{L}$-frame, which we are in the middle of constructing, but the intention should be clear.

that $\theta \notin \mathcal{T}$. Since $\mathcal{T}$ is φ-maximal, it follows that there is a finite $\Sigma \subseteq \mathcal{T}$ such that $\Sigma, \theta \Rightarrow \varphi$ has an $\mathcal{S}$-cut-free-proof from $\mathcal{S}$. Now from $\Gamma_i \Rightarrow \psi_i$ $(1 \leq i \leq n)$, $\Sigma, \theta \Rightarrow \varphi$, and $\psi_1, \ldots, \psi_n \Rightarrow \theta$ one can infer $\Gamma_1, \ldots, \Gamma_n, \Sigma \Rightarrow \varphi$ by $n+1$ $\mathcal{S}$-cuts (on $\psi_1, \ldots, \psi_n$ and θ). It follows that the last sequent has an $\mathcal{S}$-cut-free-proof from $\mathcal{S}$. Since $\Gamma_1, \ldots, \Gamma_n, \Sigma \subseteq \mathcal{T}$, this contradicts the φ-maximality of $\mathcal{T}$. □

Theorem 4. (**Soundness and Completeness**) Every canonical constructive system **G** is strongly sound and complete with respect to the semantics of **G**-legal generalized frames. In other words:

1. $T \vdash_{\mathbf{G}} \varphi$ iff $T \models_{\mathbf{G}} \varphi$.
2. $\mathcal{S} \vdash_{\mathbf{G}}^{seq} s$ iff $\mathcal{S} \models_{\mathbf{G}}^{seq} s$.

Proof. Immediate from Theorems 3 and 2, and Propositions 1, 3. □

Corollary 1. *If* **G** *is a canonical constructive system in* $\mathcal{L}$ *then* $\langle \mathcal{L}, \models_{\mathbf{G}} \rangle$ *is a logic.*

Corollary 2. (**Compactness**) *Let* **G** *be a canonical constructive system.*

1. *If* $\mathcal{S} \models_{\mathbf{G}}^{seq} s$ *then there exists a finite* $\mathcal{S}' \subseteq \mathcal{S}$ *such that* $\mathcal{S}' \models_{\mathbf{G}}^{seq} s$.
2. $\models_{\mathbf{G}}$ *is finitary.*

Theorem 5.

1. (**General Strong Cut Elimination Theorem**) *Every canonical constructive system* **G** *admits strong cut-elimination (see Definition 12).*
2. (**General Cut Elimination Theorem**) *A sequent is provable in a canonical constructive system* **G** *iff it has a cut-free proof there.*

Proof. The first part follows from Theorem 4 and Theorem 3. The second part is a special case of the first, where the set $\mathcal{S}$ of premises is empty. □

Corollary 3. *The following conditions are equivalent for a canonical single-conclusion Gentzen-type system* **G***:*

1. $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ *is a logic (by Proposition 2, this means that* $\vdash_{\mathbf{G}}$ *is consistent).*
2. **G** *is coherent.*
3. **G** *admits strong cut-elimination.*
4. **G** *admits cut-elimination.*

Proof. 1 implies 2 by Theorem 1. 2 implies 3 by Theorem 5. 3 trivially implies 4. Finally, without using cuts there is no way to derive $p_1 \Rightarrow p_2$ in a canonical Gentzen-type system. Hence 4 implies 1. □

5 Analycity and Decidability

In general, in order for a denotational semantics of a propositional logic to be useful and effective, it should be *analytic*. This means that to determine whether a formula φ follows from a theory $\mathcal{T}$, it suffices to consider *partial* valuations, defined on the set of all subformulas of the formulas in $\mathcal{T} \cup \{\varphi\}$. Now we show that the semantics of **G**-legal frames is analytic in this sense.

Definition 19. Let **G** be a canonical constructive system for $\mathcal{L}$. A **G**-legal *semiframe* is a triple $\mathcal{W}' = \langle W, \leq, v' \rangle$ such that:

1. $\langle W, \leq \rangle$ is a nonempty partially ordered set.
2. v' is a partial function from the set of formulas of $\mathcal{L}$ into the set of persistent functions from W into $\{t, f\}$ such that:
   - $\mathcal{F}'$, the domain of v', is closed under subformulas.
   - v' respects the rules of **G** on $\mathcal{F}'$ (e.g.: if r is an introduction rule for an n-ary connective $\diamond$, and $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}'$, then $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$ whenever all the premises of r are satisfied in a by a substitution σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$).

Theorem 6. *Let* **G** *be a canonical constructive system for* $\mathcal{L}$. *Then the semantics of* **G**-*legal frames is* analytic *in the following sense: If* $\mathcal{W}' = \langle W, \leq, v' \rangle$ *is a* **G**-*legal* semiframe, *then* v' *can be extended to a function* v *so that* $\mathcal{W} = \langle W, \leq, v \rangle$ *is a* **G**-*legal frame.*

Proof. Let $\mathcal{W}' = \langle W, \leq, v' \rangle$ be a **G**-legal semiframe. We recursively extend v' to a total function v. For atomic p we let $v(p) = v'(p)$ if $v'(p)$ is defined, and $v(p) = \lambda a \in W.t$ (say) otherwise. For $\varphi = \diamond(\psi_1, \ldots, \psi_n)$ we let $v(\varphi) = v'(\varphi)$ whenever $v'(\varphi)$ is defined, and otherwise we define $v(\varphi, a) = f$ iff there exists an elimination rule r with $\diamond(p_1, \ldots, p_n) \Rightarrow$ as its conclusion, and an element $b \geq a$ of W, such that all premises of r are satisfied in b (with respect to $\langle W, \leq, v \rangle$) by a substitution σ such that $\sigma(p_j) = \psi_j$ $(1 \leq j \leq n)$. Note that the satisfaction of the premises of r by σ in elements of W depends only on the values assigned by v to $\psi_1, \ldots, \psi_n$, so the recursion works, and v is well defined. From the definition of v and the assumption that $\mathcal{W}'$ is a **G**-legal semiframe, it immediately follows that v is an extension of v', that $v(\varphi)$ is a persistent function for every φ (so $\mathcal{W} = \langle W, \leq, v \rangle$ is a generalized $\mathcal{L}$-frame), and that $\mathcal{W}$ respects all the elimination rules of **G**. Hence it only remains to prove that it respects also the introduction rules of **G**. Let $r = \{\Pi_i \Rightarrow q_i\}_{1 \leq i \leq m} / \ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ be such a rule, and assume that for every $1 \leq i \leq m$, $\sigma(\Pi_i) \Rightarrow \sigma(q_i)$ is true in a with respect to $\langle W, \leq, v \rangle$. We should show that $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$.

If $v'(a, \diamond(\psi_1, \ldots, \psi_n))$ is defined, then since its domain is closed under subformulas, for every $1 \leq i \leq n$ and every $b \in W$ $v'(b, \psi_i)$ is defined. In this case, our construction ensures that for every $1 \leq i \leq n$ and every $b \in W$ we have $v'(b, \psi_i) = v(b, \psi_i)$. Therefore, since for every $1 \leq i \leq m$, $\sigma(\Pi_i) \Rightarrow \sigma(q_i)$ is locally true in every $b \geq a$ with respect to $\langle W, \leq, v \rangle$, it is also locally true with respect to $\langle W, \leq, v' \rangle$. Since v' respects r, $v'(a, \diamond(\psi_1, \ldots, \psi_n)) = t$, so $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$ as well, as required.

Now, assume $v'(a, \diamond(\psi_1, \ldots, \psi_n))$ is not defined, and assume by way of contradiction that $v(a, \diamond(\psi_1, \ldots, \psi_n)) = f$. So, there exists $b \geq a$ and an elimination rule $\{\Delta_j \Rightarrow \Sigma_j\}_{1 \leq j \leq k} / \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ such that $\sigma(\Delta_j) \Rightarrow \sigma(\Sigma_j)$ is locally true in b for $1 \leq j \leq k$. Since $b \geq a$, our assumption about a implies that $\sigma(\Pi_i) \Rightarrow \sigma(q_i)$ is locally true in b for $1 \leq i \leq m$. It follows that by defining $u(p) = v(b, \sigma(p))$ we get a valuation u in $\{t, f\}$ which satisfies all the clauses in the union of $\{\Pi_i \Rightarrow q_i \mid 1 \leq i \leq m\}$ and $\{\Delta_j \Rightarrow \Sigma_j \mid 1 \leq j \leq k\}$. This contradicts the coherence of **G**. □

The following two theorems are now easy consequence of Theorem 6 and the soundness and completeness theorems of the previous section:[7]

Theorem 7. *Let* **G** *be a canonical constructive system. Then* **G** *is strongly decidable: Given a finite set* $\mathcal{S}$ *of sequents, and a sequent* s*, it is decidable whether* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *or not. In particular: it is decidable whether* $\Gamma \vdash_{\mathbf{G}} \varphi$*, where* φ *is formula and* Γ *is a finite set of formulas.*

Proof. Let $\mathcal{F}'$ be the set of subformulas of the formulas in $\mathcal{S} \cup \{s\}$. From Theorem 6 and the proof of Theorem 3 it easily follows that in order to decide whether $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ it suffices to check all triples of the form $\langle W, \subseteq, v' \rangle$ where $W \subseteq 2^{\mathcal{F}'}$ and $v' : \mathcal{F}' \to (W \to \{t, f\})$, and see if any of them is a **G**-legal semiframe which is a model of $\mathcal{S}$ but not a model of s. □

Theorem 8. *Let* $\mathbf{G_1}$ *be a canonical constructive system in a language* $\mathcal{L}_1$*, and let* $\mathbf{G_2}$ *be a canonical constructive system in a language* $\mathcal{L}_2$*. Assume that* $\mathcal{L}_2$ *is an extension of* $\mathcal{L}_1$ *by some set of connectives, and that* $\mathbf{G_2}$ *is obtained from* $\mathbf{G_1}$ *by adding to the latter canonical rules for connectives in* $\mathcal{L}_2 - \mathcal{L}_1$*. Then* $\mathbf{G_2}$ *is a conservative extension of* $\mathbf{G_1}$ *(i.e.: if all formulas in* $\mathcal{T} \cup \{\varphi\}$ *are in* $\mathcal{L}_1$ *then* $\mathcal{T} \vdash_{\mathbf{G_1}} \varphi$ *iff* $\mathcal{T} \vdash_{\mathbf{G_2}} \varphi$*).*

Proof. Suppose that $\mathcal{T} \not\vdash_{\mathbf{G_1}} \varphi$. Then there is $\mathbf{G_1}$-legal model $\mathcal{W}$ of $\mathcal{T}$ which is not a model of φ. Since the set of formulas of $\mathcal{L}_1$ is a subset of the set of formulas of $\mathcal{L}_2$ which is closed under subformulas, Theorem 6 implies that $\mathcal{W}$ can be extended to a $\mathbf{G_2}$-legal model of $\mathcal{T}$ which is not a model of φ. Hence $\mathcal{T} \not\vdash_{\mathbf{G_2}} \varphi$. □

Note. In [7] (his famous response to [13]), Belnap suggested that the rules for a connective $\diamond$ should be *conservative*, in the sense that if $\mathcal{T} \vdash \varphi$ is derivable using them, and $\diamond$ does not occur in $\mathcal{T} \cup \varphi$, then $\mathcal{T} \vdash \varphi$ can also be derived without using the rules for $\diamond$. Now our notion of coherence provides an effective necessary and sufficient criterion for checking whether a given set of canonical rules is conservative in this sense. Moreover: Theorem 8 shows that a very strong form of Belnap's conservativity criterion is valid for canonical constructive systems, and so what a set of canonical rules defines is system-independent.

[7] The two theorems can also be proved directly from the cut-elimination theorem for canonical constructive systems. We leave this to the full paper.

6 Related and Further Works

There have been several works in the past on conditions for cut-elimination. Except for [6], the closest to the present one is [8]. The range of systems dealt with there is in fact broader than ours, since it deals with various types of structural rules, while in this paper we assume the standard structural rules of minimal logic. On the other hand, our coherence criterion is much simpler than the reductivity criterion of [8], while our strong cut-elimination is stronger then the reductive cut-elimination of [8]. Another crucial similarity is that both papers use nondeterministic semantic frameworks (in [8] this is only implicit). However, while we use the concrete framework of intuitionistic-like Kripke frames, variants of the significantly more abstract phase semantics are used in [8].

Another difference is that unlike the present work, [8] treats also systems which allow the use in derivations of negative sequents. Our next task is to extend our framework and results so they apply to systems of this sort as well.

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